ON COMPLEXES OF ABELIAN GROUPS WITH APPLICATIONS TO FRACTIONAL FACTORIAL DESIGNS*

H. Pesotan, B. L. Raktoe and W. T. Federer

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1. Introduction

Although the theory of factorial and fractional factorial designs has now a history of many years, it is only recently that some of its deeper mathematical and combinatorial aspects are being exhibited and investigated. In this connection one of the earliest papers which used an abstract algebraic system to obtain results in confounding theory in the case of the symmetrical prime power factorials and hence also fractional replication is the classical paper of Fisher [3]. Subsequently, Bose [1] utilized finite geometrical methods to discuss and resolve certain problems in the confounding and enumeration for the prime powered symmetrical factorial. These two papers are pioneering efforts from the viewpoint of utilizing abstract mathematical systems in the study of symmetrical factorials.

The present paper is not only in the spirit of the above mentioned two authors but also formulates the algebraic and combinatorial aspects of arbitrary fractional factorial designs from arbitrary factorials, using only the basic notions of group theory. In general, we have studied a universal class of fractional factorial designs obtained from selecting subsets of arbitrary cardinality from the group of treatment combinations with the aim to characterize and enumerate subclasses of designs such that the spectra of the underlying information matrices are the same with respect to an allowable set of parameters. In addition, we provide a method such that these subclasses of designs generate the whole universal class. The impetus for this development arose from a paper by Raktoe and Federer [5] and a paper by Srivastava, Raktoe and Pesotan [6] which were preceded by the fundamental paper of Paik and Federer [4].

In Section 2 we introduce the concepts of faithful and unfaithful

complexes for an arbitrary finite abelian group G and study the basic properties of these complexes. In particular, we characterize unfaithful complexes as precisely those subsets of G which are set unions of distinct cosets belonging to a nonzero subgroup of G. We also introduce here the notion of length for an unfaithful complex and obtain criteria for an unfaithful complex to be of a given length. In Section 3 we present recursive formulae which count the cardinality of the set of all unfaithful complexes of a given order ν, and we use these formulae to obtain the cardinality of the smallest set of complexes of order ν under translation. In the paper by Raktoe and Federer [5] the term "generator" was used for both the terms faithful and unfaithful complexes. In this paper we have abandoned this terminology since the term generator has a distinct meaning in group theory. In the paper by Raktoe and Federer [5] a formula is given connecting the cardinality of all the main effect plans in the s^t factorial, $s = p^{\alpha}$, p a prime, in terms of the number of generators which produce s^t plans and those which do not. This formula is not true in general, though it holds for many classes of complexes, and in Section 3 we give a counter example to this effect. In addition, in this section we give necessary and sufficient conditions under which the Raktoe-Federer formula holds. In Section 4 we show how the theory developed in Sections 2 and 3 may be applied to classify and enumerate designs. In particular, we show that the class of all designs of a distinct order ν may be generated by the operation of translation followed by replication on a minimal set $\mathcal{J}_{\nu}$ of complexes of order ν. In addition, we describe a general classification problem for designs of distinct order ν and show how the theory developed in the previous sections helps in the resolution of this problem.

2.   Characterization and properties of faithful and unfaithful complexes

Let G be a finite additive abelian group. We will denote the operation in G by the symbol $+$. We call a nonempty subset H of G a *complex* and we refer to the cardinality of the set H as the *order* of the complex H. For any g in G we mean by $H+g$ the complex $H+g = \{h+g \mid h \in H\}$. In this section we introduce the concepts of *faithful* and *unfaithful* complexes as particular types of complexes of G. We establish here that unfaithful complexes are precisely those complexes of G obtainable as set unions of distinct cosets of some non-zero subgroup of G, that is a subgroup distinct from the subgroup of G consisting only of the zero element of G. As a consequence we obtain some results on the order of unfaithful complexes of G. In this section we also introduce the notion of *length* for an unfaithful complex and study the connections between a given unfaithful complex H of G of

a given length and the complex H^c which is the set complement of H in G. Finally in this section we study the action of certain permutation groups of G on a set of complexes of a given order for the particular case where G is a finite direct product of finite additive abelian groups with the componentwise operation.

DEFINITION 2.1. A subset H of G will be called an *unfaithful subset* if and only if (i) H is the empty set or (ii) H is a complex and there exists a g in G, $g \neq 0$ (0 is the zero element [additive identity] of G) such that $H + g = H$. A nonempty *unfaithful subset* will be called an *unfaithful complex*. Further a complex H will be called a *faithful complex* if and only if for every g in G such that $g \neq 0$, we have that $H + g \neq H$.

DEFINITION 2.2. Let H be any complex of G. We say that an element g in G *subtracts in* H if and only if for each h in H there exists a k in H with $h - k = g$.

PROPOSITION 2.1. A complex H of G is an unfaithful complex if and only if there exists a g in G different from the zero element of G such that g subtracts in H.

The above proposition leads us immediately to:

COROLLARY 2.1. *A complex H of G is a faithful complex if and only if every $g \neq 0$ in G does not subtract in H.*

Let H be any complex in G. Define a binary relation ρ_H in G by $g_1 \rho_H g_2$ if and only if $H + g_1 = H + g_2$. It is easily verified that ρ_H is a congruence relation on G, that is, it is an equivalence relation on G such that $g_1 \rho_H g_2$ and $f_1 \rho_H f_2$ implies $(g_1 + f_1) \rho_H (g_2 + f_2)$ where g_1, g_2, f_1, and f_2 are in G. As a consequence it follows that the set

(2.1) $$S_H(G) = \{g \mid g \in G, \ g \rho_H 0\} ,$$

where 0 is the zero element of G, is a subgroup of G. We will refer to the subgroup $S_H(G)$ as the *subgroup induced* by H, and generally, we will write S_H for $S_H(G)$ when no ambiguity arises as to the group G in question. We observe that if the complex H contains the zero element of G, then S_H is a subset of H and, further, that if H is an unfaithful (faithful) complex of G, then since G is an abelian group, $H + g$ is an unfaithful (faithful) complex of G for any g in G.

THEOREM 2.1. *Let H be a complex of G. Then H is an unfaithful complex if and only if H is a set union of distinct cosets belonging to some nonzero subgroup of G.*

PROOF. Suppose that H is an unfaithful complex. Then by definition there exists a nonzero g in G such that $H+g=H$. Hence g belongs to S_H so that S_H is a nonzero subgroup of G. Consider the relation ρ_H restricted to H and pick $h_1, h_2, \cdots, h_n$ such that $h_i \in H$, $1 \leqq i \leqq n$, is a representative from each equivalence class in the quotient set H/ρ_H. Then we claim that $H = \bigcup_{i=1}^{n} (S_H + h_i)$. Select any h in H. Then, $h \rho_H h_i$ for exactly one i ($1 \leqq i \leqq n$). This implies that $H+h = H+h_i$, that is, $H + (h - h_i) = H$. Thus $(h - h_i)$ belongs to S_H, and it follows that h belongs to $S_H + h_i$. Hence, $H \subseteq \bigcup_{i=1}^{n} (S_H + h_i)$. Next select any k in $\bigcup_{i=1}^{n} (S_H + h_i)$ then $k = s + h_i$ for some s in S_H and some i ($1 \leqq i \leqq n$). Hence k belongs to $H+s$. But $H+s=H$ since $s \in S_H$, and, thus it follows that $H = \bigcup_{i=1}^{n} (S_H + h_i)$. Select h_i and h_j with $i \neq j$ ($1 \leqq i$, $j \leqq n$) and suppose that $(S_H + h_i) \cap (S_H + h_j) \neq \phi$. Then there exist s and t in S_H with $s + h_i = t + h_j$ and thus $(h_i - h_j) \in S_H$. It follows that $H + h_i = H + h_j$, that is, $h_i \rho_H h_j$ a contradiction to the choice of h_i ($1 \leqq i \leqq n$). Hence H is a disjoint union of cosets belonging to the subgroup S_H of G. Conversely suppose that $H = \bigcup_{i=1}^{n} (S + a_i)$ is a disjoint union of cosets belonging to some nonzero subgroup S of G. Pick any s in S such that $s \neq 0$. Then since S is a subgroup we have clearly that $H + s = \bigcup_{i=1}^{n} (S + a_i + s) = \bigcup_{i=1}^{n} (S + a_i) = H$, that is, H is an unfaithful complex and this completes the proof.

DEFINITION 2.3. We say that an unfaithful complex H of G is *based* on a nonzero subgroup S of G, or alternatively that S *underlies* H if and only if H is a set union of distinct cosets of S.

COROLLARY 2.2. *Let S be any nonzero subgroup of G of order s and index t. Then any unfaithful complex H of G based on S has order ks for some k ($1 \leqq k \leqq t$).*

THEOREM 2.2. *Let H be an unfaithful complex of G. Then the subgroup S_H induced by H underlies H. Further, H is based on a nonzero subgroup S of G if and only if S is contained in S_H.*

DEFINITION 2.4. Let l be a natural number, $l \geqq 0$. We say that an unfaithful complex H of G is of *length* l if and only if H is a disjoint union of l cosets of some nonzero subgroup S of G and H cannot be written as a disjoint union of k cosets of some nonzero subgroup of G with $k < l$. If H is an unfaithful subset and $H = \phi$ (=empty set) we define its length to be zero.

The next result gives the connection between the concept of length

for an unfaithful complex H and the subgroup S_H induced by H.

THEOREM 2.3. *Let l be a positive number. The following statements are equivalent for an unfaithful complex H of G:*
(1) *H has length l.*
(2) *H is a disjoint union of l cosets belonging to S_H.*
(3) *The order of H is $l|S_H|$, where $|S_H|$ is the order of S_H.*

The following is now an immediate consequence of Theorems 2.2 and 2.3.

COROLLARY 2.3. *Each unfaithful subset H of G has an unique length l. If $H=\phi$ then $l=0$ and if $H\neq\phi$ the number l is the number of distinct cosets of S_H whose union is H.*

THEOREM 2.4. *Let G have order n and let ν be a number such that $1\leq\nu\leq n$. Then there exists an unfaithful complex of order ν if and only if the g.c.d. $(\nu, n)>1$.*

PROPOSITION 2.2. *If H is an unfaithful complex of order ν and g.c.d. $(\nu, n)=d$, then the order of any underlying subgroup of H divides d where n is the order of G.*

Let G have order n and let ν be any number such that $1\leq\nu\leq n$. Suppose that the g.c.d. $(\nu, n)=d>1$. Then $\nu=kd$ where g.c.d. $(k, n/d)=1$. Proposition 2.2 leads us to the consideration of the following sets. Let

(2.2) $$D=\{d_1, d_2, \cdots, d_t\}$$

where each d_i $(1\leq i\leq t)$ is a divisor of d and $d_i>1$ for each i. We assume that $d_t=d$ and that $d_1<d_2<\cdots<d_t$. Further, let $d=m_i d_i$ $(1\leq i\leq t)$ and let

(2.3) $$L=\{km_i: \nu=kd;\ 1\leq i\leq t\}\ .$$

In terms of the above notation we now establish the following result.

