NOTE ON THE ESTIMATION OF CORRELOGRAM BY USING TRANSFORMED VARIABLES

MITUAKI HUZII

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1. Introduction

Let $X(n)$, $n=0, \pm 1, \pm 2, \cdots$, be a real-valued stationary Gaussian process with discrete time parameter and

$$EX(n)=0$$

$$EX(n)^2=\sigma^2$$

$$EX(n)X(n+h)=\sigma^2\rho_h\ .$$

We assume σ^2 is known and ρ_h is unknown. In this paper, we shall discuss the estimation of ρ_h. Let us assume that the process $X(n)$ is observed at $n=1, 2, \cdots, N, \cdots, N+h$. In the previous papers ([2], [3]), we have discussed, mainly, the estimates

$$\tilde{\gamma}_h=\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N} X(n)X(n+h)$$

and

$$\gamma_h=\sqrt{\frac{\pi}{2}}\frac{1}{N}\frac{1}{\sigma}\sum_{n=1}^{N} X(n)\ \mathrm{sgn}\ (X(n+h))\ ,$$

where

$$\mathrm{sgn}\ (x)=\begin{cases} 1\ ; & x>0 \\ 0\ ; & x=0 \\ -1\ ; & x<0\ . \end{cases}$$

In the following, we consider a generalization of estimates γ_h and $\tilde{\gamma}_h$. This generalized estimate is

$$R_G(h)=\frac{1}{\alpha}\frac{1}{N}\sum_{n=1}^{N} X(n)G(X(n+h))\ ,$$

where $G(x)$ is a measurable function of x satisfying some conditions and

α is a constant being independent of ρ_h. We can determine α such that $R_G(h)$ is an unbiased estimate of ρ_h. In this paper, we shall evaluate the variance of $R_G(h)$ for $h>2M$, where M is a positive integer such that $|\rho_k|<\varepsilon$ holds for a given positive number ε when $k>M$. In this case, we can easily see that

$$\mathrm{Var}\,(R_G(h))=\frac{1}{\alpha^2}\frac{\sigma^2}{N}\left\{EG(X(0))^2+2\sum_{k=1}^{\infty}\rho_k EG(X(0))G(X(k))\right\}+O(\varepsilon)\ .$$

But in this paper, we give a more precise evaluation of $\mathrm{Var}\,(R_G(h))$. Using this result, we can show, asymptotically, that

$$\mathrm{Var}\,(R_G(h))\geqq\mathrm{Var}\,(\tilde{\gamma}_h)$$

for $h\geqq 2M$. Related discussions are found in Rodemich [7] and Brillinger [1].

2. Unbiased estimates of ρ_h and their asymptotic variances

Let $G(x)$ be a real valued measurable function of x such that

$$(G, 1) \qquad G(-x)=-G(x)$$

$$(G, 2) \qquad \int_{-\infty}^{\infty} G(x)^4\varphi(x;\sigma)dx<+\infty\ ,$$

and

$$(G, 3) \qquad \int_{-\infty}^{\infty} xG(x)\varphi(x;\sigma)dx\neq 0\ ,$$

where

$$\varphi(x;\sigma)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-(x^2/2\sigma^2)}\ .$$

We shall define the process $Y(n)$ such as

$$Y(n)=G(X(n))\ .$$

Then $Y(n)$ is a strictly stationary process with $EY(n)^2<+\infty$. Let us construct the statistic

$$R_G(h)=\frac{1}{\alpha}\frac{1}{N}\sum_{n=1}^{N}X(n)G(X(n+h))\ ,$$

where

$$\alpha=\int_{-\infty}^{\infty} xG(x)\varphi(x;\sigma)dx\ .$$

It can be seen that $R_G(h)$ is an unbiased estimate of ρ_h and the variance of $R_G(h)$ is given as follows:

$$\mathrm{Var}\,(R_G(h))=\frac{1}{\alpha^2}\frac{1}{N}EX(0)^2G(X(h))^2$$
$$+2\frac{1}{\alpha^2}\frac{1}{N^2}\sum_{k=1}^{N-1}(N-k)EX(0)X(k)G(X(h))G(X(k+h))-\rho_h^2\,.$$

Hereafter we shall treat the processes which satisfy the following condition:

$(P, 1)$ $\quad \sum_{l=-\infty}^{\infty}|\rho_l|<+\infty$,

$(P, 2)$ $\quad$ for any distinct parameter values k, l, m, n, the joint distributions of $(X(k), X(l), X(m), X(n))$ is non degenerate.

