ASYMPTOTIC FORMULAS FOR THE NON-NULL DISTRIBUTIONS OF THREE STATISTICS FOR MULTIVARIATE LINEAR HYPOTHESIS*

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Summary

Asymptotic formulas for the distributions of the likelihood ratio statistic, Hotelling's statistic and Pillai's statistic for multivariate linear hypothesis are derived under the assumption of $n_e=ne$, $n_h=nh$ ($e>0$, $h>0$, $e+h=1$) and $\Omega=O(n)$, where n_e and n_h are the degrees of freedom for the error and for the hypothesis, respectively and Ω is the non-centrality matrix. New asymptotic formulas are given in terms of normal distribution function and its derivatives up to the order n^{-1}. We give also some numerical results of our asymptotic approximations.

1. Introduction

Let S_e and S_h denote the independent $p\times p$ matrices with the central Wishart distribution $W_p(n_e, \Sigma)$ and the non-central Wishart distribution $W_p(n_h, \Sigma, \Omega)$, respectively. The likelihood ratio statistic, Hotelling's statistic and Pillai's statistic for multivariate linear hypothesis are expressed by $W=-\{n_e+(n_h-p-1)/2\}\log|S_e(S_h+S_e)^{-1}|$, $T_0^2=n_e \operatorname{tr} S_hS_e^{-1}$ and $V=(n_e+n_h)\operatorname{tr} S_h(S_h+S_e)^{-1}$, respectively. Then n_e and n_h mean the degrees of freedom for the error and the hypothesis. The asymptotic expansions for the non-null distributions of W, T_0^2 and V with respect to n_e up to the order n_e^{-2}, assuming that n_h is a fixed number and Ω is a fixed matrix, are available in the literature (Sugiura and Fujikoshi [11], Siotani [9], Fujikoshi [3], Muirhead [6] and Lee [4]). Sugiura [12] has obtained the asymptotic expansions of the non-null distributions of these statistics under the assumption that $\Omega=O(n_e)$ and n_h is a fixed number.

In this paper we study the asymptotic distributions of these statistics in the situation that n_h is also relatively large. We derive the asymptotic expansions for the distributions of W, T_0^2 and V assuming that $n_e=ne$, $n_h=nh$ ($e>0$, $h>0$, $e+h=1$) and $\Omega=n\Theta$ where Θ is a fixed

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matrix. New asymptotic formulas are given in terms of normal distribution function and its derivatives up to the order n^{-1} by using the similar line as in Sugiura [12]. Here we note that it is realistic to assume that n_h is relatively large, in some multivariate linear hypothesis, e.g., in testing the interactions in multivariate multi-way classification design with relatively large levels and small cell frequencies. In Section 4 we also attempt to test numerically our asymptotic approximations. In the following we may assume $\Sigma=I$ and $\Omega=n\Theta=n\operatorname{diag}(\theta_1, \theta_2, \cdots, \theta_p)$ since we treat the distributions of invariant tests for multivariate linear hypothesis.

2. Preliminary lemmas

Let T_e and T_h be the statistics defined by

$$(2.1)\qquad T_e=\sqrt{m}\,(S_e/m-\mu eI) \qquad \text{and} \qquad T_h=\sqrt{m}\,\{S_h/m-\mu(hI+2\Theta)\}\ ,$$

for $m=\mu^{-1}n-r$ where μ and r are fixed numbers. Then, by the same technique as in Fujikoshi [2], it is easily seen that the statistics T_e and T_h converge in law to $p(p+1)/2$ variate normal distribution with mean 0 as m tends to infinity. Put

$$(2.2)\qquad \Delta_\beta^\alpha=\mu\operatorname{tr}[eA^\alpha\partial_e^\beta+\{hI+2(\alpha+\beta)\Theta\}B^\alpha\partial_h^\beta]$$

for any given diagonal matrices A and B where ∂_e and ∂_h are defined by

$$(2.3)\qquad \partial_e=\left(\frac{1}{2}(1+\delta_{ij})\partial/\partial\lambda_{ij}^{(e)}\right) \qquad \text{and} \qquad \partial_h=\left(\frac{1}{2}(1+\delta_{ij})\partial/\partial\lambda_{ij}^{(h)}\right)$$

for any symmetric matrices $\Lambda_e=(\lambda_{ij}^{(e)})$ and $\Lambda_h=(\lambda_{ij}^{(h)})$ of order p. We write Δ_0^α as d_α which does not depend on the operators ∂_e and ∂_h. We need the following lemma similar to Sugiura [12].

