# NOTE ON THE CONSTRUCTION OF PARTIALLY BALANCED ARRAYS*

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## Summary

The $p$-symbol partially balanced arrays of strength $t$, where $p=3$ and 4, are given by a method due to A. Dey, A. C. Kulshreshtha and G. M. Saha [4].


## 1. Introduction

Suppose $A=\|a_{ij}\|$ is an $n\times m$ matrix and the elements $a_{ij}$ of $A$ are symbols (or levels) $0, 1, 2, \cdots, s-1$. Consider the $s^t$ $1\times t$ matrices $X'=(x_1, x_2, \cdots, x_t)$ that can be formed by giving different values to the $x_i$'s, $x_i=0, 1, 2, \cdots, s-1$; $i=1, 2, \cdots, t$. Suppose that associated with each $t\times 1$ matrix $X$ there is a non-negative integer $\mu_{x_1x_2\cdots x_t}$ which is invariant under permutations of a given set $(x_1, x_2, \cdots, x_t)$. If, for every $t$-rowed submatrix of $A$, the $s^t$ $t\times 1$ matrices $X$ occur as columns $\mu_{x_1x_2\cdots x_t}$ times, then the matrix $A$ is called an $s$-symbol Partially Balanced (PB) array of strength $t$ with $m$ assemblies, $n$ constraints (or factors) and parameters $\mu_{x_1x_2\cdots x_t}$, which was first introduced by Chakravarti [2] as a substitute for the orthogonal array, both serving the purpose of fractional replicates of factorial experiments.

Recently, Dey, Kulshreshtha and Saha [4] have given a method of constructing three-symbol PB arrays of strength two and three. In this note, it is shown that three-symbol and four-symbol PB arrays of strength $t$ are constructed by using their method. Further, it is remarked that in general $p$-symbol PB arrays of strength $t$ are also constructed by the similar method.

## 2. Statements

A balanced incomplete block design with parameters $v$, $b$, $r$, $k$ and $\lambda_2$ is called a $t$-$(v, k, \lambda_t)$ design, if each set of $t$ different treatments occurs together in $\lambda_t$ blocks. It is well known that there exist these

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$t$-designs for many cases, see for example [1] and [5].

Let $N=\|n_{ij}\|$ be the incidence matrix of a $t$-$(v, k, \lambda_t)$ design, where

$$n_{ij}=\begin{cases} 1, & \text{if } i\text{th treatment occurs in } j\text{th block}, \\ 0, & \text{otherwise}, \end{cases}$$

and the $j$th assembly of this $v\times b$ array $N$ of (0, 1)-symbols be denoted by a column vector, $\boldsymbol{n}_j=(n_{1j}, n_{2j}, \cdots, n_{vj})'$. Then we can define a new column vector $\boldsymbol{n}_j^*$ from $\boldsymbol{n}_j$, given by

$$\text{(2.1)} \qquad \boldsymbol{n}_j^*=(n_{1j}^*, n_{2j}^*, \cdots, n_{vj}^*)', \qquad n_{ij}+n_{ij}^*=2$$

for all $i=1, 2, \cdots, v$; $j=1, 2, \cdots, b$.

Letting $N^*=\|n_{ij}^*\|$, we consider the columns of the matrix $A=[N: N^*]$ as $2b$ assemblies. It is shown by Chakravarti [3] that the matrix $N$ is a two-symbol PB array of strength $t$ with $b$ assemblies, $v$ constraints and parameters $\lambda(x_1, x_2, \cdots, x_t)$, where when $x_i=1$ for $i=1, 2, \cdots, r$ and $x_i=0$ for $i=r+1, \cdots, t$,

$$\begin{aligned}\lambda(x_1, x_2, \cdots, x_t)&=N_r-\binom{t-r}{1}N_{r+1}+\binom{t-r}{2}N_{r+2}-\cdots+(-1)^{t-r}\binom{t-r}{t-r}N_t\\ &=(-1)^{t-r}\Delta^{t-r}N_r\end{aligned}$$