From the condition $(P, 1)$, it follows

(2.1) $\quad$ for any $\varepsilon>0$ there exists a positive integer M such that $|\rho_l|<\varepsilon$ holds for $l>M$.

We discuss the estimation of ρ_h for $h>2M$, where M is defined as above.

In this section, we shall show the following result.

THEOREM 1. *Let $X(n)$ and $G(x)$ satisfy the above stated conditions. Then we have, asymptotically,*

$$\mathrm{Var}\,(R_G(h))=\frac{1}{\alpha^2}\frac{\sigma^2}{N}\left\{EG(X(0))^2+2\sum_{k=1}^{\infty}\rho_k EG(X(0))G(X(k))+C_{\varepsilon,N}\right\}$$
$$+O\left(\frac{1}{N^2}\right),$$

where $C_{\varepsilon,N}$ is a constant satisfying the following conditions;
(a) *there exists a constant C_ε such that $|C_{\varepsilon,N}|<C_\varepsilon$,*
(b) *C_ε does not depend on N,*
(c) *C_ε tends to zero as $\varepsilon\to 0$.*

In order to prove this theorem, we modify the expression of Var $(R_G(h))$ as follows;

$$\mathrm{Var}\,(R_G(h))=\frac{1}{\alpha^2}\frac{1}{N}EX(0)^2G(X(h))^2+V_1+V_2-\rho_h^2\,,$$

where

$$V_1=2\frac{1}{\alpha^2}\frac{1}{N^2}\sum_{k=1}^{M}(N-k)EX(0)X(k)G(X(h))G(X(k+h))$$

and

$$V_2=2\frac{1}{\alpha^2}\frac{1}{N^2}\sum_{k=M+1}^{N-1}(N-k)EX(0)X(k)G(X(h))G(X(k+h))\ .$$

And we shall show that $EX(0)^2G(X(h))^2/(\alpha^2N)$, V_1 and V_2 are expressed as

$$\frac{1}{\alpha^2}\frac{\sigma^2}{N}EG(X(0))^2+\frac{A_\varepsilon^0}{N}\ ,$$

$$2\frac{1}{\alpha^2}\frac{\sigma^2}{N}\sum_{k=1}^{\infty}\rho_k EG(X(0))G(X(k))+\frac{A_\varepsilon^1}{N}+O\left(\frac{1}{N^2}\right)$$

and

$$\rho_h^2+\frac{B_{\varepsilon,N}}{N}$$

respectively, where A_ε^0 and A_ε^1 are constants tending to zero as $\varepsilon\to 0$ and $B_{\varepsilon,N}$ is a constant having the same meaning as $C_{\varepsilon,N}$ in Theorem 1. These facts are obtained from the following Lemmas.

LEMMA 1. *For any k, $0\leqq k\leqq M$, it holds*

$$EX(0)X(k)G(X(h))G(X(k+h))=EX(0)X(k)EG(X(0))G(X(k))+O(\varepsilon)\ .$$

In fact, $EX(0)X(k)G(X(h))G(X(k+h))$ can be regard as a function of ρ_k, ρ_h, ρ_{h-k} and ρ_{k+h}. $|\rho_k|$ might be large, but ρ_h, ρ_{h-k} and ρ_{k+h} are less than ε. Accordingly we notice ρ_h, ρ_{h-k} and ρ_{k+h} and put

$$F_1(\rho_h, \rho_{k+h}, \rho_{h-k})=EX(0)X(k)G(X(h))G(X(k+h))\ .$$

$F_1(\rho_h, \rho_{k+h}, \rho_{h-k})$ is a differentiable function of ρ_h, ρ_{k+h} and ρ_{h-k}. So we have the above result.