LEMMA 1. *Let S_e and S_h have $W_p(n_e, I)$ and $W_p(n_h, I, \Omega)$ respectively. Suppose $f(\Lambda_e, \Lambda_h)$ is an analytic function of two positive definite matrices Λ_e and Λ_h. Then the following asymptotic formula holds:*

$$\begin{aligned}(2.4)\qquad & \mathrm{E}\,[f(S_e/m, S_h/m)\operatorname{etr}\{\phi(AT_e+BT_h)\}]\\ &=\operatorname{etr}(-d_2t^2)[1+m^{-1/2}\phi\{\Gamma_1+4d_3\phi^2/3\}+m^{-1}\{\Gamma_2+(\Gamma_3+\Gamma_1^2/2)\phi^2\\ &\quad+2(d_4+2d_3\Gamma_1/3)\phi^4+8d_3^2\phi^6/9\}+m^{-3/2}\phi\{\Gamma_4+\Gamma_1\Gamma_2+(\Gamma_5+\Gamma_1\Gamma_3\\ &\quad+4d_3\Gamma_2/3+\Gamma_1^3/6)\phi^2+2(8d_5/5+2d_3\Gamma_3/3+d_4\Gamma_1+d_3\Gamma_1^2/3)\phi^4\\ &\quad+8(d_3d_4/3+d_3^2\Gamma_1/9)\phi^6+32d_3^3\phi^8/81\}+m^{-2}\{(\Delta_2^0)^2/2+2(\Delta_1^1)^2\Delta_2^0\phi^2\\ &\quad+2(\Delta_1^1)^4\phi^4/3+\textit{the lower order derivatives}\}\\ &\quad+O(m^{-5/2})]f(\Lambda_e, \Lambda_h)|_{\Lambda_e=\mu eI,\Lambda_h=\mu(hI+2\Theta)}\ ,\end{aligned}$$

where $\phi=it$ $(i=\sqrt{-1})$ and Γ_j are defined by

(2.5) $\Gamma_1=2\Delta_1^1+rd_1$, $\quad \Gamma_2=\Delta_2^0+r\Delta_1^0$,

$\Gamma_3=4\Delta_1^2+rd_2$, $\quad \Gamma_4=4\mu\,\mathrm{tr}\,\{eA\partial_e^2+(hI+4\Theta)B\partial_h^2\}+8\mu\,\mathrm{tr}\,\Theta\partial_h B\partial_h+2r\Delta_1^1$,

$\Gamma_5=8\Delta_1^3+4rd_3/3$.

PROOF. By expanding f around $S_e/m=\mu eI$, $S_h/m=\mu(hI+2\Theta)$ in Taylor's series and evaluating the expectations with respect to S_e and S_h by using the formula (Anderson [1])

$$\mathrm{E}\,[\mathrm{etr}\,(iTS_h)]=|I-2iT|^{-n_h/2}\,\mathrm{etr}\,\{2i\Omega T(I-2iT)^{-1}\} \tag{2.6}$$

for any symmetric matrix $T=((1+\delta_{ij})t_{ij}/2)$, we can express the left-hand side of (2.4) as follows:

$$\begin{aligned}\exp\{-m^{-1/2}\phi d_1-\Delta_1^0\}\,&|I-2m^{-1/2}(\phi A+m^{-1/2}\partial_e)|^{-\mu e(m+r)/2}\\ &\cdot|I-2m^{-1/2}(\phi B+m^{-1/2}\partial_h)|^{-\mu h(m+r)/2}\,\mathrm{etr}\,[2\mu(m^{1/2}+rm^{-1/2})\Theta\\ &\cdot(\phi B+m^{-1/2}\partial_h)\{I-2m^{-1/2}(\phi B+m^{-1/2}\partial_h)\}^{-1}]\\ &\cdot f(\Lambda_e,\Lambda_h)|_{\Lambda_e=\mu eI,\Lambda_h=\mu(hI+2\Theta)}\ .\end{aligned} \tag{2.7}$$

Applying the asymptotic formulas, $-\log|I-n^{-1}A|=\sum_{\alpha=1}^{l}n^{-\alpha}\,\mathrm{tr}\,A^\alpha/\alpha+O(n^{-(l+1)})$ and $(I-n^{-1}A)^{-1}=\sum_{\alpha=1}^{l}n^{-\alpha}A^\alpha+O(n^{-(l+1)})$ which hold for large n, to (2.7), we obtain (2.4).