PROPOSITION 2.3. *If H is an unfaithful complex of order ν and length l, then l belongs to L.*

Let ν and l be any numbers such that $0\leq\nu$, $l\leq n$ where n is the order of the group G. Let S be any nonzero subgroup of G. We now introduce the following sets.

(2.4)   Let $F_\nu(G)$ be the set of all faithful complexes of order ν.

(2.5)   Let $U_\nu(G)$ be the set of all unfaithful subsets of order ν.

(2.6)   Let $C_\nu(G)$ be the set of all subsets of G of cardinality ν.

(2.7)   Let $X_{l,\nu}(S)$ be the set of all unfaithful subsets of length l and order ν which have underlying subgroup S.

It is clear that $C_\nu(G)=F_\nu(G)\cup U_\nu(G)$ and further that for each ν ($0\leq \nu\leq n$) this union is disjoint. Note that $X_{0,\nu}(S)$ and $X_{l,0}(S)$ are empty sets if ν and l are respectively nonzero numbers and that $X_{0,0}(S)$ is a nonempty set and equals $\{\phi\}$.

PROPOSITION 2.4. Let G have order n, let S be a subgroup of G, and let H be an unfaithful subset of G of length l and order ν. (i) If $H=\phi$, then the set complement H^c in G is an unfaithful complex of length 1. (ii) If $H\neq\phi$, let T be the subgroup induced by H and let t be its index. Then $H\in X_{l,\nu}(S)$ if and only if $H^c\in X_{t-l,n-\nu}(S)$. In particular, T is also the subgroup induced by H^c and the map φ given by $\varphi(H)=H^c$ establishes a one-to-one correspondence between the sets $U_\nu(G)$ and $U_{n-\nu}(G)$ such that φ carries the set $X_{l,\nu}(T)$ onto the set $X_{t-l,n-\nu}(T)$.

The following is now an immediate corollary of the previous proposition.

COROLLARY 2.4. *Let H be a faithful complex of order ν in G and let G have order n where $1\leq\nu\leq n$. Then, the set complement H^c in G is a faithful complex of order $(n-\nu)$ and the map φ given by $\varphi(H)=H^c$ establishes a one-to-one correspondence between the sets $F_\nu(G)$ and $F_{n-\nu}(G)$.*

PROPOSITION 2.5. Let G have order n. Then H is a faithful complex of order ν in G if and only if (i) the g.c.d. $(\nu,n)=1$ or (ii) the g.c.d. $(\nu,n)=d>1$ and for any nonzero subgroup S of G there exists b in H such that $S+b\nsubseteq H$.

For any g in G we define a permutation ω_g on the underlying set of G by $\omega_g(x)=x+g$. Let

$$(2.8)\qquad\qquad \tilde{\Omega}(G)=\{\omega_g\,|\,g\in G\}\ .$$

Clearly $\tilde{\Omega}(G)$ is a subgroup of the symmetric group of all permutations of G and is isomorphic to G under the map $g\to\omega_g$.

Let $G_1, G_2, \cdots, G_m$ be finite additive abelian groups of orders $s_1, s_2, \cdots, s_m$ respectively and let $s=\prod_{i=1}^{m} s_i$. Let $K=\underset{i=1}{\overset{m}{\times}} G_i$ be their direct product under the componentwise operation. For each i, ($1\leq i\leq m$), let $\Omega(G_i)$ be the symmetric group of order $s_i!$ on the underlying set of G_i and let

$$(2.9)\qquad\quad \Omega(K)=\underset{i=1}{\overset{m}{\times}}\Omega(G_i)=\{(\omega_1,\cdots,\omega_m)\,|\,\omega_i\in\Omega(G_i)\}$$

be the direct product of the groups $\Omega(G_i)$ under the componentwise operation of functional composition. For any $a=(a_1,\cdots,a_m)$ in K and $\omega=(\omega_1,\cdots,\omega_m)$ in $\Omega(K)$, we define $\omega(a)=(\omega_1(a_1),\cdots,\omega_m(a_m))$ and for any complex H of K we define $\omega(H)=\{\omega(h)\,|\,h\in H\}$. Then for any $g=(g_1,\cdots,g_m)$ in K the permutation ω_g in $\tilde{\Omega}(K)$ is given by $\omega_g(a)=a+g=(a_1+g_1,\cdots,a_m+g_m)$, and clearly $\tilde{\Omega}(K)$ is a subgroup of $\Omega(K)$. For any α in $\Omega(K)$ we denote by $\tilde{\Omega}(K)\alpha$ a right coset of $\tilde{\Omega}(K)$ in $\Omega(K)$ determined by α.

Let Ω_1 be any nonempty subset of $\Omega(K)$ and $\mathcal{S}$ a set of complexes of K of a given order ν $(1\leqq\nu\leqq s)$. Let

(2.10) $$\Omega_1^*\mathcal{S}=\{\alpha(H)\,|\,\alpha\in\Omega_1 \text{ and } H\in\mathcal{S}\}\,.$$

If $\mathcal{S}=\{H\}$, we write Ω_1^*H instead of $\Omega_1^*\mathcal{S}$.

DEFINITION 2.5. We call the set $\Omega_1^*\mathcal{S}$ the set generated by the action of Ω_1 on the set of complexes $\mathcal{S}$. If $\Omega_1=\Omega(K)$ and $\mathcal{S}=\{H\}$, we say that any two sets in $\Omega(K)^*H$ are *related by a permutation*, and if $\Omega_1=\tilde{\Omega}(K)\alpha$ with $\alpha\in\Omega(K)$, then we say that any two sets in $\tilde{\Omega}(K)\alpha^*H$ are *related by a translation*.

Given $H\in C_\nu(K)$, $(1\leqq\nu\leqq s)$, we now ask what are the sets $\mathcal{S}\subseteq C_\nu(K)$ which have the following two properties: (i) $\tilde{\Omega}(K)^*\mathcal{S}=\Omega(K)^*H$ and (ii) if $\mathcal{J}$ is any set such that $\mathcal{J}\subseteq C_\nu(K)$ with $\tilde{\Omega}(K)^*\mathcal{J}=\tilde{\Omega}(K)^*\mathcal{S}$, then $|\mathcal{S}|\leqq|\mathcal{J}|$, that is, what are the sets $\mathcal{S}\subseteq C_\nu(K)$ of least cardinality such that the class generated by the action of $\Omega(K)$ on H is the same as the class generated by the action of $\tilde{\Omega}(K)$ on $\mathcal{S}$?

Let $R=\{\alpha_1,\cdots,\alpha_r\}$ be a set of representatives of right cosets of the subgroup $\tilde{\Omega}(K)$ in $\Omega(K)$. For H in $C_\nu(K)$ and α_i, α_j in R define a relation σ_H on R by

(2.11) $\alpha_i\sigma_H\alpha_j$ if and only if there exists a g in K such that $\omega_g(\alpha_i(H))=\alpha_j(H)$.

It is clear that σ_H is an equivalence relation on R. In the following theorem using the above notation we answer the question posed above.

THEOREM 2.5. (i) *For any α in $\Omega(K)$ and H in $C_\nu(K)$ the cardinality of the class generated by the action of the right coset $\tilde{\Omega}(K)\alpha$ on the complex H is the index of the subgroup $S_{\alpha(H)}(K)$, induced by $\alpha(H)$ in K. (ii) Let $\mathcal{S}=\{\beta_1(H),\cdots,\beta_k(H)\}$ where k is the cardinality of the quotient set R/σ_H and $\{\beta_1,\cdots,\beta_k\}$ is a set of representatives, one from each equivalence class in R/σ_H. Then (a) $\Omega(K)^*H=\cup\,\{\tilde{\Omega}(K)^*\beta_i(H)\,|\,1\leqq i\leqq k\}$ and this union is disjoint and (b) If $\mathcal{J}$ is any set of complexes contained*

in $C_\nu(K)$ such that $\Omega(K)*H = \tilde{\Omega}(K)*\mathscr{J}$, then $|\mathcal{S}| \leqq |\mathscr{J}|$.

PROOF. (i) Now by definition $\tilde{\Omega}(K)\alpha*H = \{\omega_g(\alpha(H)) \,|\, g \in K\}$. Hence $\omega_{g_1}(\alpha(H)) = \omega_{g_2}(\alpha(H))$ if and only if $\alpha(H) + g_1 = \alpha(H) + g_2$, that is, if and only if $g_1 - g_2$ belongs to $S_{\alpha(H)}(K)$. It follows that the cardinality of the set of complexes which are related by a translation to H is the index of the subgroup $S_{\alpha(H)}(K)$ in K, that is $|\tilde{\Omega}(K)\alpha*H| = |K/S_{\alpha(H)}(K)|$. (ii) Pick any H_1 in $\Omega(K)*H$. This implies that there exists an α in $\Omega(K)$ such that $H_1 = \alpha(H)$. Now $\Omega(K) = \bigcup_{i=1}^{r} \tilde{\Omega}(K)\alpha_i$ and hence there exists i, $(1 \leqq i \leqq r)$, and some $g \in K$ such that $\alpha = \omega_g\alpha_i$. Thus $H_1 = \omega_g\alpha_i(H)$. Now $\alpha_i \in R$ implies $\alpha_i \sigma_H \beta_j$ for some j $(1 \leqq j \leqq k)$, that is, $\alpha_i(H) = \omega_{g_1}(\beta_j(H))$ for some g_1 in K. Hence $H_1 = \alpha(H) = \omega_{g+g_1}(\beta_j(H)) \in \tilde{\Omega}(K)*\beta_j(H)$. It follows that $\Omega(K)*H \subseteq \tilde{\Omega}(K)*\mathcal{S}$ and since the reverse inclusion is clear, we have equality. Further suppose that for some $i \neq j$ $(1 \leqq i, j \leqq k)$, we have $[\tilde{\Omega}(K)*\beta_i(H)] \cap [\tilde{\Omega}(K)*\beta_j(H)] \neq \phi$. Then for some g_1, g_2 in K we have that $\omega_{g_1}(\beta_i(H)) = \omega_{g_2}(\beta_j(H))$. This immediately implies that $\beta_i \sigma_H \beta_j$ and this contradicts the choice of the β_i $(1 \leqq i \leqq k)$. Hence $\Omega(K)*H$ is a disjoint union of the sets $[\tilde{\Omega}(K)*\beta_i(H)]$ where $1 \leqq i \leqq k$. Next suppose that $\Omega(K)*H = \tilde{\Omega}(K)*\mathscr{J}$ for some $\mathscr{J} \subseteq C_\nu(K)$. For $\beta_i(H) \in \mathcal{S}$, where $1 \leqq i \leqq k$, $\beta_i(H) = \alpha(H)$ for some α in $\Omega(K)$, and hence $\beta_i(H) = \omega_{g_1}(H_i)$ for some H_i in $\mathscr{J}$ and $g_1 \in K$. Consider the map φ from $\mathcal{S}$ into $\mathscr{J}$ which assigns to each $\beta_i(H)$ in $\mathcal{S}$ the complex H_i in $\mathscr{J}$ such that $\beta_i(H) = \omega_{g_1}(H_i)$ for some g_1 in K. Suppose that $\varphi(\beta_i(H)) = \varphi(\beta_j(H))$. Then for some g_1, g_2 in K we have that $\beta_i(H) = \omega_{g_1}(H_i) = \omega_{g_1-g_2}(\omega_{g_2}(H_j)) = \omega_{g_1-g_2}\beta_j(H)$. This immediately implies that $\beta_i \sigma_H \beta_j$ and hence by the choice of the β_j's we must have that $\beta_i = \beta_j$. Hence the map φ is one-to-one and $|\mathcal{S}| \leqq |\mathscr{J}|$. This completes the proof.