Using Lemma 1, we can easily obtain

$$\frac{1}{\alpha^2}\frac{1}{N}EX(0)^2G(X(h))^2=\frac{1}{\alpha^2}\frac{\sigma^2}{N}EG(X(0))^2+\frac{A_\varepsilon^0}{N}$$

and

$$V_1=2\frac{1}{\alpha^2}\frac{\sigma^2}{N}\sum_{k=1}^{M}\rho_k EG(X(0))G(X(k))+\frac{A_\varepsilon^2}{N}+O\left(\frac{1}{N^2}\right)\ ,$$

where A_ε^0 is a constant as stated in the above and A_ε^2 is a constant having the same meaning as A_ε^0. Now

$$\sum_{k=1}^{M} \rho_k EG(X(0))G(X(k)) = \sum_{k=1}^{\infty} \rho_k EG(X(0))G(X(k)) - \sum_{k=M+1}^{\infty} \rho_k EG(X(0))G(X(k)) .$$

For $k>M$, putting $W(\rho_k)=EG(X(0))G(X(k))$, we have

$$W(\rho_k)=W(0)+\rho_k W'(\rho)\,|_{\rho=\theta_1\rho_k} ,$$

where $0<\theta_1<1$. It holds

$$W(0)=0 \qquad \text{and} \qquad |W'(\rho)\,|_{\rho=\theta_1\rho_k}|<W ,$$

where W is a constant independent of k. Therefore

$$\left|\sum_{k=M+1}^{\infty} \rho_k EG(X(0))G(X(k))\right| \leqq \sum_{k=M+1}^{\infty} \rho_k^2 W < \varepsilon W \sum_{k=-\infty}^{\infty} |\rho_k| .$$

Combining these results, we get

$$V_1 = 2\frac{1}{\alpha^2}\frac{\sigma^2}{N}\sum_{k=1}^{\infty} \rho_k EG(X(0))G(X(k)) + \frac{A_\varepsilon^1}{N} + O\left(\frac{1}{N^2}\right) ,$$

where A_ε^1 is a constant as stated in the above.

LEMMA 2. *If $k>M$, we have*

$$EX(0)X(k)G(X(h))G(X(k+h)) = \alpha^2\rho_h^2 + F(\rho_h, \rho_k, \rho_{k+h}, \rho_{k-h}) ,$$

where $F(\rho_h, \rho_k, \rho_{k+h}, \rho_{k-h})$ is a function of $\rho_h, \rho_k, \rho_{k+h}$ and ρ_{k-h} and satisfies the following conditions:
(a) *there exists a constant F_ε such that*

$$\sum_{k=M+1}^{N-1} |F(\rho_h, \rho_k, \rho_{k+h}, \rho_{k-h})| < F_\varepsilon ,$$

(b) *F_ε is independent of N,*
(c) *F_ε tends to zero as $\varepsilon \to 0$.*

Lemma 2 is shown as follows. If we put

$$X(l) = \rho_h X(l+h) + \xi(l) ,$$

$\xi(l)$ and $X(l+h)$ are mutually independent. We have

$$\begin{aligned} &EX(0)X(k)G(X(h))G(X(k+h)) \\ &= E\{\rho_h X(h)+\xi(0)\}\{\rho_h X(k+h)+\xi(k)\}G(X(h))G(X(k+h)) \\ &= \rho_h^2 EX(h)X(k+h)G(X(h))G(X(k+h)) \\ &\quad + \rho_h EX(h)\xi(k)G(X(h))G(X(k+h)) \\ &\quad + \rho_h E\xi(0)X(k+h)G(X(h))G(X(k+h)) \\ &\quad + E\xi(0)\xi(k)G(X(h))G(X(k+h)) . \end{aligned}$$

It holds

$$EX(h)X(k+h)G(X(h))G(X(k+h))=EX(0)X(k)G(X(0))G(X(k)) .$$

And this is a function of ρ_k. So we put

$$F_2(\rho_k)=EX(0)X(k)G(X(0))G(X(k)) .$$

Then we have

$$F_2(\rho_k)=F_2(0)+\rho_k F_2'(\theta_2\rho_k) ,$$

where $0<\theta_2<1$ and

$$F_2'(\theta_2\rho_k)=\left[\frac{d}{dx}F_2(x)\right]\Big|_{x=\theta_2\rho_k} .$$

We can see that $F_2(0)=\alpha^2$ and $|F_2'(\theta_2\rho_k)|<C_2$ holds for $|\rho_k|<\varepsilon$, where C_2 is a constant being independent of k.