Let $G[h(T_e,T_h)]$ be an abbreviated notation for

$$e^{d_2t^2}\,\mathrm{E}_{T_e,T_h}\,[h(T_e,T_h)\,\mathrm{etr}\,\{\phi(AT_e+BT_h)\}]\ . \tag{2.8}$$

The following lemma is useful for our asymptotic expansions.

LEMMA 2. *Let R, S, W and Z be any fixed diagonal matrices of order p. Then the following identities hold:*

$$\begin{aligned}G[1]=1&+m^{-1/2}\phi(rd_1+4\phi^2d_3/3)+m^{-1}\phi^2\{r(d_2+rd_1^2/2)\\ &+2\phi^2(d_4+2rd_1d_3/3)+8\phi^4d_3^2/9\}+O(m^{-3/2})\ ,\end{aligned} \tag{2.9}$$

$$\begin{aligned}&G[\mathrm{tr}\,RT_eST_e](\mu e)^{-1}\\ &\quad=a(R,S)+4\phi^2\mu e\alpha+m^{-1/2}\phi[2c(R,S;A)+r\{d_1a(R,S)\\ &\qquad+4\mu e\,\mathrm{tr}\,RSA\}+4\phi^2\{d_3a(R,S)/3+4\mu e\,\mathrm{tr}\,RSA^3+rd_1\mu e\alpha\}\\ &\qquad+16\phi^4\mu ed_3\alpha/3]+O(m^{-1})\ ,\end{aligned}$$

$$\begin{aligned}&G[\mathrm{tr}\,RT_eST_h](\mu^2e)^{-1}\\ &\quad=4\phi^2\beta+2m^{-1/2}\phi[r\,\mathrm{tr}\,RS(AL+BM)\\ &\qquad+2\phi^2\{2\,\mathrm{tr}\,RSAB(AM+BN)+rd_1\beta\}+8\phi^4d_3\beta/3]+O(m^{-1})\ ,\end{aligned}$$

$$\begin{aligned}&G[\mathrm{tr}\,RT_hST_h]\mu^{-1}\\ &\quad=c(R,S;M)/2+4\phi^2\mu\gamma+m^{-1/2}\phi[2\{c(R,S;BM)\end{aligned}$$

$$+2a(R\Theta, SB)+2a(RB, S\Theta)\}+r\{d_1c(R, S; M)/2 +4\mu \operatorname{tr} RSBLM\}+4\phi^2\{d_3c(R, S; M)/6 +4\mu \operatorname{tr} RSB^3MN+rd_1\mu\gamma\}+16\phi^4\mu d_3\gamma/3]+O(m^{-1}) ,$$

$$G[\operatorname{tr} RT_eST_eWT_e](\mu e)^{-2} =2\phi b(R, S, W; A)+8\phi^3\mu e \operatorname{tr} RSWA^3+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} RT_eST_eWT_h](\mu^2 e)^{-1} =2\phi a(RWBM, S)+8\phi^3\mu e \operatorname{tr} RSWA^2BM+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} RT_eST_hWT_h](\mu^2 e)^{-1} =2\phi c(RSA, W; M)+8\phi^3\mu \operatorname{tr} RSWAB^2M^2+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} RT_hST_hWT_h]\mu^{-2} =\phi\{b(R, S, W; BM^2)+c(RM, SM, WM; BM^{-1})\} +8\phi^3\mu \operatorname{tr} RSWB^3M^3+O(m^{-1/2}) ,$$

$$G[(\operatorname{tr} RT_eST_e)^2](\mu e)^{-2} =a(R, S)^2+2a(R^2, S^2)+2a(RS, RS)+8\phi^2\mu e\{4 \operatorname{tr} (RSA)^2 +\alpha a(R, S)\}+16\phi^4(\mu e\alpha)^2+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} T_e^2 \operatorname{tr} (RT_e)^2](\mu e)^{-2} =(p^2+p+4)a(R, R)+4\phi^2\mu e\{(p^2+p+8) \operatorname{tr} (RA)^2 +a(R, R) \operatorname{tr} A^2\}+16\phi^4(\mu e)^2 \operatorname{tr} A^2 \operatorname{tr} (RA)^2+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} RT_eST_e \operatorname{tr} WT_eZT_h](\mu^3 e^2)^{-1} =4\phi^2\{4 \operatorname{tr} RSWZABM+\tilde{\beta}a(R, S)\}+16\phi^4\mu e\alpha\tilde{\beta}+O(m^{-1/2}) ,$$

