PROPERTIES OF DUALITY IN TANDEM QUEUEING SYSTEMS

GENJI YAMAZAKI AND HIROTAKA SAKASEGAWA

(Received Dec. 19, 1974)

1. Introduction and summary

Duality in the queueing theory has been discussed by Finch [2], Foster [3], Ghosal [4] and others in connection with storage problems since the late 1950's. For example, Foster considered in his descriptive paper queues with maximum of N waiting-spaces and derived duality principles to relate the number of customers to the number of waiting spaces, since the number of waiting spaces decreased by one as the number of customers increased by one. In relationship between the number of customers and the number of waiting spaces, the arrival and service processes are simply reversed. This is the reason why the dual system in a single server queueing system has commonly been defined by interchanging the arrival process and the service process in the primary system. On the other hand, duality in tandem queueing systems has been applied only to the following special cases. Gordon and Newell [5] considered cyclic queueing systems where N customers advanced sequentially in clockwise. They studied the relationship between a customer and a waiting space as follows: when a customer completed his service at the ith stage and advanced to the $(i+1)$th stage, waiting spaces at the $(i+1)$th stage were decreased by one and those at the ith stage were increased by one just like one of waiting spaces at the $(i+1)$th stage completed her 'service', that is, waiting spaces advanced sequentially in counterclockwise. Therefore they defined the dual system as reversing the order of service in the primary system. Makino [8] analyzed the reversibility for some two and three stage tandem queueing systems where he compared the ordinary system with the reversed order-of-service system. The above authors treated negative exponential service processes and have applied the standard method of the queueing analysis to these problems. When all service time distributions are negative exponential, the process of queue lengths becomes Markovian so the balance equation method can be applied though there still remains insurmountable computational problems. However, when service processes are more complicated and realistic, an alternative method must be developed to analyze the system.

In this paper we treat more realistic case where service time distributions of all stages are arbitrary. For the duality in tandem queueing systems we accept the definition of Gordon et al. (cited above) and derive three equalities concerning the duality. The first equality is on the time required for N customers in batch arrival completing their services; the second equality is concerning with the interdeparture time between the nth and the $(n+1)$th customers when the system starts from such a state that a large number of customers are waiting in front of the first stage but no customers in another place; third and the last equality is most important and usuful and which is on the capacity of the system. Now if we consider such systems where the order of performing services can be changed (for example, scheduling and similar problems), the optimal order of service is the most important and should be researched. When we discuss this optimal ordering, the capacity is taken up as one of good measures. Therefore it is desirable that we select the order of service which maximizes this capacity. For such an optimization of the system, theorems proved here show that analyses can be reduced by half. As an example of the above problem, we derive in Section 4 that in certain three stage tandem queueing systems the maximum capacity is obtainable by arranging the server with minimum expected service time at the second stage.

2. Descriptions of systems

In this section we give descriptions of systems and some notations which are used in this paper. First, the ordinary tandem queueing system each stage of which is consisted of a single server is considered as follows:

Model I: There are K servers (namely $A_1, A_2, \cdots, A_K$) arranged in tandem (see Fig. 1 (a)). Customers arrive at the system and queue up for service by the server A_1. Each customer receives service from the server A_1 in order of arrival, and next from the second server A_2 and so forth until the service by the last server A_K is completed. The service discipline is first-come and first-served at each server. There are no customers defections at any point. The initial queue length may grow unlimitedly, but on and after the second queue, a finite number of customers (N_k-1 in front of the server A_{k+1}) are permitted to wait. If there are N_k-1 customers waiting in front of the server A_{k+1} and the service by the server A_k is completed, the customer just completed his service cannot advance any more and he is compelled to wait and idles the server A_k until one waiting space opens up (so-called blocking state).

We define in the next place the dual system of Model I by reversing the order of service as stated in the preceding section.

Model II: There are K servers ($A_1, A_2, \cdots, A_K$ same as Model I) arranged in tandem in reversed order (that is, $A_K, A_{K-1}, \cdots, A_1$). There may be permitted N_k-1 customers waiting in front of the server A_k. The others are the same as Model I.