3. On counting the number of faithful and unfaithful complexes

Throughout this section the following notation will be used: (i) We will denote the order of the abelian group G by n and ν will be a fixed number such that $1 \leqq \nu \leqq n$. If X is any set, we will denote the cardinality of X by $|X|$. (ii) For any number q, $1 \leqq q \leqq n$, such that q divides n, let $K_q(G)$ denote the set of all subgroups of G of order q. When it is clear from the context which abelian group G is being considered, we shall write K_q for $K_q(G)$. For any numbers q_1, q_2 such that $1 \leqq q_1 < q_2 \leqq n$ and for a given subgroup $T \in K_{q_2}$ let $R(K_{q_1}, T) = \{S \,|\, S \in K_{q_1}$ and S is a subgroup of $T\}$, if q_1 divides q_2, and, let $R(K_{q_1}, T) = \phi$ otherwise. Further, let $\psi_G(q) = |K_q|$ and let $\psi_G(q_1, T) = |R(K_{q_1}, T)|$. Note that $\psi_G(q_1, T) = 0$ if q_1 does not divide q_2. (iii) For any numbers s and l, $(1 \leqq s, l \leqq n)$, we define the set $X_{l,\nu}(K_s) = \{H \,|\, H \in U_\nu(G), H$ has length l

and an underlying subgroup $S \in K_s$}, that is, the set of all unfaithful complexes of order ν and length l which have an underlying subgroup of order s. Since the number ν is fixed throughout this section we will write $X_l(K_s)$ for $X_{l,\nu}(K_s)$. Similarly we write $X_l(S)$ for the set $X_{l,\nu}(S)$ defined by (2.7).

We introduce a binary relation τ in the set of complexes of order ν, namely in $C_\nu(G)$, as follows:

(3.1) $H_1 \tau H_2$ if and only if there exists a g in G such that $\omega_g(H_1)$
        $= H_2$ where ω_g is the mapping introduced in (2.8).

It is clear that τ is an equivalence relation on $C_\nu(G)$. In this section we will consider restrictions of τ to various subsets of $C_\nu(G)$. When no confusion is likely as to the group G in question we will write F_ν, U_ν, and C_ν respectively for the sets $F_\nu(G)$, $U_\nu(G)$, $C_\nu(G)$ and we will denote the corresponding quotient sets by F_ν/τ, U_ν/τ, and C_ν/τ respectively.

We denote an equivalence class under τ by $[H]_\tau$. Then, we have clearly that

(3.2) $\tilde{\Omega}(G)^* H = [H]_\tau$,

that is, the class $[H]_\tau$ consists of complexes of order ν which are related by translation to H, indeed, $[H]_\tau = \{H' \mid H' \in C_\nu,\ H' = H + g,\ g \in G\}$. Further, it is clear that $[H]_\tau$ belongs to exactly one of F_ν/τ or U_ν/τ according as H belongs to F_ν or U_ν respectively.

Let $\mathcal{S} = \{H_1, H_2, \cdots, H_c\}$ be a set of representatives one from each equivalence class of the quotient set C_ν/τ. Then, clearly, $\mathcal{S}$ is the set with the smallest cardinality such that $\tilde{\Omega}(G)^* \mathcal{S} = C_\nu(G)$, that is, (a) $C_\nu(G)$ $= \{H_i + g \mid 1 \leq i \leq c,\ g \in G\}$ and (b) if $\tilde{\Omega}(G)^* \mathcal{J} = C_\nu(G)$ for some set $\mathcal{J} \subseteq C_\nu(G)$ then $|\mathcal{S}| \leq |\mathcal{J}|$.

The main objectives in this section are the following:
(1) We provide recursive formulae which give the cardinality of the set of unfaithful complexes of order ν which have a prescribed length and underlying subgroup. In general these formulae require knowing the cardinalities of the sets of all subgroups of G of a given order containing, respectively contained in, a fixed subgroup of G.
(2) In the special case such that (i) G is a cyclic group or (ii) G is a direct product, $G = (C_{p_1} \times \cdots \times C_{p_1}) \times (C_{p_2} \times \cdots \times C_{p_2}) \times \cdots \times (C_{p_m} \times \cdots \times C_{p_m})$ where C_{p_i}, $(1 \leq i \leq m)$, is a cyclic group of order p_i and $p_1, p_2, \cdots, p_m$ are distinct primes we show that the formulae mentioned in (1) reduce to recursive formulae which allow one to calculate the cardinality c of the quotient set C_ν/τ. Hence we have a method in the case G has the particular forms indicated to compute the cardinality of the set $\mathcal{S}$ of complexes satisfying the properties (a) and (b) mentioned above. In

the general case, that is, for an arbitrary abelian group G, the cardinality c of the quotient set C_{ν}/τ could be similarly computed if the cardinalities of the sets of subgroups of a given order containing, respectively contained in, a fixed subgroup of G are known

(3) We state a formula given in Raktoe and Federer [5] which connects the cardinalities of the sets $U_{\nu}(G)/\tau$, $F_{\nu}(G)/\tau$, and $C_{\nu}(G)/\tau$ for the special case where G is a direct product, $G=C_{s_1}\times\cdots\times C_{s_t}$ where for each i, $(1\leq i\leq t)$, $s_i=p^\alpha$, p is a fixed prime and C_{s_i} a cyclic group of order s_i and $\nu=t(p^\alpha-1)+1$. We show by an example that, in general, the R-F formula is false. We then provide necessary and sufficient conditions under which the R-F formula holds.

We begin by providing the basic setting in which the results of this section will be formulated.

Suppose that the g.c.d. $(\nu, n)=d$ and let $D=\{d_1, d_2,\cdots, d_t\}$, $L=\{km_i: 1\leq i\leq t\}$ be the sets introduced in (2.2) and (2.3). Recall that $1<d_1<d_2<\cdots<d_t$, and $d_t=d$, and $d=m_id_i$ for each i $(1\leq i\leq t)$. Set $b_i=km_i$, $(1\leq i\leq t)$. Then by Proposition 2.3 the various possible lengths for unfaithful complexes of order ν occur amongst the set $\{b_1, b_2, \cdots, b_t\}$. Let $r_i=n/d_i$, $(1\leq i\leq t)$, and set $r_t=r$, $b_t=b$. Note that $b_1>b_2>\cdots>b_t$.

LEMMA 3.1. *For each i, $(1\leq i\leq t)$, we have the following: (a) The set $X_{b_i}(S)$ is precisely the set of all unfaithful complexes of order ν and length b_i which have induced subgroup S if and only if the order of S is d_i. (b) The set $X_{b_i}(K_{d_i})=\cup\{X_{b_i}(S)|S\in K_{d_i}\}$ and this set union is pairwise disjoint. In particular the set $X_{b_i}(K_{d_i})$ is precisely the set of all unfaithful complexes of order ν and length b_i which have an induced subgroup of order d_i. (c) If $H\in X_{b_i}(S)$, where S is some nonzero subgroup of G, then $[H]_\tau\subseteq X_{b_i}(S)$.*

LEMMA 3.2. *The set of all unfaithful complexes of order ν, namely, $U_\nu=\cup\{X_{b_i}(K_{d_i})|1\leq i\leq t\}$ and this set union is pairwise disjoint.*

LEMMA 3.3. *For each i, $(1\leq i\leq t)$, and a given subgroup S of G such that $|S|=d_i$, let $D_i(S)$ be the set of all unfaithful complexes of length l, where $l\leq b_i$, and underlying subgroup S,   Then we have that*

$$(3.3)\qquad D_i(S)=X_{b_i}(S)\cup[\cup\{X_{b_j}(T_{jy})|i+1\leq j\leq t \text{ and } 1\leq y\leq x_j\}]$$

where for each j, $(i+1\leq j\leq t)$, the set $\{T_{j1}, T_{j2},\cdots, T_{jx_j}\}$ is the set of all subgroups of order d_j which contain S. Moreover, this representation of $D_i(S)$ as a set union is pairwise disjoint.

LEMMA 3.4. *Let T be a subgroup of G of order d_j and let i be any number such that $1\leq i<j\leq t$. Each unfaithful complex $H\in X_{b_j}(T)$ gives*

rise to exactly $\phi_G(d_i, T) = |R(K_{d_i}, T)|$ *representations of* H *as disjoint unions of cosets belonging to some subgroup of order* d_i. *Hence the total number of distinct representations of the members of the set* $X_{b_j}(K_{d_j})$ *as disjoint set unions of cosets belonging to subgroups* S *of order* d_i *is given by* $\sum\limits_{y=1}^{\phi_G(d_j)} \phi_G(d_i, T_{jy}) |X_{b_j}(T_{jy})|$, *where* $\{T_{jy} | 1 \leq y \leq \phi_G(d_j)\}$ *is the set of all subgroups of* G *of order* d_j.

We now in the next result derive a recursive formula giving the cardinality of the set of complexes of order ν and length l which have a fixed subgroup S as their induced subgroup.

THEOREM 3.1. *For each* i, $(1 \leq i \leq t)$, *the cardinality of the set of all unfaithful complexes of order* ν *and length* b_i *which have an induced subgroup* S *of order* d_i, *that is, the cardinality of the set* $X_{b_i}(S)$ *where* S *is a subgroup of order* d_i, *is given by the following recursive formula: For* $i = t$ *so that* $d_t = d$, $b_t = b$ *and* $r_t = r$ *we have*

$$(3.4) \qquad\qquad |X_b(S)| = C_b^r$$

and for $1 \leq i \leq t-1$ *we have*

$$(3.5) \qquad\qquad |X_{b_i}(S)| = C_{b_i}^{r_i} - \sum_{j=i+1}^{t} \sum_{y=1}^{x_j} |X_{b_j}(T_{jy})|$$

where for each j, $(i+1 \leq j \leq t)$, $\{T_{j_1}, \cdots, T_{jx_j}\}$ *is the set of all subgroups of* G *of order* d_j *which contain* S.