$$G[(\operatorname{tr} RT_eST_h)^2](\mu^2 e)^{-1} =c(R, S, R, S; M)+8\phi^2\mu \operatorname{tr} R^2S^2(eA^2+B^2M)M +16\phi^4\mu^2 e\beta^2+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} RT_eST_h \operatorname{tr} WT_eZT_h](\mu^2 e)^{-1} =c(R, S, W, Z; M)+8\phi^2\mu \operatorname{tr} RSWZ(eA^2+B^2M)M +16\phi^4\mu^2 e\beta\tilde{\beta}+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} RT_eST_e \operatorname{tr} WT_hZT_h](\mu^2 e)^{-1} =a(R, S)c(W, Z; M)/2+2\phi^2\mu\{e\alpha c(W, Z; M)+2a(R, S)\tilde{\gamma}\} +16\phi^4\mu^2 e\alpha\tilde{\gamma}+O(m^{-1/2}) ,$$

$$G[\operatorname{tr} RT_eST_h \operatorname{tr} WT_hST_h](\mu^3 e)^{-1} =2\phi^2\{8 \operatorname{tr} RSWZABM^2+\beta c(W, Z; M)\}+16\phi^4\mu\beta\tilde{\gamma}+O(m^{-1/2}) ,$$

$$G[(\mathrm{tr}\, RT_hST_h)^2]\mu^{-2}$$
$$=\{c(R, S; M)^2/2+a(R^2M^2, S^2)+a(R^2, S^2M^2)\}/2$$
$$+a(R^2M, S^2M)+a(RSM, RSM)+a(RSM^2, RS)$$
$$+4\phi^2\mu\{8\,\mathrm{tr}\, R^2S^2B^2M^3+\gamma c(R, S; M)\}+16\phi^4(\mu\gamma)^2+O(m^{-1/2})\ ,$$

where $\alpha=\mathrm{tr}\, RSA^2$, $\beta=\mathrm{tr}\, RSABM$, $\tilde{\beta}=\mathrm{tr}\, WZABM$, $\gamma=\mathrm{tr}\, RSB^2M^2$, $\tilde{\gamma}=\mathrm{tr}\, WZB^2M^2$, $L=hI+2\Theta$, $M=hI+4\Theta$, $N=hI+6\Theta$, $a(R, S)=\mathrm{tr}\, RS+\mathrm{tr}\, R\cdot\mathrm{tr}\, S$, $b(R, S, W; Q)=3\,\mathrm{tr}\, RSWQ+\mathrm{tr}\, RSQ\,\mathrm{tr}\, W+\mathrm{tr}\, SWQ\,\mathrm{tr}\, R+\mathrm{tr}\, RWQ\,\mathrm{tr}\, S$, $c(R, S; Q)=c(R, S, I, I; Q)$ *and* $c(R, S, W, Z; Q)=\{4\,\mathrm{tr}\, RSWZQ+\mathrm{tr}\, RZQ\cdot\mathrm{tr}\, SW+\mathrm{tr}\, RWQ\,\mathrm{tr}\, SZ+\mathrm{tr}\, SZQ\,\mathrm{tr}\, RW+\mathrm{tr}\, SWQ\,\mathrm{tr}\, RZ\}/2$.

PROOF. The first identity is obtained by putting $f(\Lambda_e, \Lambda_h)=1$ in Lemma 1. Putting $f(\Lambda_e, \Lambda_h)=\mathrm{tr}\, R(\Lambda_e-\mu eI)S(\Lambda_e-\mu eI)$ in Lemma 1, we can reduce $G[\mathrm{tr}\, RT_eST_e](\mu e)^{-1}$ as follows:

$$\text{(2.10)}\quad [\mathrm{tr}\,\partial_e^2+2\phi^2\mu e(\mathrm{tr}\, A\partial_e)^2+m^{-1/2}\phi\{4\,\mathrm{tr}\, A\partial_e^2+rd_1\,\mathrm{tr}\,\partial_e^2+2r\mu e\,\mathrm{tr}\, A\partial_e\,\mathrm{tr}\,\partial_e$$
$$+2\phi^2(2d_3\,\mathrm{tr}\,\partial_e^2/3+4\,\mu e\,\mathrm{tr}\, A\partial_e\,\mathrm{tr} A^2\partial_e+rd_1\mu e(\mathrm{tr}\, A\partial_e)^2)$$
$$+8\phi^4d_3\mu e(\mathrm{tr}\, A\partial_e)^2\}+O(m^{-1})]\,\mathrm{tr}\, R\Lambda_eS\Lambda_e\ .$$