In the next place, we consider $EX(h)\xi(k)G(X(h))G(X(k+h))$. It holds

$$EX(h)^2=EX(k+h)^2=\sigma^2 ,$$
$$E\xi(k)^2=\sigma^2(1-\rho_h^2) ,$$
$$EX(h)\xi(k)=\sigma^2(\rho_{k-h}-\rho_h\rho_k) ,$$
$$EX(h)X(k+h)=\sigma^2\rho_k ,$$
$$E\xi(k)X(k+h)=0 .$$

So $EX(h)\xi(k)G(X(h))G(X(k+h))$ is a function of σ^2, ρ_h, ρ_k and ρ_{k-h}. But we are now paying attention to the small value of correlation and the showing that

$$\sum_{k=M+1}^{N-1} EX(h)\xi(k)G(X(h))G(X(k+h))$$

is bounded, independently of N. Therefore we consider $EX(h)\xi(k)\cdot G(X(h))G(X(k+h))$ is a function of ρ_k only and we put

$$F_3(\rho_k)=EX(h)\xi(k)G(X(h))G(X(k+h)) .$$

It holds

$$F_3(\rho_k)=F_3(0)+\rho_k F_3'(\rho)\,|_{\rho=\theta_3\rho_k} ,$$

where $0<\theta_3<1$. We have $F_3(0)=0$ and $|F_3'(\theta_3\rho_k)|<C_3$ for $|\rho_k|<\varepsilon$, where C_3 is a constant being independent of k. Similarly putting

$$F_4(\rho_k, \rho_{k+h})=E\xi(0)X(k+h)G(X(h))G(X(k+h)) ,$$

we have

$$F_4(\rho_k, \rho_{k+h}) = F_4(0, 0) + \rho_k F_4^{(1)}(\theta_4\rho_k, \theta_4\rho_{k+h}) + \rho_{k+h} F_4^{(2)}(\theta_4\rho_k, \theta_4\rho_{k+h}) ,$$

where

$$F_4^{(1)}(\theta_4\rho_k, \theta_4\rho_{k+h}) = \left[\frac{\partial}{\partial x} F_4(x, y)\right]\Big|_{(x,y)=(\theta_4\rho_k, \theta_4\rho_{k+h})} ,$$

$$F_4^{(2)}(\theta_4\rho_k, \theta_4\rho_{k+h}) = \left[\frac{\partial}{\partial y} F_4(x, y)\right]\Big|_{(x,y)=(\theta_4\rho_k, \theta_4\rho_{k+h})} ,$$

and

$$0 < \theta_4 < 1 .$$

In the above expression, we have

$$F_4(0, 0) = 0$$

and

$$|F_4^{(1)}(\theta_4\rho_k, \theta_4\rho_{k+h})| < C_4^{(1)} , \qquad |F_4^{(2)}(\theta_4\rho_k, \theta_4\rho_{k+h})| < C_4^{(2)}$$

for $|\rho_k| < \varepsilon$ and $|\rho_{k+h}| < \varepsilon$, where $C_4^{(1)}$ and $C_4^{(2)}$ are constants and independent of k.

Finally we consider $E\xi(0)\xi(k)G(X(h))G(X(k+h))$. Joint distribution of $(\xi(0), \xi(k), X(h), X(k+h))$ is a 4-dimensional Gaussian distribution with zero mean and covariances

$$\begin{aligned}
&E\xi(0)\xi(k) = \sigma^2(\rho_k - \rho_h\rho_{k-h} - \rho_h\rho_{k+h} + \rho_h^2\rho_k) \\
&E\xi(0)X(h) = 0 \\
&E\xi(0)X(k+h) = \sigma^2(\rho_{k+h} - \rho_h\rho_k) \\
&E\xi(k)X(h) = \sigma^2(\rho_{k-h} - \rho_h\rho_k) \\
&E\xi(k)X(k+h) = 0 \\
&EX(h)X(k+h) = \sigma^2\rho_k .
\end{aligned}$$