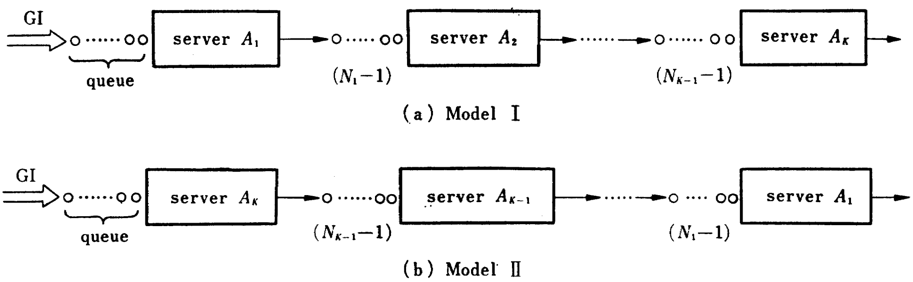

Fig. 1

We assume that $\{a_n\}_{n=1,2,\dots}$ and $\{S_{k,n}, S^*_{k,n}\}_{k=1,2,\cdots,K;\, n=1,2,\dots}$ are sequences of random variables defined on a probability space (Ω, B, P). Let a_n be the nth arrival epoch in Model I and Model II. $S_{k,n}$ ($S^*_{k,n}$) shows the nth service time of the server A_k in Model I (Model II), and $\{S_{k,n}, S^*_{k,n}\}_{n=1,2,\dots}$ are mutually independent and are identically distributed (i.i.d.). Let $T_{k,n}$ ($T^*_{k-1,n}$) be a time epoch at which the nth customer leaves the server A_k in Model I (Model II). Let $T_{0,n}$ ($T^*_{K,n}$) be a time epoch at which the server A_1 (A_K) begins her service to the nth customer in Model I (Model II). When a_n's and $S_{k,n}$'s ($S^*_{k,n}$'s) are given, $T_{k,n}$'s ($T^*_{k,n}$'s) are perfectly determined in a recursive manner. For example, if all the N_k's are equal to 1, $T_{k,n}$'s are written as follows (with initial conditions $T_{0,1}=a_1$, $T_{k,1}=S_{1,1}+\cdots+S_{k,1}$):

$$
\begin{aligned}
(2.1)\qquad & T_{0,n}=\max(T_{1,n-1}, a_n), \\
& T_{k,n}=\max(T_{k+1,n-1}, T_{k-1,n}+S_{k,n}) \qquad (k=1, 2, \cdots, K-1), \\
& T_{K,n}=T_{K-1,n}+S_{K,n}.
\end{aligned}
$$

When we consider such tandem queueing systems that all the service time distributions are arbitrary and only finite customers are permitted to wait between servers, we can identify them with those systems where none is permitted to wait between servers by considering suitable number of servers of 0-service, which is due to Avi-Itzhak and Yadin [1]. Accordingly from now on, we assume all the N_k's are equal to one, without loss of generality.

3. Duality

We first suppose that N customers arrive in batch at the system and they find the entire system empty. Considering time required for servicing N customers in both systems, we can obtain the following

LEMMA 1. *When the order of servicing customers in Model* II *is reversed in contrast to Model* I, *two quantities of time required for a batch of N customers to complete service in both models are equal, that is, the following identity is valid for any K, N and $\omega \in \Omega$.*

$$T^*_{0,N}(\omega) - T^*_{K,1}(\omega) = T_{K,N}(\omega) - T_{0,1}(\omega) . \tag{3.1}$$

PROOF. We suppress explicit dependence of ω in the following for the convenience of notation. $T_{k,n}$'s are defined recursively as (2.1) where all a_n's are equal to 0. $T^*_{k,n}$'s are also defined recursively in the same manner. Reversing the order of servicing customers for two models, the following is valid for any k and n.

$$S^*_{k,n} = S_{k,N-n+1} . \tag{3.2}$$

First we define $k(N), k(N-1), \cdots, k(2)$ as follows:

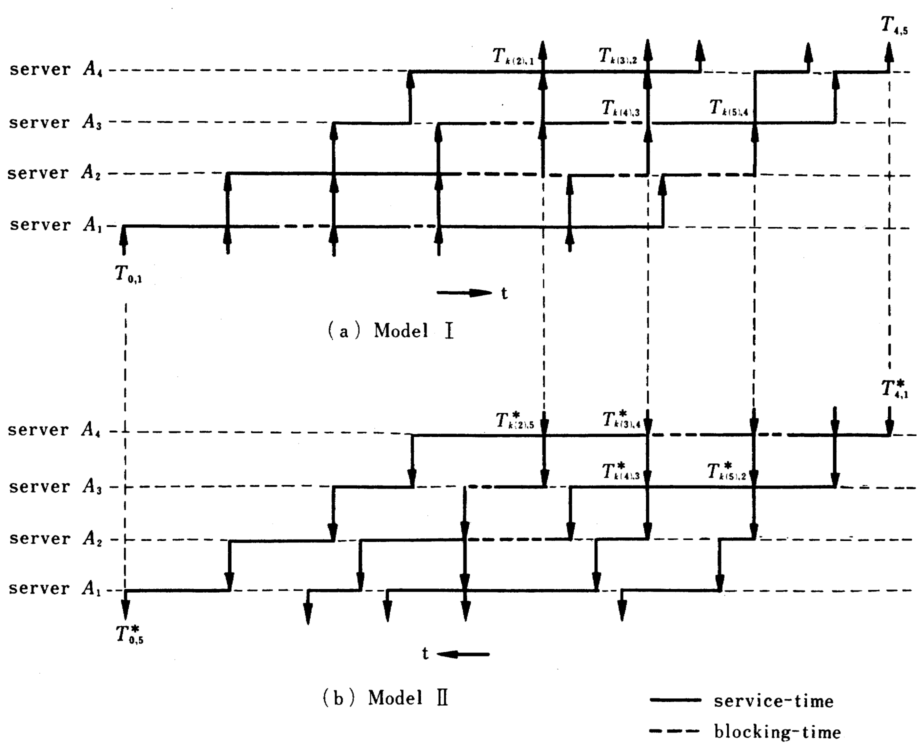

Fig. 2 A sample path (K=4, N=5).

(3.3) $$k(N)=\max(k;\ T_{k,N-1}=T_{k-1,N}),$$

$$k(n)=\max(k\leqq k(n+1)+1;\ T_{k,n-1}=T_{k-1,n})$$
$$(\text{for } n=N-1, N-2, \cdots, 2).$$

Existence of these $k(n)$'s is assured by the fact that for any n, $T_{1,n-1}=T_{0,n}$. Then we shall prove, by means of double mathematical induction, the following inequalities.

(3.4) $$T^*_{k,N-n+2}-T^*_{K,1}\leqq T_{K,N}-T_{k,j-1}$$
(equality holds at least for $k=k(j)$.)

(a) In case of $j=N$: From the definition of $k(N)$, the Nth customer advances from the $k(N)$th stage to the last stage without blocking, accordingly $T_{K,N}-T_{k-1,N}=S_{k,N}+\cdots+S_{K,N}$ $(k\geqq k(N))$. For k less than $k(N)$, clearly $T_{K,N}-T_{k-1,N}\geqq S_{k,N}+\cdots+S_{K,N}$. On the other hand, the first customer in Model II does not take any blocking state at all, so $T^*_{k-1,1}-T^*_{K,1}=S^*_{k,1}+\cdots+S^*_{K,1}$ for any k. Using (3.2) the next (3.5) is established.

(3.5) $$T^*_{k-1,1}-T^*_{K,1}=T_{K,N}-T_{k-1,N}\qquad (k\geqq k(N)),$$

$$T^*_{k-1,1}-T^*_{K,1}\leqq T_{K,N}-T_{k-1,N}\qquad (k<k(N)).$$

Now we prove (3.4) by means of the mathematical induction. First, clearly, $T^*_{K,2}-T^*_{K,1}\leqq T_{K,N}-T_{K,N-1}$. For common $k>0$,