Let E_{ij} be the $p\times p$ matrix defined by $(1/2)(1+\delta_{ij})(\partial/\partial\lambda_{ij}^{(e)})\Lambda_e$. We have

$$\text{(2.11)}\quad \mathrm{tr}\,\Theta\partial_e^2\,\mathrm{tr}\, R\Lambda_eS\Lambda_e=2\sum_{i,j=1}^{p}\theta_i\,\mathrm{tr}\, RE_{ij}SE_{ij}=c(R, S; \Theta)/2\ ,$$

$$\text{(2.12)}\quad \mathrm{tr}\,\Theta\partial_e\,\mathrm{tr}\,\Gamma\partial_e\,\mathrm{tr}\, R\Lambda_eS\Lambda_e=2\sum_{i,j=1}^{p}\theta_i\gamma_j\,\mathrm{tr}\, RE_{ii}SE_{jj}=2\,\mathrm{tr}\, R\Theta S\Gamma\ ,$$

where $\Theta=\mathrm{diag}\,(\theta_1, \theta_2, \cdots, \theta_p)$ and $\Gamma=\mathrm{diag}\,(\gamma_1, \gamma_2, \cdots, \gamma_p)$. Hence we obtain the second identity. Similarly we can derive the other identities by using Lemma 1 and Lemma 2.2 in Sugiura [12].

3. Derivation of the asymptotic expansions

We shall obtain the asymptotic expansions for the distributions of (i) $W=-m\log|S_e(S_h+S_e)^{-1}|$, (ii) $T_0^2=m\,\mathrm{tr}\, S_hS_e^{-1}$ and (iii) $V=m\,\mathrm{tr}\, S_h(S_h+S_e)^{-1}$ with respect to $m=\mu^{-1}n-r$ assuming $\Omega=n\Theta$ where Θ is a fixed matrix. In derivation of expansions we assume only that the correction factors μ and r are fixed numbers. The expansions for the distributions for traditional statistics are obtained by choosing (i) $\mu^{-1}=(1+e)/2$, $r=(p+1)/2$, (ii) $\mu^{-1}=e$, $r=0$ and (iii) $\mu^{-1}=1$, $r=0$, respectively.

3.1. *Expansion for the distribution of W*

From (2.1) S_e and S_h+S_e are expressed in terms of T_e and T_h as $m\mu e\{I+m^{-1/2}(\mu e)^{-1}T_e\}$ and $m\mu(I+2\Theta)\{I+m^{-1/2}\mu^{-1}(I+2\Theta)^{-1}(T_h+T_e)\}$ respectively. Hence we can express the characteristic function of $\{W-m\cdot\log|e^{-1}(I+2\Theta)|\}/\sqrt{m}$ as follows:

$$\text{(3.1)} \qquad \mathrm{E}\,[\operatorname{etr}\{\phi(AT_e+BT_h)\}\{1+m^{-1/2}\phi q_1(T)+m^{-1}\phi(q_2(T) + \phi q_1(T)^2/2)\}]+O(m^{-3/2}) ,$$

where $A=B-(\mu e)^{-1}I$, $B=\mu^{-1}(I+2\Theta)^{-1}$ and

$$\text{(3.2)} \qquad q_1(T)=\frac{1}{2}[(\mu e)^{-2}\operatorname{tr} T_e^2-\operatorname{tr}\{B(T_e+T_h)\}^2] ,$$

$$q_2(T)=-\frac{1}{3}[(\mu e)^{-3}\operatorname{tr} T_e^3-\operatorname{tr}\{B(T_e+T_h)\}^3] .$$

With the help of Lemma 2 and noting $d_1=0$, we can simplify (3.1), getting

$$\exp(-\tau^2t^2/2)[1+m^{-1/2}\{\phi v_1+\phi^3 v_3\}+m^{-1}\sum_{\alpha=1}^{3}\phi^{2\alpha}w_{2\alpha}+O(m^{-3/2})] ,$$

where $\tau^2=2\mu^{-1}(p/e-s_2)$ for $s_j=\operatorname{tr}(I+2\Theta)^{-j}$ and

$$\text{(3.3)} \qquad v_1=(2\mu)^{-1}\{p(p+1)/e-2(p+1)s_1+s_1^2+s_2\} ,$$

$$v_3=2(3\mu^2)^{-1}\{p/e^2+2s_3-3s_4\} ,$$

$$w_2=v_1^2/2+(2\mu^2)^{-1}\{p(p+1)/e^2+2(p+1)s_2-4s_1s_2-4(p+2)s_3+s_2^2 + 4s_1s_3+5s_4\}+r\mu^{-1}\{-p/e+s_2\} ,$$