For simplicity, we put

$$\begin{aligned}
&E\xi(0)\xi(k) = \sigma^2\rho^{(1)} \\
&E\xi(0)X(k+h) = \sigma^2\rho^{(2)} \\
&E\xi(k)X(h) = \sigma^2\rho^{(3)} .
\end{aligned}$$

Then we can express

$$\begin{aligned}
F_5(\rho^{(1)}, \rho^{(2)}, \rho_k) &= E\xi(0)\xi(k)G(X(h))G(X(k+h)) \\
&= F_5(0, 0, 0) + \rho^{(1)}\frac{\partial F_5}{\partial \rho^{(1)}}\Big|_{(0,0,0)} + \rho^{(2)}\frac{\partial F_5}{\partial \rho^{(2)}}\Big|_{(0,0,0)}
\end{aligned}$$

$$+\rho_k \frac{\partial F_5}{\partial \rho_k}\bigg|_{(0,0,0)} + \frac{1}{2}\bigg\{(\rho^{(1)})^2 \frac{\partial^2 F_5}{\partial (\rho^{(1)})^2}\bigg|_\theta + (\rho^{(2)})^2 \frac{\partial^2 F_5}{\partial (\rho^{(2)})^2}\bigg|_\theta$$

$$+\rho_k^2 \frac{\partial^2 F_5}{\partial \rho_k^2}\bigg|_\theta + 2\rho^{(1)}\rho^{(2)} \frac{\partial^2 F_5}{\partial \rho^{(1)}\partial \rho^{(2)}}\bigg|_\theta + 2\rho^{(1)}\rho_k \frac{\partial^2 F_5}{\partial \rho^{(1)}\partial \rho_k}\bigg|_\theta$$

$$+ 2\rho^{(2)}\rho_k \frac{\partial^2 F_5}{\partial \rho^{(2)}\partial \rho_k}\bigg|_\theta\bigg\} ,$$

where $\theta=(\theta_5\rho^{(1)}, \theta_5\rho^{(2)}, \theta_5\rho_k)$ and $0<\theta_5<1$.

In the above expression, we have

$$F_5(0, 0, 0)=0 .$$

When $\rho^{(1)}=\rho^{(2)}=\rho_k=0$, it holds

$$\begin{aligned} &E\xi(0)^2\xi(k)^2G(X(h))G(X(k+h)) \\ &=E\xi(0)^2\xi(k)X(h)G(X(h))G(X(k+h)) \\ &=E\xi(0)\xi(k)^2X(h)G(X(h))G(X(k+h)) \\ &=E\xi(0)\xi(k)^2G(X(h))X(k+h)G(X(k+h)) \\ &=E\xi(0)\xi(k)X(h)G(X(h))X(k+h)G(X(k+h)) \\ &=0 \end{aligned}$$

and

$$E\xi(0)^2\xi(k)G(X(h))X(k+h)G(X(k+h))=A(k)(\rho_{k-h}-\rho_h\rho_k) ,$$

where $|A(k)|<A_0$ and A_0 is a constant being independent of k. Therefore we can express

$$\frac{\partial F_5}{\partial \rho^{(1)}}\bigg|_{(0,0,0)}=A_1(k)(\rho_{k-h}-\rho_h\rho_k)$$

$$\frac{\partial F_5}{\partial \rho^{(2)}}\bigg|_{(0,0,0)}=A_2(k)(\rho_{k-h}-\rho_h\rho_k)$$

$$\frac{\partial F_5}{\partial \rho_k}\bigg|_{(0,0,0)}=A_3(k)(\rho_{k-h}-\rho_h\rho_k)$$

where $|A_i(k)|<A$ and A is a constant being independent of k. And furthermore, the absolute values of

$$B_1(k)=\frac{1}{2}\frac{\partial^2 F_5}{\partial (\rho^{(1)})^2}\bigg|_\theta,\ B_2(k)=\frac{1}{2}\frac{\partial^2 F_5}{\partial (\rho^{(2)})^2}\bigg|_\theta,\cdots,\ B_6(k)=\frac{1}{2}\frac{\partial^2 F_5}{\partial \rho^{(2)}\partial \rho_k}\bigg|_\theta$$

are all less than a constant B, which is independent of k.