(3.6) $$\begin{aligned}T^*_{k,2}-T^*_{K,1}&=\max(T^*_{k+1,2}+S^*_{k+1,2},\ T^*_{k-1,1})-T^*_{K,1}\\&=\max(T^*_{k+1,2}-T^*_{K,1}+S^*_{k+1,2},\ T^*_{k-1,1}-T^*_{K,1})\\&\leqq\max(T_{K,N}-T_{k+1,N-1}+S_{k+1,N-1},\ T_{K,N}-T_{k-1,N})\\&=T_{K,N}-\min(T_{k+1,N-1}-S_{k+1,N-1},\ T_{k-1,N})\\&\leqq T_{K,N}-T_{k,N-1}\end{aligned}$$

because $T_{k,N-1}\leqq T_{k-1,N}$ and $T_{k,N-1}+S_{k+1,N-1}\leqq T_{k+1,N-1}$. For $k=0$,

$$T^*_{0,2}-T^*_{K,1}=T^*_{1,2}+S^*_{1,2}-T^*_{K,1}\leqq T_{K,N}-T_{1,N-1}+S_{1,N-1}\leqq T_{K,N}-T_{0,N-1}.$$

For the equality in (3.4), it is sufficient to show $T^*_{k(N),2}-T^*_{K,1}\geqq T_{K,N}-T_{k(N),N-1}$, which is derived by using the definition (3.3), the equality (3.5) and the fact that $T^*_{k(N),2}-T^*_{K,1}\geqq T^*_{k(N)-1,1}-T^*_{K,1}$. Together with the inequality (3.6) for $k=k(N)$, the equality in (3.4) is concluded.

(b) In case of $j=n$ (for j greater than n, we assume that (3.4) holds as the induction hypothesis): For $k=K$,

$$T^*_{K,N-n+2}-T^*_{K,1}=T^*_{K-1,N-n+1}-T^*_{K,1}\leqq T_{K,N}-T_{K-1,n}\leqq T_{K,N}-T_{K,n-1}.$$

For common $k>0$,

$$(3.7)\quad \begin{aligned} T^*_{k,N-n+2}-T^*_{K,1} &= \max\left(T^*_{k+1,N-n+2}+S^*_{k+1,N-n+2},\ T^*_{k-1,N-n+1}\right)-T^*_{K,1} \\ &\leqq \max\left(T_{K,N}-T_{k+1,n-1}+S_{k+1,n-1},\ T_{K,N}-T_{k-1,n}\right) \\ &= T_{K,N}-\min\left(T_{k+1,n-1}-S_{k+1,n-1},\ T_{k-1,n}\right) \\ &\leqq T_{K,N}-T_{k,n-1}\ . \end{aligned}$$

For $k=0$,

$$\begin{aligned} T^*_{0,N-n+2}-T^*_{K,1} &= T^*_{1,N-n+2}+S^*_{1,N-n+2}-T^*_{K,1} \\ &\leqq T_{K,N}-T_{1,n-1}+S_{1,n-1}\leqq T_{K,N}-T_{0,n-1}\ . \end{aligned}$$

On the other hand, $T^*_{k,N-n+2}-T^*_{K,1}\geqq T^*_{k-1,N-n+1}-T^*_{K,1}$, we can derive the equality in (3.4) for $k=k(n)=k(n+1)+1$ as follows:

$$\begin{aligned} T_{K,N}-T_{k(n),n-1}\geqq T^*_{k(n),N-n+2}-T^*_{K,1} &\geqq T^*_{k(n)-1,N-n+1}-T^*_{K,1} \\ &= T_{K,N}-T_{k(n)-1,n}=T_{K,N}-T_{k(n),n-1}\ . \end{aligned}$$

For $k=k(n)\leqq k(n+1)$, nth customer does not take any blocking state at the $k(n)$th, $\cdots$, $k(n+1)$th stage in Model I by the definition of $k(n)$, $T_{k,n}$ equals to $T_{k(n)-1,n}+S_{k(n),n}+S_{k(n)+1,n}+\cdots+S_{k,n}$ for any k such that $k(n)\leqq k\leqq k(n+1)$. In a meanwhile,

$$\begin{aligned} T^*_{k,N-n+1}-T^*_{K,1} &\geqq T^*_{k(n+1),N-n+1}+S^*_{k(n+1),N-n+1}+\cdots+S^*_{k+1,N-n+1}-T^*_{K,1} \\ &= T_{K,N}-T_{k(n+1),n}+S_{k(n+1),n}+\cdots+S_{k+1,n} \\ &= T_{K,N}-T_{k,n}\qquad (k(n)-1\leqq k\leqq k(n+1))\ . \end{aligned}$$

Together with the hypothesis of the induction, i.e.,

$$T^*_{k,N-n+1}-T^*_{K,1}\leqq T_{K,N}-T_{k,n}\ ,$$

equalities in (3.4) hold for $k(n)-1\leqq k\leqq k(n+1)$. Especially using the equality for $k=k(n)-1$

$$\begin{aligned} T^*_{k(n),N-n+2}-T^*_{K,1}\geqq T^*_{k(n)-1,N-n+1}-T^*_{K,1} &= T_{K,N}-T_{k(n)-1,n} \\ &= T_{K,N}-T_{k(n),n-1}\ . \end{aligned}$$

Using (3.7), equality for $k=k(n)$ in (3.4) is deduced, and this completes the induction.