THEOREM 3.2. *For each* i, $(1 \leq i \leq t)$, *the cardinality of the set of all unfaithful complexes of order* ν *and length* b_i *which have an induced subgroup of order* d_i, *that is, the cardinality of the set* $X_{b_i}(K_{d_i})$ *is given by the following recursive formula: For* $i = t$, *so that* $d_t = d$, $b_t = b$ *and* $r_t = r$:

$$(3.6) \qquad\qquad |X_b(K_d)| = \phi_G(d) C_b^r$$

and for i *such that* $1 \leq i \leq t-1$:

$$(3.7) \qquad |X_{b_i}(K_{d_i})| = \phi_G(d_i) C_{b_i}^{r_i} - \sum_{j=i+1}^{t} \sum_{y=1}^{\phi_G(d_j)} \phi_G(d_i, T_{jy}) |X_{b_j}(T_{jy})|$$

where $\{T_{jy} | 1 \leq y \leq \phi_G(d_j)\}$ *is the set of all subgroups of* G *of order* d_j *and* $|X_{b_j}(T_{jy})|$ $(i+1 \leq j \leq t)$ *is given by* (3.4) *and* (3.5).

For each i, $(1 \leq i \leq t)$, let $c_i = |X_{b_i}(K_{d_i})/\tau|$, that is, let c_i be the number of equivalence classes into which the set $X_{b_i}(K_{d_i})$ is decomposed under the relation τ. We let $c_i = 0$ in case $X_{b_i}(K_{d_i}) = \phi$.

THEOREM 3.3. *For each* i, $(1 \leq i \leq t)$, *the number of equivalence classes*

c_i into which the set $X_{b_i}(K_{d_i})$ is decomposed under τ is given by $c_i = (1/r_i) \cdot |X_{b_i}(K_{d_i})|$. Hence the least number of unfaithful complexes of order ν, length b_i and induced subgroup of order d_i, which will generate the set $X_{b_i}(K_{d_i})$ under the action of $\tilde{\Omega}(G)$ is given by $(1/r_i)|X_{b_i}(K_{d_i})|$.

COROLLARY 3.1.   *The cardinality of the quotient set U_ν/τ is given by* $\sum_{i=1}^{t} \frac{1}{r_i} |X_{b_i}(K_{d_i})|$ *where for each i, $(1 \leq i \leq t)$, the number $|X_{b_i}(K_{d_i})|$ is given by the recursive formulae (3.6) and (3.7).*

THEOREM 3.4.   *The cardinality f of the quotient set F_ν/τ is given by*

$$(3.8) \qquad f = \frac{1}{n}\left[C_\nu^n - \sum_{i=1}^{t} c_i r_i \right] = \frac{1}{n}[C_\nu^n - |U_\nu|] \,.$$

Thus, in particular, the set consisting of faithful complexes of order ν one from each equivalence class in F_ν/τ which generate all of F_ν under the action of the group $\tilde{\Omega}(G)$ has cardinality f, given by formula (3.8).

COROLLARY 3.2.   *The cardinality c of the quotient set C_ν/τ is given by*

$$(3.9) \qquad c = \frac{1}{n}\left[C_\nu^n - \sum_{i=1}^{t} c_i r_i \right] + \sum_{i=1}^{t} c_i$$

where for each i, $(1 \leq i \leq t)$, $c_i = (1/r_i)|X_{b_i}(K_{d_i})|$ and the number $|X_{b_i}(K_{d_i})|$ is given by the recursive formulae (3.6) and (3.7). Thus, in particular, the set consisting of complexes of order ν, one from each equivalence class in C_ν/τ which generates all of C_ν under the action of the group $\tilde{\Omega}(G)$, has cardinality c given by (3.9).

In the next result we obtain a formula which counts the number of complexes of order ν which contain the zero element of G.

THEOREM 3.5.   (i) *The number of unfaithful complexes of order ν ($\nu \geq 2$) in G which contain the zero element of G is given by* $\sum_{i=1}^{t} c_i b_i$.   (ii) *The number of faithful complexes of order ν ($\nu \geq 2$) in G which contain the zero element of G is given by $f\nu$ where $f = |F_\nu/\tau|$. Hence the total number of complexes of order ν ($\nu \geq 2$) in G which contain the zero of G is given by $\sum_{i=1}^{t} c_i b_i + f\nu$ and in particular $\sum_{i=1}^{t} c_i b_i + f\nu = C_{\nu-1}^{n-1}$.*

PROOF.   (i) Pick any unfaithful complex H in U_ν containing the zero element of G. Then by Lemma 3.2, $H \in X_{b_i}(K_{d_i})$ for exactly one i ($1 \leq i \leq t$). Thus there exists a group S of order d_i which is the group induced by H such that $H = S \cup (S + a_1) \cup \cdots \cup (S + a_{b_i - 1})$, a disjoint union. Clearly the sets $H, H + (-a_1), H + (-a_2), \cdots, H + (-a_{b_i - 1})$ are all distinct;

otherwise, in case $H+(-a_i)=H+(-a_j)$, it follows that $a_j-a_i \in S=S_H$ and thus $S+a_i=S+a_j$, a contradiction to the choice of the a_i. Further these sets $H, H+(-a_1), \cdots, H+(-a_{b_i-1})$ all belong to $[H]_r$. Hence there are exactly b_i sets in each equivalence class $[H]_r$ which contain the zero of G. Now by definition $c_i=|X_{b_i}(K_{a_i})/\tau|$ and hence it follows that there are exactly $c_i b_i$ unfaithful complexes in $X_{b_i}(K_{a_i})$ which contain the zero of G. Using Lemma 3.2 we immediately obtain that there are $\sum_{i=1}^{t} c_i b_i$ unfaithful complexes of order ν containing the zero element of G. (ii) Pick any H in F_ν, and suppose $H=\{a_1, a_2, \cdots, a_\nu\}$. Then clearly the sets $H+(-a_1), H+(-a_2), \cdots, H+(-a_\nu)$ are all distinct, since H is faithful, and each contains the zero element of G. Hence in each equivalence class $[H]_r$ with H in F_ν there are precisely ν faithful complexes of order ν each containing the zero of G. Since $|F_\nu/\tau|=f$, it follows that there are precisely $f\nu$ faithful complexes of order ν in G containing the zero element of G, and (ii) is established. Finally consider the set $G'=G-\{0\}$ where 0 is the zero element of G. There are exactly $C_{\nu-1}^{n-1}$ sets of order $\nu-1$ in G'. If $\nu \geq 2$ then each of these sets in nonempty and lacks the zero of G. Adding the zero element of G to each of these sets gives $C_{\nu-1}^{n-1}$ sets of order ν in G containing the zero element of G. It follows that $\sum_{i=1}^{t} c_i b_i + f\nu = C_{\nu-1}^{n-1}$.

COROLLARY 3.3. *If the g.c.d. $(\nu, n)=1$ and $\nu \geq 2$ then $U_\nu=\phi$ and the cardinality f of the quotient set F_ν/τ is $f=C_{\nu-1}^{n-1}/\nu$. If $\nu=1$, then $f=1$. In particular if the g.c.d. $(\nu, n)=1$, then ν divides $C_{\nu-1}^{n-1}$ and n divides C_ν^n.*

Suppose that K is an abelian group such that K is a direct product, $K=\overset{\alpha}{\underset{i=1}{\times}} C_i$, where for each i $(1 \leq i \leq \alpha)$ C_i is a cyclic group and the order of each C_i is a fixed prime p. Then $|K|=p^\alpha$. The following formula giving the number of subgroups of K of order p^β is well known, for example, see Carmichael [2]:

$$(3.10) \qquad \phi_K(p^\beta)=\frac{(p^\alpha-1)(p^\alpha-p)\cdots(p^\alpha-p^{\beta-1})}{(p^\beta-1)(p^\beta-p)\cdots(p^\beta-p^{\beta-1})} .$$

In the next result G_0 will stand for either a cyclic group of order n or an abelian group of order n of the form $G_0=(C_{p_1} \times \cdots \times C_{p_1}) \times (C_{p_2} \times \cdots \times C_{p_2}) \times \cdots \times (C_{p_m} \times \cdots \times C_{p_m})$ where for each i, $(1 \leq i \leq m)$, C_{p_i} is a cyclic group of order p_i, $p_1, \cdots, p_m$ are distinct primes and the group C_{p_i} occurs α_i times. Thus in this case $n=p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$.

Let $G_i=C_{p_i} \times C_{p_i} \times \cdots \times C_{p_i}$ so that $G_0=G_1 \times G_2 \times \cdots \times G_m$. Then it is clear that the number of subgroups of G_0 of order $s=p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ is the number $\phi_{G_0}(s)=\phi_{G_1}(p_1^{\beta_1}) \phi_{G_2}(p_2^{\beta_2}) \cdots \phi_{G_m}(p_m^{\beta_m})$ where $\phi_{G_i}(p_i^{\beta_i})$ is given by the formula (3.10).

Note further that any two subgroups of G_0 of the same order are isomorphic. For each i, $(1 \leq i \leq t)$, let S_i be a subgroup of G_0 of order d_i and for any j such that $1 \leq i < j \leq t$ let $\psi_{G_0}(d_i, S_j) = a_{ij}$. Then by the remark just made if T is any subgroup of G_0 of order d_j we have immediately that $\psi_{G_0}(d_i, S_j) = \psi_{G_0}(d_i, T) = a_{ij}$ for each i, j $(1 \leq i < j \leq t)$. In the next result we show that the formulae (3.6) and (3.7) reduce to formulae involving the numbers a_{ij} in the case the group G has the form of G_0.

THEOREM 3.6. *For each* i $(1 \leq i \leq t)$ *the cardinality of the set of all unfaithful complexes of* G_0 *of order* ν *and length* b_i *which have an induced subgroup of order* d_i *is given by the following formula*: *For* $i = t$ *so that* $d_t = d$, $b_t = b$ *and* $r_t = r$,

$$(3.11) \qquad\qquad |X_b(K_d)| = \psi_{G_0}(d)C_b^r$$

and for $1 \leq i \leq t-1$

$$(3.12) \qquad |X_{b_i}(K_{d_i})| = \psi_{G_0}(d_i)C_{b_i}^{r_i} - \sum_{j=i+1}^{t} a_{ij}|X_{b_j}(K_{d_j})|$$

and $a_{ij} = \psi_{G_0}(d_i, S_j)$.