$$w_4=v_1v_3+2(3\mu^3)^{-1}\{p/e^3-3s_4+12s_5-10s_6\} ,$$

$$w_6=v_3^2/2 .$$

Inverting this characteristic function, we have,

$$\text{(3.4)} \qquad \mathrm{P}(\{W-m\log|e^{-1}(I+2\Theta)|\}/(\tau\sqrt{m})<x) =\Phi(x)-m^{-1/2}\{v_1\Phi^{(1)}(x)/\tau+v_3\Phi^{(3)}(x)/\tau^3\} +m^{-1}\sum_{\alpha=1}^{3}w_{2\alpha}\Phi^{(2\alpha)}(x)/\tau^{2\alpha}+O(m^{-3/2}) ,$$

where $\Phi^{(j)}(x)$ denotes the jth derivative of the standard normal distribution function $\Phi(x)$. The coefficients v_j and w_j are given by (3.3).

The null distribution of W is obtained by putting $\Theta=0$ in (3.4) and the result is

$$\text{(3.5)} \qquad \mathrm{P}(\{W+mp\log e\}/(\tilde{\tau}\sqrt{m})<x) =\Phi(x)-m^{-1/2}\{\tilde{v}_1\Phi^{(1)}(x)/\tilde{\tau}+\tilde{v}_3\Phi^{(3)}(x)/\tilde{\tau}^3\} +m^{-1}\sum_{\alpha=1}^{3}\tilde{w}_{2\alpha}\Phi^{(2\alpha)}(x)/\tilde{\tau}^{2\alpha}+O(m^{-3/2}) ,$$

where $\tilde{\tau}=2ph(\mu e)^{-1}$ and

$$(3.6)\quad \tilde{v}_1 = p(p+1)h(2\mu e)^{-1}, \qquad \tilde{v}_3 = 2ph(1+e)(\mu e)^{-2}/3,$$
$$\tilde{w}_2 = p(p+1)h\{p(p+1)+4(1+e)\}(\mu e)^{-2}/8 - rph(\mu e)^{-1},$$
$$\tilde{w}_4 = ph\{p(p+1)(1+e)h+2(1+e+e^2)\}(\mu e)^{-3}/3,$$
$$\tilde{w}_6 = \tilde{v}_3^2/2.$$

3.2. *Expansion for the distribution of* T_0^2

The characteristic function of $\{T_0^2 - me^{-1}\operatorname{tr}(hI+2\Theta)\}/\sqrt{m}$ can be expressed by (3.1) for $A=-(\mu e^2)^{-1}(hI+2\Theta)$, $B=(\mu e)^{-1}I$ and

$$(3.7)\quad q_1(T) = -(\mu e)^{-2}\{\operatorname{tr} T_h T_e + \mu e \operatorname{tr} AT_e^2\},$$
$$q_2(T) = (\mu e)^{-3}\{\operatorname{tr} T_h T_e^2 + \mu e \operatorname{tr} AT_e^3\}.$$

By using Lemma 2, we have after inversion and arranging each term with respect to $t_j = \operatorname{tr}(I+2\Theta)^j$

$$(3.8)\quad \begin{aligned} &\mathrm{P}(\{T_0^2 - me^{-1}\operatorname{tr}(hI+2\Theta)\}/(\tau\sqrt{m}) < x) \\ &= \Phi(x) - m^{-1/2}\{v_1\Phi^{(1)}(x)/\tau + v_3\Phi^{(3)}(x)/\tau^3\} \\ &\quad + m^{-1}\sum_{\alpha=1}^{3} w_{2\alpha}\Phi^{(2\alpha)}(x)/\tau^{2\alpha} + O(m^{-3/2}), \end{aligned}$$

where $\tau^2 = 2(\mu e^3)^{-1}\{-pe+t_2\}$ and

$$(3.9)\quad v_1 = (\mu e^2)^{-1}(p+1)(-pe+t_1),$$
$$v_3 = 4(3\mu^2 e^5)^{-1}\{pe^2 - 3et_1 + 2t_3\},$$
$$\begin{aligned} w_2 = {} & v_1^2/2 + (\mu^2 e^4)^{-1}\{p(p+1)e(e-3) - 2(p+1)et_1 + t_1^2 + (3p+4)t_2\} \\ & + r(\mu e^3)^{-1}\{pe - t_2\}, \end{aligned}$$
$$w_4 = v_1 v_3 + 2(\mu^3 e^7)^{-1}\{pe^2(2-e) + 4e^2 t_1 - 10et_2 + 5t_4\},$$
$$w_6 = v_3^2/2.$$