For the proof of the lemma, let's take $j=2$ in (3.4), then

$$T^*_{k(2),N}-T^*_{K,1}=T_{K,N}-T_{k(2),1}\ ,$$

accordingly,

$$\begin{aligned} T^*_{0,N}-T^*_{K,1} &\geqq T^*_{k(2),N}+S^*_{1,N}+\cdots+S^*_{k(2),N}-T^*_{K,1} \\ &= T_{K,N}-T_{k(2),1}+S_{1,1}+\cdots+S_{k(2),1} \\ &= T_{K,N}-T_{0,1}\ . \end{aligned}$$

On the other hand, also from (3.4),

$$T^*_{0,N}-T^*_{K,1}\leqq T_{K,N}-T_{0,1}\ .$$

These two inequalities assure to hold (3.1), which completes the proof.

The following theorem is derived from the preceding lemma.

THEOREM 1. *The distribution of time required for a batch of N customers completing services for Model* II *is identical with the one for Model* I.

PROOF. Let $D_N=T_{K,N}-T_{0,1}$ and $D^*_N=T^*_{0,N}-T^*_{K,1}$. Since $T_{k,n}$'s can be expressed recursively as (2.1) with suitable initial conditions, D_N can be expressed as a function of $\{S_{k,n}\}_{k=1,2,\cdots,K;\,n=1,2,\cdots,N}$. In the same manner, D^*_N can also be expressed as a function of $S_{k,n}$'s such as

$$D^*_N=f(S_{K,N},\,S_{K-1,N},\cdots,\,S_{1,N},\,S_{K,N-1},\cdots,\,S_{1,1})\ . \tag{3.8}$$

Let D^{**} be a random variable defined as

$$D^{**}_N=f(S_{K,1},\,S_{K-1,1},\cdots,\,S_{1,1},\,S_{K,2},\cdots,\,S_{1,N})\ . \tag{3.9}$$

Since $\{S_{k,n}\}_{n=1,2,\cdots,N}$ are i.i.d., D^*_N and D^{**}_N are identically distributed. Therefore D_N and D^{**}_N are also identically distributed according to Lemma 1. Since D^{**}_N is defined as the time required for N customers completing services in Model II whose order of customers is the same as in Model I, the proof is completed.

COROLLARY. *In tandem queueing systems described above, the distribution of interdeparture time between the nth and* $(n+1)$*th customers remain unchanged even though the order of service is reversed.*

The capacity of tandem queueing systems which has been discussed by Hunt [7], Makino [8], Hildebrand [6] and others becomes a maximum input rate. In other words, this maximum rate provides us the criterion for existence of a limiting probability distribution of a waiting time. These authors have defined the capacity as the reciprocal of expected service time plus blocking time of the first stage in its limiting state. If we define the capacity as the reciprocal of the limiting expected interdeparture time from the last stage, the following lemma assures the coincidence of the two definitions.

LEMMA 2. *In the tandem queueing system with no intermediate queue with a large number of customers waiting in front of the first stage, we can get*

$$\lim_{n\to\infty}\mathrm{E}\,(T_{k,n}-T_{k,n-1})=\alpha<\infty \tag{3.10}$$

$$(\textit{for } k=1,2,\cdots,K;\ \alpha \textit{ is constant for any } k)\ .$$

PROOF. Let $B_{k,n}$ be the time that the nth customer is blocked in the server A_k and let $V_{k,n}$ be the time that the server A_k waits for the nth customer after servicing the $(n-1)$th customer. Using these random variables, $T_{k,n}-T_{k,n-1}$ is represented by $V_{k,n}+S_{k,n}+B_{k,n}$, so

$$\lim_{n\to\infty} \mathrm{E}\,(T_{k,n}-T_{k,n-1})=\lim_{n\to\infty} \mathrm{E}\,(V_{k,n})+\lim_{n\to\infty} \mathrm{E}\,(S_{k,n}+B_{k,n}) \,. \tag{3.11}$$