We illustrate some of the ideas developed thus far in this section with the following example:

Example 3.1. Let $G = \overset{7}{\underset{i=1}{\times}} C_i$ be the direct product of seven cyclic groups C_i each of order 2. Then $|G| = n = 128$. Let $\nu = 8$ so that the g.c.d. $(\nu, n) = d = 8$. Hence the set of divisors of d distinct from 1 are $D = \{2, 4, 8\}$. Let $d_1 = 2$, $d_2 = 4$, $d_3 = 8$, then $b_1 = 4$, $b_2 = 2$, $b_3 = 1$ so that the set $L = \{1, 2, 4\}$, and $r_1 = 64$, $r_2 = 32$, and $r_3 = 16$. Let S_1 and S_2 be subgroups of G of orders 4 and 8 respectively. Using formula (3.10) we now have that (i) $\psi_G(2) = 127$, $\psi_G(4) = 2667$, $\psi_G(8) = 11,811$ and that (ii) $\psi_G(2, S_1) = 3$, $\psi_G(4, S_2) = 7$, $\psi_G(2, S_2) = 7$. Now using Theorem 3.6 the number of unfaithful complexes of length 1 is $|X_1(K_d)| = |X_1(K_8)| = \psi_G(8)C_1^{16}$. The number of unfaithful complexes of length 2 is $|X_2(K_4)| = (\psi_G(4))C_2^{32} - [\psi_G(8)C_1^{16}]\psi_G(4, S_2) = 0$. The number of unfaithful complexes of length 4 is $|X_4(K_2)| = \psi_G(2)C_4^{64} - |X_2(K_4)|\psi_G(2, S_1) - |X_1(K_8)|\psi_G(2, S_2) = 79,369,920$. Hence using Lemma 3.2 the total number of unfaithful complexes of order 8, that is, $|U_8| = |X_1(K_8)| + |X_2(K_4)| + |X_4(K_2)| = 79,558,896$. Using formula (3.8) of Theorem 3.4 the cardinality f of the quotient set F_s/τ is $f = 1/128[C_8^{128} - |U_8|] = 11,168,930,418$. Using Corollary 3.1 the cardinality u of the quotient set U_8/τ is $u = |X_1(K_8)|/16 + |X_2(K_4)|/32 + |X_4(K_2)|/64 = 1,251,966$. Finally using formula (3.9) of Corollary 3.2 the cardinality c of the quotient set C_8/τ is $c = f + u = 11,170,182,384$. Hence if one selects c sets one from each equivalence class in C_8/τ the resulting

set is the set of least cardinality such that the action of $\tilde{\Omega}(G)$ on it generates all of C_8.

Paik and Federer [4] proposed the name "generator" for both faithful and unfaithful complexes of a group and this term was subsequently used by Raktoe and Federer [5]. We have departed in this paper from this terminology because the term generator has a distinct meaning in group theory. Raktoe and Federer [5] gave the following formula relating the cardinalities of C_ν, F_ν, and U_ν:

$$(3.13) \qquad\qquad (\phi_G(\nu))a + f \cdot s^t = C_\nu^{s^t}$$

where (1) $G = \overset{t}{\underset{i=1}{\times}} C_{s_i}$, a direct product of t cyclic groups C_{s_i}, each of order $s_i = s$ and $s = s_i = p^\alpha$ for each i, p is a fixed prime, and $\alpha \geqq 1$ is a fixed number, (2) $\nu = t(s-1)+1$, (3) $a =$ index of a subgroup of order ν, and (4) $f =$ cardinality of the quotient set F_ν/τ.

The theory developed thus far shows that the R-F formula is not true in general, since the formula assumes that all unfaithful complexes of order ν are of length one, that is, it assumes that all unfaithful complexes of order ν are subgroups of G or cosets belonging to a subgroup of G of order ν. Since unfaithful complexes of length other than one do exist in general, it is clear that the number f in formula (3.13) merely provides an upper bound to the cardinality of F_ν/τ. A comparison of formula (3.13) with formula (3.8) which gives the cardinality of F_ν/τ illustrates that those unfaithful complexes of order ν and length other than one have not been taken into account in formula (3.13). We now present a counter example to formula (3.13) by exhibiting an unfaithful complex of length other than one in a particular case.

Example 3.2. Let $G = \overset{5}{\underset{i=1}{\times}} C_i$ be the direct product of cyclic groups C_i where for each i, $C_i = \{0, 1\}$ under addition mod. 2. Hence in this case $s^t = 2^5$ and $\nu = 6$. Since there are no subgroups of order six in G, we have that $\phi_G(6) = 0$ so that according to formula (3.13) $f(2^5) = C_6^{2^5}$ which implies that $2^5 = 32$ divides C_6^{32}. This, however, is impossible. Further there are unfaithful complexes of order six. For example, take $H = \{(0, 0, 0, 0, 0), (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (1, 1, 1, 1, 1), (0, 1, 1, 1, 1), (1, 1, 0, 0, 0)\}$. Then H is an unfaithful complex of length three whose unique induced subgroup is $\{(0, 0, 0, 0, 0), (1, 0, 0, 0, 0)\}$.

We now give necessary and sufficient conditions under which the formula (3.13) holds where we remove the restriction placed on ν as given in (2) and allow ν to be any given number such that $1 \leqq \nu \leqq s^t$. The following lemmas will be useful.

LEMMA 3.5. *Suppose that $C_s = \{0, 1, 2, \cdots, s-1\}$ is a cyclic group under addition mod. s and let $s = p^\alpha$ where p is a prime and α is any number such that $\alpha > 1$. Then for each i, $1 < i < \alpha$, there exists an unfaithful complex of order p^i and length p in C_s.*

LEMMA 3.6. *Let $G = \overset{m}{\underset{i=1}{\times}} C_{s_i}$ be a direct product of cyclic groups C_{s_i} of order s_i where for each i, $1 \leq i \leq m$, $s_i = s = p^\alpha$, p a fixed prime and α any number such that $\alpha > 1$. Suppose that $m > 1$. Then for any k, $1 \leq k < m$, there exists an unfaithful complex of order s^k and an unfaithful complex of order $s^k p$ in G each of length p.*

LEMMA 3.7. *Let $G = G_1 \times G_2$ be the direct product of two nonzero finite abelian groups G_1 and G_2. Let K be any subgroup of G_1 and let H be any unfaithful complex of length l in G_2. Let S be the subgroup induced by H. Then $H_1 = K \times H$ is an unfaithful complex of length l and the subgroup induced by H_1 is $K \times S$.*

PROPOSITION 3.1. Let $G = \overset{t}{\underset{m=1}{\times}} C_{s_m}$ be a direct product of cyclic groups C_{s_m} each of order $s = s_m$ where for each m, $s = s_m = p^\alpha$, p a fixed prime, $t \geq 1$, and α is any number such that $\alpha > 1$. Then for each i such that $1 < i < \alpha\, t$ there exists an unfaithful complex of G of order p^i and length p.

LEMMA 3.8. *Let $G = \overset{m}{\underset{j=1}{\times}} C_{p_j}$ be a direct product of cyclic groups C_{p_j} of order $p = p_j$, where for each j, $p_j = p$ and p is a fixed prime. Suppose that $m \geq 2$ and let K be any subgroup of G such that $K \neq G$. Then for any $x \notin K$ the cyclic subgroup S of G generated by x is such that $S \cap K = \{0\}$.*

PROPOSITION 3.2. Let $G = \overset{m}{\underset{j=1}{\times}} C_{p_j}$ be a direct product of cyclic groups C_{p_j} each of order $p_j = p$ where for each j, $p_j = p$ is a fixed prime such that $p \neq 2$. Suppose that $m \geq 3$. Then for each i such that $1 < i < m$ there exists an unfaithful complex in G of order p^i and length p.

PROPOSITION 3.3. Let $G = \overset{m}{\underset{j=1}{\times}} C_j$ be a direct product of cyclic groups C_j, where for each j, $C_j = \{0, 1\}$ is a cyclic group of order two under addition mod. 2. Then (i) there exists no unfaithful complexes in G of length two, (ii) if $m \geq 4$ then for each i such that $2 < i < m$ there exists an unfaithful complex of order 2^i and length four, (iii) if $1 \leq m \leq 3$, then every unfaithful complex of G is either of length one, that is, it is a coset belonging to some subgroup of G, or it is of length three.

In the next result we provide criteria under which formula (3.13)

holds when the restriction placed on ν is removed and we allow ν to be any number such that $1 \leqq \nu \leqq s^t$.

THEOREM 3.7. *Formula (3.13) holds for a given ν such that $1 \leqq \nu \leqq s^t$ if and only if one of the following conditions is satisfied*: (i) *the g.c.d.* $(\nu, p) = 1$, (ii) $\nu = s^t$, (iii) $\nu = p$, (iv) $G = C_2 \times C_2 \times C_2$ *and* $\nu \neq 6$ *where* (a) $s = p^\alpha$, p *a fixed prime*, $\alpha \geqq 1$, $t \geqq 1$ *and* (b) $C_2 = \{0, 1\}$ *is the cyclic group of order two under addition* mod. 2.

PROOF. Let ν be a number such that $1 \leqq \nu \leqq s^t$. Suppose that formula (3.13) holds, that is, suppose that $\phi_G(\nu) a + f s^t = C_\nu^{s^t}$ where $f = |F_\nu / \tau|$, a is the index of a subgroup of G of order ν and $G = C_s \times \cdots \times C_s$ (t factors). Then since $f s^t = |F_\nu|$ and $C_\nu^{s^t} = |C_\nu|$, it follows that $\phi_G(\nu) a = |U_\nu|$ so that if (3.13) holds then every unfaithful complex of G of order ν must be a coset belonging to some subgroup of G of order ν. Thus if (3.13) holds then every unfaithful complex of order ν has length one. Now suppose that the conditions (i), (ii) and (iii) fail to hold. We deduce then that (iv) must hold. Suppose then that the g.c.d. $(\nu, p) = d > 1$. Let $\nu = m p^i$ where p^i is the highest power of p dividing ν. Then the g.c.d. $(m, p) = 1$. Suppose now that $m > 1$. Let K be any subgroup of G of order p^i. Since we assume that the condition (ii) fails, we have that $\nu = m p^i < s^t = p^{\alpha t}$. Hence the index of K is larger than m. Let $H = K \cup (K + a_1) \cup \cdots \cup (K + a_{m-1})$, a union of distinct cosets belonging to K. Then H is an unfaithful complex of order $\nu = m p^i$. Further H cannot be a subgroup of G since $m > 1$ implies that ν does not divide the order of G. Thus H has length $m > 1$ and this is a contradiction to our assumption that (3.13) holds. Hence $m = 1$ and as a consequence $\nu = p^i$. Again since $\nu \neq p$, $\nu \neq s^t = p^{\alpha t}$ we must have that $1 < i < \alpha t$. If $\alpha > 1$ then by Proposition 3.1 there exists an unfaithful complex of order p^i and length p which would contradict our assumption that (3.13) holds. Thus $\alpha = 1$ and $1 < i < t$, which means that $t \geqq 3$. Again if $p \neq 2$, then by Proposition 3.2 there exists an unfaithful complex of order $\nu = p^i$ and length p contradicting our assumption that (3.13) holds. Hence we must have that $p = 2$, so that $\nu = 2^i$, $t \geqq 3$, and $1 < i < t$. Now if $t \geqq 4$, then by Proposition 3.3 there exists an unfaithful complex of order 2^t and length 4 which would contradict our assumption that (3.13) holds. Hence we must have that $t = 3$ so that $1 < i < t$ implies that $i = 2$. This means finally that $G = C_2 \times C_2 \times C_2$ and $\nu = 2^i = 2^2$, that is, $\nu \neq 6$. Hence condition (iv) has been deduced. Conversely, if conditions (ii) or (iii) hold, it is clear that (3.13) holds. If condition (i) holds, that is, if the g.c.d. $(\nu, p) = 1$, then by Proposition 2.4 we have that $U_\nu = \phi$ so that $C_\nu = F_\nu$ and it follows immediately that (3.13) holds. Finally suppose that $G = C_2 \times C_2 \times C_2$ and $\nu \neq 6$. Then $|G| = 8$ so that if ν is any number such that $1 \leqq \nu \leqq 8$ and the g.c.d. $(\nu, 2) = 1$, then again by Proposition 2.4 $U_\nu = \phi$ so