We shall arrange the above results:

$$\begin{aligned} &EX(0)X(k)G(X(h))G(X(k+h)) \\ &=\alpha^2\rho_h^2+\rho_h^2\rho_kF_2'(\theta_2\rho_k)+\rho_h\rho_kF_3'(\theta_3\rho_k) \end{aligned}$$

$$\begin{aligned}
&+\rho_h\{\rho_k F_4^{(1)}(\theta_4\rho_k,\, \theta_4\rho_{k+h})+\rho_{k+h}F_4^{(2)}(\theta_4\rho_k,\, \theta_4\rho_{k+h})\} \\
&+(\rho_{k-h}-\rho_k\rho_h)\{\rho^{(1)}A_1(k)+\rho^{(2)}A_2(k)+\rho_k A_3(k)\} \\
&+(\rho^{(1)})^2B_1(k)+(\rho^{(2)})^2B_2(k)+\rho_k^2B_3(k) \\
&+\rho^{(1)}\rho^{(2)}B_4(k)+\rho^{(1)}\rho_k B_5(k)+\rho^{(2)}\rho_k B_6(k) \\
=&\,\alpha^2\rho_h^2+F(\rho_h,\, \rho_k,\, \rho_{k+h},\, \rho_{k-h}) \;.
\end{aligned}$$

From the above result and the properties $(P, 1)$ and $(2, 1)$, we get the assertion of Lemma 2.

Using Lemma 2, we can easily obtain

$$V_2=\rho_h^2+B_{\varepsilon,N}/N \;,$$

where $B_{\varepsilon,N}$ is a constant having the same meaning as $C_{\varepsilon,N}$ in Theorem 1.

Combining the above results and putting $C_{\varepsilon,N}=(A_\varepsilon^0+A_\varepsilon^1+B_{\varepsilon,N})\alpha^2/\sigma^2$, we get Theorem 1.

3. A minimum variance estimate

In this section, we ignore the terms $C_{\varepsilon,N}/N$ and $O(1/N^2)$ in the expression of $\mathrm{Var}\,(R_G(h))$ in Theorem 1 and consider the main part

$$\mathrm{Var}_N\,(R_G(h))=\frac{1}{\alpha^2}\frac{\sigma^2}{N}\left\{EG(X(0))^2+2\sum_{k=1}^{\infty}\rho_k EG(X(0))G(X(k))\right\} \;.$$

Now we shall prove the following theorem.

THEOREM 2. *Let $X(n)$ and $G(x)$ satisfy $(P, 1)$ and $(P, 2)$ and $(G, 1)$, $(G, 2)$ and $(G, 3)$, respectively. Then we have, asymptotically,*

$$\mathrm{Var}_N\,(R_G(h))\geqq \mathrm{Var}_N\,(\tilde{\gamma}_h) \;.$$

PROOF. Putting $G_0(x)=x$, we have

$$\begin{aligned}
\mathrm{Var}_N\,(\tilde{\gamma}_h)&=\mathrm{Var}_N\,(R_{G_0}(h)) \\
&=\frac{1}{(EX(0)^2)^2}\frac{\sigma^2}{N}\left\{EX(0)^2+2\sum_{k=1}^{\infty}\rho_k EX(0)X(k)\right\} \\
&=\frac{1}{N}\left\{1+2\sum_{k=1}^{\infty}\rho_k^2\right\} \;.
\end{aligned}$$

We shall compare $\sigma^2EG(X(0))^2/\alpha^2$ with 1 and $\sigma^2\rho_k EG(X(0))G(X(k))/\alpha^2$ with ρ_k^2 respectively. Obviously we have

$$\frac{\sigma^2EG(X(0))^2}{\alpha^2}=\frac{\sigma^2EG(X(0))^2}{(EX(0)G(X(0)))^2}\geqq 1 \;.$$