Now, generally, the following relation is true for any k and n.

$$S_{k,n}+B_{k,n}+V_{k,n+1}=V_{k-1,n+1}+S_{k-1,n+1}+B_{k-1,n+1} \,. \tag{3.12}$$

If we start the system with a large number of customers in front of the first stage but no customer in another place, we can use the result that $\mathrm{E}\,(S_{1,n}+B_{1,n})$ and $\mathrm{E}\,(V_{k,n})$ have finite limiting values, say, α and v_k, respectively, which is due to Hildebrand [6]. Clearly $v_1=0$ on account of the assumption. Using this result, (3.11) is true for $k=1$. If we assume that (3.11) is true for any k less than j, we can derive (3.11) for $k=j$ as follows:

$$\begin{aligned}\lim_{n\to\infty} \mathrm{E}\,(T_{j,n}-T_{j,n-1})&=\lim_{n\to\infty} \mathrm{E}\,(V_{j,n})+\lim_{n\to\infty} \mathrm{E}\,(S_{j,n}+B_{j,n})\\&=v_j+\lim_{n\to\infty} \mathrm{E}\,(S_{j-1,n+1}+B_{j-1,n+1}+V_{j-1,n+1}-V_{j,n+1})\\&=\lim_{n\to\infty} \mathrm{E}\,(V_{j-1,n})+\lim_{n\to\infty} \mathrm{E}\,(S_{j-1,n}+B_{j-1,n})\\&=\lim_{n\to\infty} \mathrm{E}\,(T_{j-1,n}-T_{j-1,n-1})=\alpha<\infty\end{aligned}$$

this completes the proof.

THEOREM 2. *In the tandem queueing system with a finite intermediate queue, the capacity of the dual system is equal to that of the primary system.*

PROOF. Using the previous lemma,

$$\lim_{n\to\infty} \mathrm{E}\,(T_{K,n}-T_{K,n-1})=\lim_{n\to\infty} \mathrm{E}\,(T_{1,n}-T_{1,n-1}) \,.$$

On the other hand, $T_{K,n}-T_{K,n-1}$ and $T^*_{0,n}-T^*_{0,n-1}$ have the same distribution function owing to the corollary to Theorem 1, accordingly,

$$\lim_{n\to\infty} \mathrm{E}\,(T^*_{K-1,n}-T^*_{K-1,n-1})=\lim_{n\to\infty} \mathrm{E}\,(T^*_{0,n}-T^*_{0,n-1})=\lim_{n\to\infty} \mathrm{E}\,(T_{1,n}-T_{1,n-1}) \,.$$

Reminding that the leftmost quantity shows the reciprocal of the capacity in the dual system, the proof is completed.

4. Optimal order of service

In Section 3 we have showed that in our tandem queueing system both primary and its dual systems are identical on the basis of the capacity. However, if we consider such systems that the order of service is permitted to be changed, there are many possible orderings besides a reversing order. In such a case the optimal ordering of service is the most important problem from the point of view of the system design. For the measure of optimality, we take the capacity discussed in Section 3, that is, it is desirable that we select such order of service that maximizes the capacity of the system.

Here we consider some three stage tandem queueing system like that of Model I which is described below in detail.

Model A: The system corresponds to the special case of Model I where $K=3$ and $N_1=N_2=1$; that is, there are three servers A_1, A_2 and A_3 arranged in tandem in this order and no customer is permitted to wait in front of both the server A_2 and A_3 (see Fig. 3 (a)).

Model B: This system is obtained from Model A by interchanging server A_1 and A_2 (see Fig. 3 (b)).