that $F_\nu = C_\nu$ and (3.13) holds. If $\nu = 2$ then all unfaithful complexes of G of order two are of length 1 and if $\nu = 8$, then $F_\nu = \phi$ so that in either case (3.13) holds. Finally if $\nu = 4$, then again by Proposition 3.3 all unfaithful complexes of G of order four are of length one and the formula (3.13) holds. This completes the proof.

4. Application to fractional factorial designs

In this section we show how the theory developed in the previous two sections can be used to classify and enumerate designs. The main objectives of this section are the following: (i) to show how designs may be identified with the concepts of faithful and unfaithful complexes and to use this idea to exhibit a method for the construction and enumeration of a class of designs and (ii) to describe a classification problem for a class of designs and then to show, in general, to what extent the theory developed in the previous two sections helps towards a resolution of this problem. We conclude with a practical example to illustrate the ideas mentioned in (i) and (ii) above.

We now provide the basic setting in which the ideas of this section will be formulated.

Let $G = \overset{m}{\underset{i=1}{\times}} C_{s_i}$ be a direct product of groups where for each i, $1 \leq i \leq m$, $C_{s_i} = \{0, 1, 2, \cdots, s_i - 1\}$ is a cyclic group under addition mod. s_i and let $n = \pi s_i$. We call the elements of G *treatments* and by a *design* D of G we mean a *collection* of treatments of G where it is understood that any given treatment in D may possibly, though not necessarily, be repeated more than once. With each design D of G we associate a complex $\bar{D}$ of G consisting of the set of distinct treatments in D. We call $\bar{D}$ the *underlying complex* of the design D. It is clear under these definitions that any complex of G is a design and further that any given complex H of G generates an infinite class of designs D whose underlying complex is H. It is also clear that the class of designs D with underlying complex H is known once H is given. We say that a design D has *distinct order* ν, $1 \leq \nu \leq n$, if and only if the underlying complex $\bar{D}$ of D has order ν. It now follows from the foregoing that the construction and enumeration of the class of designs of distinct order ν reduces to the study of the set C_ν consisting of the complexes of G of order ν. The study of the set C_ν of complexes of order ν in G can be further reduced by the consideration of a subset $\mathcal{J}$ of C_ν, consisting of complexes of order ν one from each equivalence class in the quotient set C_ν / τ. Such a set $\mathcal{J}$, as remarked earlier is characterized by the properties that (a) $\tilde{\Omega}(G) * \mathcal{J} = C_\nu$ and (b) if $\mathcal{S} \subseteq C_\nu$ such that $\tilde{\Omega}(G) * \mathcal{S} = C_\nu$ then $|\mathcal{J}| \leq |\mathcal{S}|$.

The following two general remarks can be made: (a) recall that $C_\nu = F_\nu \cup U_\nu$ a disjoint union of the set F_ν of faithful complexes of order ν and the set U_ν of unfaithful complexes of order ν. Thus, clearly, we can write the set $\mathcal{J}$ mentioned above as a disjoint union $\mathcal{J} = \mathcal{J}_0 \cup \bar{\mathcal{J}}_0$ where $\mathcal{J}_0 \subseteq U_\nu$ and consists of unfaithful complexes of order ν one from each equivalence class of the quotient set U_ν/τ, and, $\bar{\mathcal{J}}_0 \subseteq F_\nu$ and consists of faithful complexes of order ν one from each equivalence class of the quotient set F_ν/τ, (b) in constructing the subsets $\mathcal{J}_0$ and $\bar{\mathcal{J}}_0$ of U_ν and F_ν respectively, it is clearly enough to list complexes in $\mathcal{J}_0$ and $\bar{\mathcal{J}}_0$ respectively which contain the zero element of G.

Let ν be any number such that $1 \le \nu \le n$. Then the set C_ν has cardinality C_ν^n and in the light of the previous two sections the construction of the sets $\mathcal{J}_0$ and $\bar{\mathcal{J}}_0$ mentioned above may be considered under the following two cases depending on the number ν.

Case 1. Suppose that the g.c.d. $(\nu, n) = 1$. Then by Proposition 2.5 there are no unfaithful complexes of order ν so that $U_\nu = \phi$ and $C_\nu = F_\nu$. Let $f = |F_\nu/\tau|$. Select f faithful complexes one from each equivalence class in F_ν/τ and let $\bar{\mathcal{J}}_0 = \{H_1, H_2, \cdots, H_f\}$ be the resulting set. Then clearly every complex H in C_ν is of the form $H = H_i + h$ for exactly one h in G and exactly one i such that $1 \le i \le f$. Finally by Corollary 3.3 the number $f = |F_\nu/\tau|$ is given by $f = C_{\nu-1}^{n-1}/\nu$ if $\nu \ge 2$ and $f = 1$ if $\nu = 1$.

Case 2. Suppose now that the g.c.d. $(\nu, n) = d > 1$. We first construct the class U_ν of unfaithful complexes of order ν. Let $D = \{d_1, d_2, \cdots, d_t\}$ be the set given in (2.2), namely, the set of all divisors of d distinct from one and suppose that $d_1 < d_2 < \cdots < d_t$. Let $\nu = kd$, $d = m_i d_i$, $r_i = n/d_i$ and $b_i = km_i$ where $1 \le i \le t$. Then by Proposition 2.3 the various possible lengths for unfaithful complexes of order ν is in the set $L = \{b_1, b_2, \cdots, b_t\}$ defined in (2.3). Note that $b_1 > b_2 > \cdots > b_t$. Also by Lemma 3.2 $U_\nu = \cup \{X_{b_i}(K_{d_i}) \mid 1 \le i \le t\}$, where the sets $X_{b_i}(K_{d_i})$, consisting of unfaithful complexes of length b_i and induced subgroup of order d_i, are pairwise disjoint. Thus to build the set U_ν it is enough to restrict ourselves to constructing the set $X_{b_i}(K_{d_i})$. Let $1 \le i \le t$ and recall that $c_i = |X_{b_i}(K_{d_i})/\tau|$. We begin by listing the set K_{d_i} of subgroups of G of order d_i. Next list all those unfaithful complexes H of G which are unions of b_i distinct cosets belonging to some subgroup in the set K_{d_i}. This is precisely the set of all unfaithful complexes of G of order ν whose underlying subgroup belongs to the set K_{d_i}. Reduce this set by removing those unfaithful complexes in it which have length smaller than b_i. From the resulting set which is $X_{b_i}(K_{d_i})$, construct the set $\mathcal{J}_i$ of cardinality c_i, where by Theorem 3.3 $c_i = (1/r_i)|X_{b_i}(K_{d_i})|$, consisting of unfaithful complexes of order ν one from each equivalence class in the quotient

set $X_{b_i}(K_{d_i})/\tau$. Let $\mathscr{G}_0 = \bigcup_{i=1}^{t} \mathscr{G}_i$, then $\mathscr{G}_0$ has cardinality $u = \sum_{i=1}^{t} c_i$ and the set $\mathscr{G}_0 = \{H_1, H_2, \cdots, H_u\}$ generates U_ν under the action of the group $\tilde{\Omega}(G)$. Next consider the set $C_\nu - U_\nu = F_\nu$ and let $\bar{\mathscr{G}}_0 = \{H_1', H_2', \cdots, H_f'\}$ be a set of faithful complexes one from each equivalence class in the quotient set F_ν/τ where $f = |F_\nu/\tau|$. Finally the set $\mathscr{G} = \mathscr{G}_0 \cup \bar{\mathscr{G}}_0$ consisting of $c = f + u$ complexes of order ν is the set with least cardinality such that the action of the group $\tilde{\Omega}(G)$ upon it generates the set C_ν.

In addition let $\bar{\mathscr{G}}$ be the set consisting of the complexes of order $n - \nu$ which are set complements in G of the complexes in the set $\mathscr{G}$ mentioned above. Then according to Proposition 2.4 we must have that the set $\bar{\mathscr{G}}$ is the subset of $C_{n-\nu}$ of smallest cardinality which generates the set $C_{n-\nu}$ under the action of the group $\tilde{\Omega}(G)$. Thus the enumeration and construction of the set $\mathscr{G}$ which generates C_ν under the action of the group $\tilde{\Omega}(G)$ also leads to the construction of the set $\bar{\mathscr{G}}$ which generates the set $C_{n-\nu}$ under the action of the group $\tilde{\Omega}(G)$, and conversely.

Let ν be any number such that $1 \leq \nu \leq n$. Let H be a complex of order ν, and let $(r_h)_{h \in H}$ be any sequence of positive numbers. We say that the design D of G is *obtained by replication* from the pair $(H, (r_h)_{h \in H})$ if and only if the underlying complex of D is H and each treatment h in H is repeated exactly r_h times. In this case we write $D = (H, (r_h)_{h \in H})$. It is now clear that each design D of G of distinct order ν is of the form $D = (\bar{D}, (r_{\bar{d}})_{\bar{d} \in \bar{D}})$ and conversely given a pair $(H, (r_h)_{h \in H})$, where H is a complex of order ν, there exists exactly one design D of distinct order ν such that $D = (H, (r_h)_{h \in H})$. The process of obtaining the unique design D from the pair $(H, (r_h)_{h \in H})$, we will call the *replication operation*. Note, in particular, that if H is any complex of order ν, then $H = (H, (r_h)_{h \in H})$ with $r_h = 1$ for each $h \in H$.