Similarly we have under $\Omega=0$,

$$(3.10)\quad \begin{aligned} &\mathrm{P}(\{T_0^2 - mphe^{-1}\}/(\tilde{\tau}\sqrt{m}) < x) \\ &= \Phi(x) - m^{-1/2}\{\tilde{v}_1\Phi^{(1)}(x)/\tilde{\tau} + \tilde{v}_3\Phi^{(3)}(x)/\tilde{\tau}^3\} \\ &\quad + m^{-1}\sum_{\alpha=1}^{3} \tilde{w}_{2\alpha}\Phi^{(2\alpha)}(x)/\tilde{\tau}^{2\alpha} + O(m^{-3/2}), \end{aligned}$$

where $\tilde{\tau}^2 = 2ph(\mu e^3)^{-1}$ and

$$(3.11)\quad \tilde{v}_1 = p(p+1)h(\mu e^2)^{-1}, \qquad \tilde{v}_3 = 4ph(2-e)(3\mu^2 e^5)^{-1},$$
$$\tilde{w}_2 = p(p+1)\{(p^2+p+8)h^2/2 + 3he\}(\mu^2 e^4)^{-1} - rph(\mu e^3)^{-1},$$
$$\tilde{w}_4 = 2ph\{2p(p+1)h(2-e)/3 + e^2 - 5e + 5\}(\mu^3 e^7)^{-1},$$

$$\tilde{w}_6=\tilde{v}_3^2/2\ .$$

3.3. *Expansion for the distribution of V*

The characteristic function of $\{V-m\,\mathrm{tr}\,(hI+2\Theta)(I+2\Theta)^{-1}\}/\sqrt{m}$ can be expressed by (3.1) for $A=\mu^{-1}\{eC^2-C\}$, $B=\mu^{-1}eC^2$ and

$$q_1(T)=-\mu^{-1}\,\mathrm{tr}\,(AT_e+BT_h)C(T_e+T_h)\ , \tag{3.12}$$

$$q_2(T)=\mu^{-2}\,\mathrm{tr}\,(AT_e+BT_h)\{C(T_e+T_h)\}^2\ ,$$

where $C=(I+2\Theta)^{-1}$. By the same technique as in the case of W and T_0^2 we have

$$\begin{aligned}\mathrm{P}\,(\{V-m\,\mathrm{tr}\,(hI+2\Theta)(I+2\Theta)^{-1}\}/(\tau\sqrt{m})<x)\\ =\Phi(x)-m^{-1/2}\{v_1\Phi^{(1)}(x)/\tau+v_3\Phi^{(3)}(x)/\tau^3\}\\ +m^{-1}\sum_{\alpha=1}^{3}w_{2\alpha}\Phi^{(2\alpha)}(x)/\tau^{2\alpha}+O(m^{-3/2})\end{aligned} \tag{3.13}$$

and in a particular case $\Omega=0$,

$$\begin{aligned}\mathrm{P}(\{V-mph\}/(\tilde{\tau}\sqrt{m})<x)\\ =\Phi(x)-m^{-1/2}\tilde{v}_3\Phi^{(3)}(x)/\tilde{\tau}^3+m^{-1}\sum_{\alpha=1}^{3}\tilde{w}_{2\alpha}\Phi^{(2\alpha)}(x)/\tilde{\tau}^{2\alpha}+O(m^{-3/2})\ ,\end{aligned} \tag{3.14}$$

where $\tau^2=2e\mu^{-1}\{s_2-es_4\}$ for $s_j=\mathrm{tr}\,C^j$, $\tilde{\tau}^2=2phe\mu^{-1}$ and