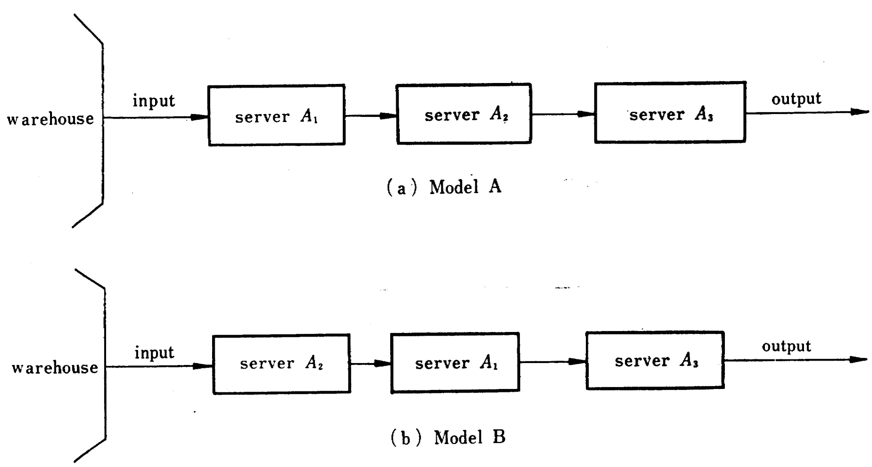

Fig. 3

We assume in both model that a large number of customers are waiting in front of the first stage initially. In the following, let $U_{k,n}$ and $U_{k,n}^*$ denote service time plus blocking time of the nth customer in the server A_k in Model A and Model B, respectively. In our context, limiting values of $\mathrm{E}(U_{1,n})$ and $\mathrm{E}(U_{2,n}^*)$ or their reciprocals are compared with each other.

For initial conditions we take

$$U_{1,0}=S_{1,0}\,,\qquad U_{2,0}=S_{2,0}\,,\qquad U_{2,0}^*=S_{2,0}^*\,,\qquad U_{1,0}^*=S_{1,0}^*\,. \tag{4.1}$$

For each n greater than 0 the following relationships are easily obtained.

$$U_{1,n}=\max(S_{1,n},\, U_{2,n-1})\,, \tag{4.2}$$

$$U_{2,n}=\max(S_{2,n},\, U_{2,n-1}+U_{3,n-1}-U_{1,n})\,, \tag{4.3}$$

$$U_{2,n}^*=\max(S_{2,n}^*,\, U_{1,n-1}^*)\,, \tag{4.4}$$

$$U_{1,n}^*=\max(S_{1,n}^*,\, U_{1,n-1}^*+U_{3,n-1}^*-U_{2,n}^*)\,. \tag{4.5}$$

Furthermore, at the third stage there does not occur any blocking state, $U_{3,n}$ and $U_{3,n}^*$ are always equal to $S_{3,n}$ and $S_{3,n}^*$, respectively. Needless to say that $\{S_{k,n}\}_{n=0,1,\dots}$ and $\{S_{k,n}^*\}_{n=0,1,\dots}$ are i.i.d. and indicate service time by the server A_k in Model A and Model B, respectively. Then we get the following

THEOREM 3. *If $S_{1,0}$ is stochastically larger than $S_{2,0}$, that is,* $\Pr(S_{1,0}\leqq x)\leqq\Pr(S_{2,0}\leqq x)$, *then next inequalities hold.*

$$\Pr(U_{1,n}\leqq x)\geqq\Pr(U_{2,n}^*\leqq x)\,, \tag{4.6}$$

$$\lim_{n\to\infty}\mathrm{E}(U_{1,n})\leqq\lim_{n\to\infty}\mathrm{E}(U_{2,n}^*)\,. \tag{4.7}$$

PROOF. In order to prove this theorem, first we establish the next

$$\Pr(U_{2,n}\leqq x)\geqq\Pr(U_{1,n}^*\leqq x)\qquad(\text{for any } n)\,. \tag{4.8}$$

These inequalities can be ascertained by means of the mathematical induction. First, using (4.2) and (4.4), we rewrite (4.3) and (4.5) follows:

$$U_{2,n}=\max(S_{2,n},\, S_{3,n-1}+0\wedge(U_{2,n-1}-S_{1,n}))\,, \tag{4.9}$$

$$U_{1,n}^*=\max(S_{1,n}^*,\, S_{3,n-1}^*+0\wedge(U_{1,n-1}^*-S_{2,n}^*))\,, \tag{4.10}$$

where $X\wedge Y$ denotes the minimum of X and Y. For $n=0$, according to (4.1) and the assumption of the theorem, (4.8) holds. For the nth step of the induction, $\Pr(U_{2,n-1}\leqq x)\geqq\Pr(U_{1,n-1}^*\leqq x)$ and $\Pr(S_{1,n}\leqq x)\leqq\Pr(S_{2,n}^*\leqq x)$ hold because of hypotheses of the induction and the theorem, we can derive

$$\Pr(0\wedge(U_{2,n-1}-S_{1,n})\leqq x)\geqq\Pr(0\wedge(U_{1,n-1}^*-S_{2,n}^*)\leqq x)\,. \tag{4.11}$$