In terms of the replication operation we may summarize the above ideas in the following way. Let $\varDelta$ be the class of all designs of G and for each ν, $1 \leq \nu \leq n$, let $\varDelta_\nu$ be the class of all designs of G of distinct order ν. Then, clearly, $\varDelta = \bigcup \{\varDelta_\nu | 1 \leq \nu \leq n\}$ and this set union is pairwise disjoint. It follows then that the study of the set $\varDelta$ reduces to the study of each of the sets $\varDelta_\nu$. From the above we have seen that the study of the class of designs of distinct order ν may be reduced to the consideration of the set $\mathscr{G}_\nu$ of complexes of order ν which has least cardinality with respect to the property that $\tilde{\Omega}(G) * \mathscr{G}_\nu = C_\nu$. Further, the number c which is the cardinality of the quotient set C_ν/τ is given by formula (3.9) of Corollary 3.2. Now, let D be any design in $\varDelta$. Then, D belongs to $\varDelta_\nu$ for exactly one ν such that $1 \leq \nu \leq n$, and D is obtained by replication from its underlying complex $\bar{D}$. Of course, $\bar{D}$ belongs

to C_ν and hence, there exists exactly one H in $\mathscr{G}_\nu$ and exactly one h in G such that $\bar{D}=H+h$, that is, $\bar{D}$ is obtained by translation from a unique complex in $\mathscr{G}_\nu$. Hence, we have established that the class of all designs $\varDelta$ of G is generated by the operation of translation followed by the operation of replication on the set $\overset{n}{\underset{\nu=1}{\cup}} \mathscr{G}_\nu$ of complexes of G.

Let D_1 and D_2 be any two designs in $\varDelta_\nu$. Let p be a fixed set of parameters and let $M_{D_1,p}$, $M_{D_2,p}$ be the information matrices of D_1 and D_2 respectively with respect to p. We say that D_1 is *similar* to D_2 with respect to p, in symbols, $D_1 \cong_p D_2$ if and only if the information matrices $M_{D_1,p}$ and $M_{D_2,p}$ have the same spectra. A general classification problem for the designs in $\varDelta_\nu$ would be to obtain a description of the equivalence class $[D]_{\cong_p}$ of the quotient set $\varDelta_{\nu/\cong_p}$ consisting of all designs similar to D with respect to p. While no such description is available at the moment, the theory developed so far combined with a result in a recent paper by Srivastava, Raktoe and Pesotan [6] helps in giving a description for some of the members, though not necessarily all, of the equivalence class $[D]_{\cong_p}$ with respect to a suitable choice of parameters p.

For each j, $1 \leq j \leq m$, let ω_j be a permutation on the underlying set of the cyclic group C_{s_j} of order s_j and consider the group $\Omega(G)$, introduced in general in (2.9), consisting of permutations $\omega=(\omega_1, \cdots, \omega_m)$ on G defined by $\omega(b_1, \cdots, b_m)=(\omega_1(b_1), \cdots, \omega_m(b_m))$ for any $(b_1, \cdots, b_m)$ in G. For any complex $H \subseteq G$ let $\omega(H)=\{\omega(h) | h \in H\}$ and let a design D in $\varDelta_\nu$ be given, Suppose that $D=(\bar{D}, (r_{\bar{d}})_{\bar{d} \in \bar{D}})$ where $\bar{D}$ is the underlying complex of D. We now define

(4.1) $\Omega(G)^*D=\{D_1 | D_1 \in \varDelta_\nu,\ D_1=(\omega(\bar{D}), (r_{\omega(\bar{d})})_{\bar{d} \in \bar{D}})$ and $\omega \in \Omega(G)\}$.

We call the set $\Omega(G)^*D$, the set of designs of distinct order ν generated by the action of the group $\Omega(G)$ on the design D in $\varDelta_\nu$. It is now clear that by the replication operation applied in turn to each member of the set $\Omega(G)^*\bar{D}$ one obtains the set $\Omega(G)^*D$. Hence the study of the set $\Omega(G)^*D$ reduces to the study of the set $\Omega(G)^*\bar{D}$ where $\bar{D}$ is the underlying complex of D. Now, $\Omega(G)^*\bar{D}$ is a subset of C_ν. Hence using the fact that $\tilde{\Omega}(G)^*\mathscr{G}_\nu=C_\nu$ and Theorem 2.5 we know that there exists a set of complexes $\mathcal{S}_{\bar{D}}$, depending on $\bar{D}$, contained in $\mathscr{G}_\nu$ such that $\tilde{\Omega}(G)^*\mathcal{S}_{\bar{D}}=\Omega(G)^*\bar{D}$. Thus we have established that for each $D \in \varDelta_\nu$ there exists a set $\mathcal{S}_{\bar{D}} \subseteq \mathscr{G}_\nu$ such that $\tilde{\Omega}(G)^*\mathcal{S}_{\bar{D}}=\Omega(G)^*\bar{D}$. Now, suppose that $\mathscr{G}_\nu=\{H_1, H_2, \cdots, H_c\}$; let $\bar{\mathscr{G}}_\nu=\{H_{i_1}, \cdots, H_{i_k}\}$ be a subset of $\mathscr{G}_\nu$ of least cardinality such that $\Omega(G)^*\bar{\mathscr{G}}_\nu=C_\nu$. Then, by the foregoing, for each j, $1 \leq j \leq k$, we have that there exists a set $\mathcal{S}_{H_{i_j}} \subseteq \mathscr{G}_\nu$ such that

$\Omega(G)*H_{i_j}=\tilde{\Omega}(G)*\mathcal{S}_{H_{i_j}}$. It follows that

$$(4.2) \qquad C_\nu=\Omega(G)*\bar{\mathcal{J}}_\nu=(\tilde{\Omega}(G)*\mathcal{S}_{H_{i_1}})\cup\cdots\cup(\tilde{\Omega}(G)*\mathcal{S}_{H_{i_k}})$$

and this union is clearly pairwise disjoint. Further, we have that $\mathcal{J}_\nu$ $=\mathcal{S}_{H_{i_1}}\cup\cdots\cup\mathcal{S}_{H_{i_k}}$ a pairwise disjoint union. The importance of the decomposition of the set C_ν of complexes of order ν as a disjoint union of the sets $\tilde{\Omega}(G)*\mathcal{S}_{H_{i_j}}$, $1\leq j\leq k$, depends on the result obtained recently in a paper by Srivastava, Raktoe and Pesotan [6]. We give here a brief summary of this result.

Let us associate with each of the cyclic groups C_{s_i} a formal symbol A_i and with each treatment combination $(i_1,i_2,\cdots,i_m)$ in G let us associate the formal symbol $A_1^{i_1}A_2^{i_2}\cdots A_m^{i_m}$ which we call an *effect* and a formal symbol $Y_{i_1\cdots i_m}$ which we call an *observation*. With any design D of G we associate an *observation vector* Y_D which is a column vector whose entries are the observations $Y_{i_1\cdots i_m}$ with $(i_1,\cdots,i_m)$ in D arranged in some arbitrary order. Let p_0 be the set of all effects. We place the lexicographic order on G and induce it on p_0, that is, we define $A_1^{i_1}A_2^{i_2}\cdots A_m^{i_m}<A_1^{j_1}A_2^{j_2}\cdots A_m^{j_m}$ if and only if for the first l, $1\leq l\leq m$ such that $i_l\neq j_l$ we have that $i_l<j_l$. If p is any nonempty set of effects, we denote by $\boldsymbol{p}$ the column vector obtained by arranging the effects in p in the order defined on p_0. Let $X=N_1\otimes\cdots\otimes N_m$ be the Kronecker product of matrices $N_1,N_2,\cdots,N_m$ where for each i, $1\leq i\leq m$, N_i satisfy the following requirements: (a) N_i is a real orthogonal $s_i\times s_i$ matrix, (b) the first column of N_i has the same entry, namely $1/\sqrt{s_i}$, so that (c) the sum of all terms in any column of N_i but the first is zero. Further, let $\tau:G\to\{1,2,\cdots,n\}$ be a map given by $\tau(k_1,\cdots,k_m)=j$ if and only if $(k_1,\cdots,k_m)$ is the jth element of G in the lexicographic order in G.

Let D be a given design of G and $\boldsymbol{p}$ a column vector of effects. Suppose that the observation vector Y_D is displayed as

$$(4.3) \qquad Y_D=\begin{bmatrix} y_{k_1^1 k_2^1\cdots k_m^1}\\ y_{k_1^2 k_2^2\cdots k_m^2}\\ \vdots\\ y_{k_1^N k_2^N\cdots k_m^N}\end{bmatrix}.$$

Then we define for any permutation ω in $\Omega(G)$ a new observation vector determined by D and ω as

$$(4.4) \qquad Y_{D_\omega}=\begin{bmatrix} y_{l_1^1 l_2^1\cdots l_m^1}\\ y_{l_1^2 l_2^2\cdots l_m^2}\\ \vdots\\ y_{l_1^N l_2^N\cdots l_m^N}\end{bmatrix}$$

where $(l_1^j, \cdots, l_m^j) = (\omega_1(k_1^j), \cdots, \omega_m(k_m^j))$ for $1 \leq j \leq N$ and $\omega = (\omega_1, \cdots, \omega_m)$.

Further, suppose that the column vector of effects $\boldsymbol{p}$ is displayed as below:

$$(4.5) \qquad \boldsymbol{p} = \begin{bmatrix} A_1^{j_1^1} \cdots A_m^{j_m^1} \\ A_1^{j_1^2} \cdots A_m^{j_m^2} \\ \vdots \qquad \vdots \\ A_1^{j_1^K} \cdots A_m^{j_m^K} \end{bmatrix}.$$

We now define the matrix $X_{D,p}$ as an $N \times K$ submatrix of X in the following way: Corresponding to the entry $y_{k_1^i \cdots k_m^i}$ in Y_D and $A_1^{j_1^i} A_2^{j_2^i} \cdots A_m^{j_m^i}$ in $\boldsymbol{p}$ with $\tau(k_1^i, \cdots, k_m^i) = n_i$, $1 \leq i \leq N$, and $\tau(j_1^l, \cdots, j_m^l) = t_l$, $1 \leq l \leq K$, define the (i, l)th entry of $X_{D,p}$ as the (n_i, t_l)th entry of X.

For any design D and a given column vector of effects $\boldsymbol{p}$ we call the matrix $M_{D,p} = X'_{D,p} X_{D,p}$ the *information matrix* of D with respect to the set of effects p where for any matrix A the matrix A' is the transpose of A.