(noting that $N_0=b$, $N_1=r$, $N_i=\lambda_i$ $(i\geqq 2)$)

as defined in (3.2) of [3]. Denote by $\mu_{x_1x_2\cdots x_t}$ the frequency of the ordered $t$-plet $(x_1, x_2, \cdots, x_t)$ as a column in any $t$-rowed submatrix of $A$. It follows that

when $x_i=0$ or 1 for $i=1, 2, \cdots, t$,

$$\text{(2.2)} \qquad \mu_{x_1x_2\cdots x_t}=\lambda(x_1, x_2, \cdots, x_t) \qquad \text{defined above};$$

when $x_i=1$ or 2 for $i=1, 2, \cdots, t$,

$$\text{(2.3)} \qquad \mu_{x_1x_2\cdots x_t}=\mu_{\varepsilon(x_1)\varepsilon(x_2)\cdots\varepsilon(x_t)},$$

where

$$\varepsilon(x_i)=\begin{cases} 1, & \text{if } x_i=1, \\ 0, & \text{if } x_i=2; \end{cases}$$

in particular,

$$\text{(2.4)} \qquad \mu_{11\cdots 1}=2\lambda_t;$$

when $x_i=0$ and $x_j=2$ for some $i, j$ $(=1, 2, \cdots, t)$,

$$\text{(2.5)} \qquad \mu_{x_1x_2\cdots x_t}=0.$$

From the definition of matrices $N$ and $N^*$, it is clear that $\mu_{x_1x_2\cdots x_t}$ is invariant under permutations of its arguments. Therefore we have the following:

THEOREM. *The existence of a* $t$-$(v, k, \lambda_t)$ *design implies the existence of a three-symbol PB array of strength* $t$ *with* $2b$ *assemblies,* $v$ *constraints and parameters* $\mu_{x_1x_2\cdots x_t}$ *as given in* (2.2), (2.3), (2.4) *and* (2.5).

The special cases of this theorem when $t=2$ and 3 are shown by Dey, Kulshreshtha and Saha [4].

## 3. Concluding remarks

In the above construction of a three-symbol PB array of strength $t$ with $2b$ assemblies, $v$ constraints and parameters $\mu_{x_1x_2\cdots x_t}$, if $n^*_{ij}$ in (2.1) may be defined by $n_{ij}+n^*_{ij}=3$, then it is clear that there exists a four-symbol PB array of strength $t$ with $2b$ assemblies, $v$ constraints and the following parameters $\mu_{x_1x_2\cdots x_t}$:

when $x_i=0$ or 1 for $i=1, 2, \cdots, t$,

$$\mu_{x_1x_2\cdots x_t}=\lambda(x_1, x_2, \cdots, x_t) ;$$

when $x_i=2$ or 3 for $i=1, 2, \cdots, t$,

$$\mu_{x_1x_2\cdots x_t}=\mu_{\varepsilon(x_1)\varepsilon(x_2)\cdots\varepsilon(x_t)} ,$$

where

$$\varepsilon(x_i)=\begin{cases} 1, & \text{if } x_i=2 , \\ 0, & \text{if } x_i=3 ; \end{cases}$$

when $x_i=0$ or 1 and $x_j=2$ or 3 for some $i, j$ $(i\neq j)=1, 2, \cdots, t$,

$$\mu_{x_1x_2\cdots x_t}=0 .$$

Generally, if $n^*_{ij}$ in (2.1) may be defined by $n_{ij}+n^*_{ij}=l-1$, then there exists an $l$-symbol PB array. In the $l$-symbol PB array constructed by this method, however, the levels $2, 3, \cdots, l-3$ do not appear completely among the $l$ levels. On the other hand, for the purpose of the use of all the levels (for example, $l=6$ levels), if we consider a matrix $B=[N : N^* : N^{**}]$ with $3b$ columns as assemblies defined by $N^{**}=\|n^{**}_{ij}\|$, $n_{ij}+n^*_{ij}=3$ and $n^*_{ij}+n^{**}_{ij}=7$, then the array $B$ contains all the levels 0, 1, 2, 3, 4, 5. Nevertheless, since the number of assemblies of the array $B$ increases, we cannot get the point of this method so much for the large value of symbols in an array.

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