Applying (4.11) to (4.9) and (4.10), (4.8) is easily deduced. For the inequality of (4.6), we rewrite $U_{1,n}$ and $U_{2,n}^*$ as

$$\max(S_{1,n},\, S_{2,n-1},\, S_{3,n-2}+0\wedge(U_{2,n-2}-S_{1,n-1}))\qquad\text{and}$$

$$\max(S_{2,n}^*,\, S_{1,n-1}^*,\, S_{3,n-2}^*+0\wedge(U_{1,n-2}^*-S_{2,n-1}^*))\,,$$

respectively, by (4.9) and (4.10), and using the logic by which (4.11) was deduced and (4.8) we get (4.6). (4.7) is a direct consequence from (4.6) by the fact that $U_{1,n}$ and $U_{2,n}^*$ have limiting d.f.'s (due to Hildebrand [6]).

Using Theorem 2, we can get the following

COROLLARY. *If $S_{1,0}$ and $S_{3,0}$ are both stochastically larger than $S_{2,0}$, the maximum capacity of the system is obtained by arranging the server A_2 to the second stage.*

5. Remarks

The definition of a dual system which is adopted in this paper is not unique, of course, for example if we reverse whole processes including the arrival process, that is, the service process of the last server in the primary system becomes the arrival process in its dual system, and the arrival process in the primary system becomes the service process in the dual system, another dual system can be derived. This is just like a $GI/G/1$ case where a dual system is obtained by interchanging the service and the arrival processes. Theorems we've obtained in the paper are also true for this manner of a dual system with some modification.

The capacity as a measure of the optimality of the system is one of a good measures, but is not unique, also. For example if flow rate which is defined as the reciprocal of the expected total time in system is considered as the measure of optimality, some different aspects may be presented, which will be discussed later.

Acknowledgement

We are grateful to Dr. T. Makino and Dr. R. Kato for their kind advices and also to Mr. T. Kawashima for his helpful suggestions.

KOGAKUIN UNIVERSITY
INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] Avi-Itzhak, B. and Yadin, M. (1965). A sequence of two servers with no intermediate queue, *Management Sci.*, **11**, 553-564.

[2] Finch, P. D. (1960). Deterministic customer impatience in the queueing system *GI/M/*1, *Biometrika*, **47**, 45-52.

[3] Foster, F. G. (1959). A unified theory for stock, storage and queue control, *Operat. Res. Quart.*, **10**, 121-130.

[4] Ghosal, A. (1960). Emptiness in the finite dam, *Ann. Math. Statist.*, **31**, 803-808.

[5] Gordon, W. J. and Newell, G. F. (1967). Cyclic queueing systems with restricted length queues, *Operat. Res.*, **15**, 266-277.
[6] Hildebrand, D. K. (1967). Stability of finite queue, tandem server systems, *J. Appl. Prob.*, **4**, 571-583.
[7] Hunt, G. C. (1956). Sequential arrays of waiting lines, *Operat. Res.*, **4**, 674-683.
[8] Makino, T. (1964). On the mean passage time concerning some queueing problems of the tandem type, *J. Operat. Res. Japan*, **7**, 17-47.

Ann. Inst. Statist. Math.
29 (1977), Part A, 509

CORRECTION TO "PROPERTIES OF DUALITY IN TANDEM QUEUEING SYSTEMS"

GENJI YAMAZAKI AND HIROTAKA SAKASEGAWA

In the corollary to Theorem 1 of the above paper (this Annals 27 (1975), pp. 201–212), "distribution" should be read as "expectation." The correct statement of the corollary becomes as follows.

COROLLARY. *In tandem queueing systems described above, the expectation of interdeparture time between the nth and the* $(n+1)$*th customers remain unchanged even though the order of service is reversed.*