In the next place, we shall compare $\sigma^2\rho_k EG(X(0))G(X(k))/\alpha^2$ with ρ_k^2. For simplicity, we put

$$X(0)=X\,, \qquad X(k)=Y\,, \qquad \rho_k=\rho\,.$$

Let us consider the function

$$Q_G(\rho)=\sigma^2\rho EG(X)G(Y)-\rho^2(EXG(X))^2\,.$$

In the following, we shall show $Q_G(\rho)\geqq 0$ for $0\leqq|\rho|<1$. For this purpose, we assume, firstly, $G(x)$ is a simple function and has a carrier included in a finite interval I. We denote this simple function as $H(x)$. $H(x)$ can be expressed as, for some positive integer J,

$$H(x)=\begin{cases}\alpha_i;\ x\in Z_i\,, & i=1,2,\cdots,J\,,\\ 0;\ x\notin \bigcup\limits_{i=1}^{J} Z_i\,, & \end{cases}$$

where $\{Z_i\}$ are disjoint measurable sets included in I and $\{\alpha_i\}$ are finite real numbers. Putting

$$S_H(\rho)=EH(X)H(Y)\,,$$

$S_H(\rho)$ and

$$Q_H(\rho)=\sigma^2\rho S_H(\rho)-\rho^2(EXH(X))^2$$

are infinitely differentiable with respect to ρ in $0\leqq|\rho|<1$. We shall express $Q_H(\rho)$ in the form of Maclaurin's expansion of ρ. We shall put

$$\tilde{H}(t)=\int_{-\infty}^{\infty} H(x)\exp(-2\pi itx)dx=\sum_{j=1}^{J}\alpha_j\int_{Z_j}\exp(-2\pi itx)dx\,,$$

and

$$\sigma_1=2\pi\sigma\,.$$

Using these representations, we have

$$\begin{aligned}S_H(\rho)&=\int_{t=-\infty}^{\infty}\int_{s=-\infty}^{\infty}\tilde{H}(t)\tilde{H}(s)\exp(-\sigma_1^2(t^2+2\rho ts+s^2)/2)dtds\\ &=\sum_{j=0}^{\infty}\frac{(-1)^j\rho^j\sigma_1^{2j}}{j!}\left(\int_{-\infty}^{\infty}\tilde{H}(t)t^j\exp(-\sigma_1^2t^2/2)dt\right)^2.\end{aligned}$$

Putting

$$P_j=\frac{(-1)^j\rho^j\sigma_1^{2j}}{j!}\left(\int\tilde{H}(t)t^j\exp(-\sigma_1^2t^2/2)dt\right)^2,$$

we get

$$\begin{aligned}P_j=\frac{4(-1)^{j+1}\rho^j\sigma_1^{2j}}{j!}&\Bigg[\int_{t=0}^{\infty}\left(\int_{x=0}^{\infty}H(x)\sin 2\pi txdx\right)\\ &\cdot(1+(-1)^{j+1})t^j\exp(-\sigma_1^2t^2/2)dt\Bigg]^2.\end{aligned}$$

If j is even, we have

$$P_j=0$$

and if j is odd,

$$P_j \geqq 0 \qquad \text{for } 0 \leqq \rho < 1 .$$

And, especially,

$$P_1=\frac{\rho}{\sigma^2}(EXH(X))^2 .$$

Therefore, we have

$$Q_H(\rho)=\sigma^2\rho P_1+\sigma^2\rho \sum_{m=1}^{\infty} P_{2m+1}-\rho^2(EXH(X))^2 \geqq 0 .$$

Nextly, for a general measurable function $G(x)$, we can find a sequence of simple functions $\{H_l(x); l=1, 2, \cdots\}$, which have the same properties as $H(x)$ in the above discussion, such that

$$\int_{-\infty}^{\infty} |G(x)-H_l(x)|^2\varphi(x; \sigma)dx \to 0$$

as $l \to \infty$. Therefore we have $Q_G(\rho) \geqq 0$ for $1>\rho \geqq 0$. If $-1<\rho<0$, we put $\rho=-|\rho|$. Then

$$Q_G(\rho)=-\sigma^2|\rho| S_G(-|\rho|)-|\rho|^2 (EXG(X))^2 .$$

And we can find

$$S_G(-|\rho|)=-S_G(|\rho|)$$

from the fact that $G(-x)=-G(x)$, we have also $Q_G(\rho)=Q_G(|\rho|)$. Finally we obtain $Q_G(\rho) \geqq 0$ for $0 \leqq |\rho| < 1$. This means

$$\frac{\sigma^2\rho EG(X)G(Y)}{(EXG(X))^2} \geqq \rho^2$$

and, therefore, we can get the result of Theorem 2.

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