A collection of effects p will be called *allowable* if and only if whenever $A_1^{i_1} A_2^{i_2} \cdots A_m^{i_m}$ belongs to p and $i_j \neq 0$, $1 \leq j \leq m$, then $A_1^{i_1} \cdots A_{j-1}^{i_{j-1}} A_j^l A_{j+1}^{i_{j+1}} \cdots A_m^{i_m}$ belongs to p for all $l \neq 0$ in C_{s_j}. In the paper by Srivastava, Raktoe and Pesotan [6] the term "admissible" was used in place of allowable. We have abandoned the use of the former term here, since the term admissible has special meaning in statistics.

In the paper by Srivastava, Raktoe and Pesotan [6] the following theorem was established:

THEOREM. *Let Y_D be an observation vector associated with a design D and let $\boldsymbol{p}$ be a column vector associated with an allowable set of effects p. Then each matrix in the set $\{M_{D_\omega,p} \mid \omega \in \Omega(G)\}$ of information matrices generated by the design D and ω in $\Omega(G)$ has the same spectrum.*

Applying the above theorem we observe that if p is any allowable set of effects and D is any design in $\varDelta_\nu$ then the set $\Omega(G)^*D$ defined in (4.1) is such that $\Omega(G)^*D \subseteq [D]_{\cong_p}$, the equivalence class in the quotient set $\varDelta_{\nu/\cong_p}$ determined by D and p. Further, applying this theorem to the decomposition of C_ν given in (4.2) we observe that for each j, $1 \leq j \leq k$, the information matrices corresponding to the designs in the set $\tilde{\Omega}(G)^* S_{H_{i_j}}$ with respect to an allowable set of effects p have the same spectra.

The following example illustrates some of the ideas mentioned above.

Example. Let $G = C_2 \times C_3 \times C_3$ be a direct product of cyclic groups where $C_2 = \{0, 1\}$ is a cyclic group of order 2 under addition mod. 2 and $C_3 = \{0, 1, 2\}$ is a cyclic group of order 3 under addition mod. 3. For a

saturated main effect plan we need to take $\nu=(2-1)+(3-1)+(3-1)+1$ $=6$ treatments of G. Then since $|G|=18$ we have that the g.c.d. (6, 18)$=d=6$. The divisors of 6 distinct from 1 are then $d_1=2$, $d_2=3$, d_3 $=6$ and the various possible lengths for unfaithful complexes of order 6, using Proposition 2.3, are given by $b_1=3$, $b_2=2$, $b_3=1$. Now the cardinality of the set of complexes of G of order $\nu=6$ is $|C_\nu|=|C_6|=C_6^{18}=$ 18,564. Using formula (3.10) we see that the number of subgroups of G of orders 2, 3 and 6, respectively, are $\psi_G(2)=1$, $\psi_G(3)=4$ and $\psi_G(6)$ $=4$. Using formula (3.12) of Theorem 3.6 we now obtain the following: (i) the number of unfaithful complexes of order 6 and length 1, namely, $|X_1(K_6)|=|X_{b_3}(K_{d_3})|=(\psi_G(6))C_1^3=12$, (ii) the number of unfaithful complexes of order 6 and length 2, namely, $|X_2(K_3)|=|X_{b_2}(K_{d_2})|=(\psi_G(3))C_2^6-$ $(12)(1)=48$, (iii) the number of unfaithful complexes of order 6 and length 3, namely, $|X_3(K_{d_2})|=|X_{b_1}(K_{d_1})|=(\psi_G(2))C_3^9-(0)(48)-(1)(12)=72$. Hence the number of unfaithful complexes of order 6, namely, $|U_6|=$ $\sum_{i=1}^{3}|X_{b_i}(K_{d_i})|=132$ and by Corollary 3.1 we have that $|U_6/\tau|=20$. Further using formula (3.8) we have that the cardinality of the quotient set $|F_6/\tau|=(C_6^{18}-132)/18=1024$. Let $\mathscr{G}_6\subseteq C_6$ be a set consisting of complexes of order 6 one from each equivalence class of the quotient set C_6/τ. Then by formula (3.9) we know that $|\mathscr{G}_6|=1024+20=1044$. Thus the 1044 complexes of order 6 in $\mathscr{G}_6$ will generate under translation the 18,564 complexes in C_6 and then further by replication all the designs of G of distinct order 6. Let us write $\mathscr{G}_6=\mathscr{G}_6^0\cup\mathscr{G}_6^1$ where $\mathscr{G}_6^0$ consists of all unfaithful complexes of order 6 and $\mathscr{G}_6^1$ is the remaining complexes in $\mathscr{G}_6$ which must necessarily be faithful. Then from the above $|\mathscr{G}_6^0|=$ 20 and $|\mathscr{G}_6^1|=1024$. We follow the procedure outlined earlier in this section to give a listing of the 20 unfaithful complexes in $\mathscr{G}_6^0$. There is one subgroup of order two in G, namely, $S=\{(0, 0, 0), (1, 0, 0)\}$. There are four subgroups of order three, namely, $S_1=\{(0, 0, 0), (0, 1, 0), (0, 2, 0)\}$, $S_2=\{(0, 0, 0), (0, 0, 1), (0, 0, 2)\}$, $S_3=\{(0, 0, 0), (0, 1, 1), (0, 2, 2)\}$, and $S_4=\{(0, 0, 0), (0, 1, 2), (0, 2, 1)\}$. Finally, there are four subgroups of order six, namely,

$$T_1=\{(0, 0, 0), (0, 1, 0), (0, 2, 0), (1, 0, 0), (1, 1, 0), (1, 2, 0)\}\ ,$$

$$T_2=\{(0, 0, 0), (1, 0, 1), (1, 0, 2), (1, 0, 0), (0, 0, 1), (0, 0, 2)\}\ ,$$

$$T_3=\{(0, 0, 0), (0, 1, 1), (0, 2, 2), (1, 0, 0), (1, 1, 1), (1, 2, 2)\}\ ,$$

$$T_4=\{(0, 0, 0), (0, 1, 2), (0, 2, 1), (1, 0, 0), (1, 1, 2), (1, 2, 1)\}\ .$$

Then the set $\{T_1, T_2, T_3, T_4\}$ is a set of complexes of order 6 and length 1 which generates the set $X_{b_3}(K_{d_3})=X_1(K_6)$ under the action of the group $\tilde{\Omega}(G)$. We now list 8 unfaithful complexes of order 6 and length 2 with

induced subgroups of order 3 which generate the set $X_{b_2}(K_{d_2}) = X_2(K_3)$ under the action of the group $\tilde{\Omega}(G)$. These are,

$$T_5 = \{(0, 0, 0), (0, 1, 0), (0, 2, 0), (0, 0, 1), (0, 1, 1), (0, 2, 1)\} \ ,$$

$$T_6 = \{(0, 0, 0), (0, 1, 0), (0, 2, 0), (0, 1, 2), (0, 2, 2), (0, 0, 2)\} \ ,$$

$$T_7 = \{(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 2), (0, 1, 0), (0, 1, 1)\} \ ,$$

$$T_8 = \{(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 2, 1), (0, 2, 2), (0, 2, 0)\} \ ,$$

$$T_9 = \{(0, 0, 0), (0, 1, 2), (0, 2, 1), (0, 0, 1), (0, 1, 0), (0, 2, 2)\} \ ,$$

$$T_{10} = \{(0, 0, 0), (0, 1, 2), (0, 2, 1), (0, 1, 1), (0, 2, 0), (0, 0, 2)\} \ ,$$

$$T_{11} = \{(0, 0, 0), (0, 1, 1), (0, 2, 2), (0, 1, 2), (0, 2, 0), (0, 0, 1)\} \ ,$$

$$T_{12} = \{(0, 0, 0), (0, 1, 1), (0, 2, 2), (0, 2, 1), (0, 0, 2), (0, 1, 0)\} \ .$$

Finally we list 8 unfaithful complexes of order 6 and length 3 which generate the set $X_{b_1}(K_{d_1}) = X_3(K_2)$ under the action of the group $\tilde{\Omega}(G)$. These are,

$$T_{13} = \{(0, 0, 0), (1, 0, 0), (1, 0, 2), (0, 0, 2), (1, 2, 0), (0, 2, 0)\} \ ,$$

$$T_{14} = \{(0, 0, 0), (1, 0, 0), (1, 0, 2), (0, 0, 2), (0, 1, 0), (1, 1, 0)\} \ ,$$

$$T_{15} = \{(0, 0, 0), (1, 0, 0), (1, 0, 2), (0, 0, 2), (1, 2, 1), (0, 2, 1)\} \ ,$$

$$T_{16} = \{(0, 0, 0), (1, 0, 0), (1, 0, 2), (0, 0, 2), (1, 2, 2), (0, 2, 2)\} \ ,$$

$$T_{17} = \{(0, 0, 0), (1, 0, 0), (1, 0, 1), (0, 0, 1), (1, 2, 0), (0, 2, 0)\} \ ,$$

$$T_{18} = \{(0, 0, 0), (1, 0, 0), (1, 0, 1), (0, 0, 1), (0, 1, 0), (1, 1, 0)\} \ ,$$

$$T_{19} = \{(0, 0, 0), (1, 0, 0), (1, 0, 1), (0, 0, 1), (1, 2, 1), (0, 2, 1)\} \ ,$$

$$T_{20} = \{(0, 0, 0), (1, 0, 0), (1, 0, 1), (0, 0, 1), (1, 2, 2), (0, 2, 2)\} \ .$$

Hence the set $\mathscr{G}_6^0 = \{T_i \mid 1 \leq i \leq 20\}$ will generate the set of all unfaithful complexes of order six under the action of the group $\tilde{\Omega}(G)$. Further, to obtain a complete listing of the set $\mathscr{G}_6$ one would have to list 1024 faithful complexes of order 6 which are not related by a translation; each such faithful complex of order 6 in $\mathscr{G}_6$ will produce 18 complexes under translation. Since we selected the parameters p to be the main effects and the mean then, clearly, p is an allowable set of effects. Thus if D is any design of distinct order 6, then the class of designs generated by the action of the group $\Omega(G)$ on D, namely the class $\Omega(G)*D$, is spectra invariant with respect to the set of main effects p. Also, in the list $\mathscr{G}_6$ of 1044 faithful and unfaithful complexes if we

select a complex H and obtain through it a design $D=(H, (r_h)_{h \in H})$, then the class of designs generated by the action of $\tilde{\Omega}(G)$ on D is as well spectrum invariant relative to the set of main effects p. Finally, one might measure the efficiency of this process by observing that we have discarded 17,520 complexes of order 6 so that this process has a discarding efficiency of 94.2% in this case.

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* Visiting Assistant Professor (Summer 1972) and Visiting Professor (Sept. 1, 1971-Sept. 1, 1972), Biometrics Unit, Cornell University. Currently at the Asian Statistical Institute, Tokyo, Japan.