$$\begin{aligned}v_1&=e\mu^{-1}\{-(p+1)s_2+s_1s_2+s_3\}\ ,\\ v_3&=(4/3)e\mu^{-2}\{-s_3+3es_5+e^2s_6-3e^2s_7\}\ ,\\ w_2&=v_1^2/2+e\mu^{-2}[es_1^2/2-es_1s_2+2(p+1)s_3-2s_1s_3-s_2^2\\ &\quad+\{(p+1)e-3\}s_4-es_2s_3-4e(p+2)s_5-2es_1s_4+4es_1s_5+3es_2s_4\\ &\quad+es_3^2/2+8es_6]-re\mu^{-1}(s_2-es_4)\ ,\\ w_4&=v_1v_3+2e\mu^{-3}[s_4-6es_6-4e^2s_7-e^2(e-14)s_8+8e^3s_9-12e^3s_{10}]\ ,\\ w_6&=v_3^2/2\ ,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\tilde{v}_3&=(4/3)phe\mu^{-2}(e-h)\ ,\\ \tilde{w}_2&=-phe\mu^{-2}(p+1)-rphe\mu^{-1}\ ,\\ \tilde{w}_4&=2phe\mu^{-3}(e^2+h^2-3he)\ ,\qquad \tilde{w}_6=\tilde{v}_3^2/2\ .\end{aligned} \tag{3.16}$$

Sugiura [10] and Nagao [7] have given the expansions similar to (3.5), (3.10) and (3.14), which are obtained by putting $\mu=1$, $r=0$ in (3.5), (3.10) and (3.14), respectively, in connection with the expansions for the distributions of test criteria for equality of two covariance matrices.

4. Numerical accuracy of the approximations

In a special case of $p=2$ Pillai and Jayachandran [8] have computed the exact 5% points of $\lambda_1=|S_e(S_h+S_e)^{-1}|^{1/2}$, $\lambda_2=\mathrm{tr}\, S_hS_e^{-1}$ and $\lambda_3=\mathrm{tr}\, S_h(S_h+S_e)^{-1}$ and their powers under certain alternatives for some n_e and n_h. Hence it is possible to test the accuracies of our asymptotic results. Table 1 gives the numerical comparison between the exact and approximate 5% points. Our asymptotic percentage points were computed from the formulas (c.f. Sugiura [10]) obtained by applying the general inverse expansion formula to (3.5), (3.10) and (3.14) with the correction factors such that (i) $\mu^{-1}=(1+e)/2$, $r=(p+1)/2$ for W, (ii) $\mu^{-1}=e$, $r=p+1$ for T_0^2 and (iii) $\mu^{-1}=1$, $r=0$ for V. The values in the brackets ( ) for λ_2 mean the percentage points when the traditional correction factors $\mu^{-1}=e$, $r=0$ were used. Table 1 shows that to choose $\mu^{-1}=e$, $r=p+1$ for the correction factors of T_0^2 can be recommended rather than the traditional correction factors. Table 2 gives the numerical comparison between the exact and approximate powers when $p=2$. Our approximate powers were computed from (3.4), (3.8) and (3.13) by choos-

Table 1 Comparison of approximations to the upper 5% points of λ_1, λ_2 and λ_3 for $p=2$

n_e	n_h	λ_1 exact	λ_1 approx.	λ_2 exact	λ_2 approx.	λ_3 exact	λ_3 approx.
13	7	.4460	.4458	2.880	2.851(2.614)	1.039	1.042
	13	.3194	.3196	4.941	4.874(4.475)	1.035	1.035
33	13	.5959	.5959	1.424	1.419(1.399)	.7863	.7871

Table 2 Comparison of approximations to the powers of W, T_0^2 and V for $p=2$ and significance level, $\alpha=0.05$

n_e	n_h	ω_1	ω_2	W exact	W approx.	T_0^2 exact	T_0^2 approx.	V exact	V approx.
13	7	0	1	.0826	.0843	.081	.091	.0826	.0831
		.5	.5	.0839	.0857	.081	.089	.0858	.0863
	13	0	1	.0694	.0704	.068	.055	.0698	.0700
		.5	.5	.0699	.0710	.068	.054	.0710	.0712
33	13	.5	.5	.0784	.0787	.077	.081	.0788	.0785
		0	1.5	.0941	.0942	.094	.099	.0934	.0931
63	7	0	4	.3242	.3242	.330	.332	.3167	.3159
		2.5	2.5	.4241	.4241	.419	.422	.4279	.4273
	13	2	2	.231	.2328	.230	.233	.234	.2344
		0	5	.284	.2852	.292	.294	.265*	.2753

(* It seems that this value is incorrect.)

ing the same correction factors as in Table 1, and using the exact significant points given in [8]. Exact powers for $n_e=63$ and $n_h=7$ in Table 2 were extracted from Lee [5]. From Tables 1 and 2 we can see that our approximations except for T_0^2 are good still in the case of the small values of n_h and ω_j. Further, it may be pointed that our approximations are excellent when n_e and n_h are both large and ω_j are large.

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