SOME SELECTION PROCEDURES BASED ON *U*-STATISTICS FOR THE LOCATION AND SCALE PROBLEMS

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1. Summary

This paper offers some selection procedures based on U-statistics related to Bhapkar's V and W tests [3], [4] and Deshpande's L and D tests [4], [5] for the several samples problems. Their efficiencies in comparison with 'standard' procedures are obtained on the lines of Lehmann [10] for equal sample sizes, and the results are extended to the case of unequal sample sizes.

2. Introduction

Selection procedures for choosing the 'best' among C populations according to a location parameter have been studied extensively in the statistical literature (e.g. 1, 11); very few, however, have been discussed for the scale parameter problem (e.g. 2, 7). We consider here some procedures based on generalized U-statistics, related to the V, W, L and D statistics, for selecting the 'best' amongst C populations. For convenience, we define the best as that population which has the smallest value of the characterizing parameter. Obvious modifications would give procedures for selecting the population with largest value of the parameter.

Suppose we have C independent random samples $(x_{i1}, x_{i2}, \cdots, x_{in_i})$, of size n_i from the ith population π_i with a continuous cumulative distribution function (c.d.f.) F_i, $i=1, 2, \cdots, C$. We form C-plets by taking one observation from each sample, and define functions ϕ_i from the set of C-plets as follows:

$$\phi_i(x_1, x_2, \cdots, x_C)=\begin{cases} 1 & \text{if } x_i<x_j,\ \forall j\neq i \\ 0 & \text{otherwise}. \end{cases}$$

Let

$$U_i=\frac{1}{\prod_j n_j}\sum_{\alpha_1=1}^{n_1}\cdots\sum_{\alpha_C=1}^{n_C}\phi_i(x_{1\alpha_1},\cdots,x_{C\alpha_C}) , \qquad \text{for } i=1,2,\cdots,C .$$

These generalized U-Statistics form the basis of the V-test. Similarly with

$$\phi_i^{(1)}(x_1, x_2,\cdots,x_C)=\begin{cases} 1 & \text{if } x_i>x_j,\ \forall j\neq i \\ -1 & \text{if } x_i<x_j,\ \forall j\neq i \\ 0 & \text{otherwise ,}\end{cases}$$

we get U-Statistics (defined as before) leading to the L-Statistics. If

$$\phi_i^{(2)}(x_1, x_2,\cdots,x_C)=\begin{cases} 1 & \text{if } x_i<x_j,\ \forall j\neq i \text{ or } x_i>x_j,\ \forall j\neq i \\ 0 & \text{otherwise ,}\end{cases}$$

or

$$\phi_i^{(3)}(x_1, x_2,\cdots,x_C)=r-1 ,$$
where x_i has rank r among x_j, $j=1,2,\cdots,C$,

the corresponding U-Statistics lead to the D and W-Statistics respectively.

The L and W statistics had been offered for testing the hypothesis

$$H_0 : F_1=F_2=\cdots=F_C ,$$

against the location alternatives

$$F_i(x)=F(x-\theta_i) , \tag{2.1}$$

not all θ's being equal, for symmetrical F; V was offered against (2.1) or the scale alternatives

$$F_i(x)=F([x-\mu]\theta_i) , \tag{2.2}$$

not all θ's being equal, for skew distributions F, while D was offered against (2.2) for symmetrical F.

We now use the related U-Statistics for the selection problem.

3. Location problem, equal sample sizes

Consider first the location problem with

$$F_i(x)=F(x-\theta_i) , \qquad i=1,2,\cdots,C .$$

Let $\theta_{[1]}\leqq\theta_{[2]}\leqq\cdots\leqq\theta_{[C]}$ be the ranked θ's. Suppose

$$n_1=n_2=\cdots=n_C=n , \qquad \text{say} .$$

In the 'standard' setup with F normal c.d.f. Bechhofer proposed selecting the population corresponding to the smallest $\bar{x}_i$, the mean of the ith sample, and the problem discussed was to determine the sample size 'n' required so that

$$P \text{ (correct selection)} \geqq \gamma , \tag{3.1}$$

where

$$\theta_{[1]} \leqq \theta_{[2]} - \Delta , \qquad \text{with } \gamma \text{ and } \Delta$$

pre-specified constants. It was pointed out by Lehmann [10] that in the nonparametric setup we can have an asymptotic solution if we consider the sequence of distributions

$$F_i^{(n)}(x) = F(x - \theta_i^{(n)}) , \qquad \theta_i^{(n)} = n^{-1/2}\theta_i , \tag{3.2}$$

and require the guarantee that

$$\lim_{n\to\infty} P \text{ (correct selection)} \geqq \gamma \tag{3.3}$$

when

$$\theta_{[1]} \leqq \theta_{[2]} - \Delta . \tag{3.4}$$

Such procedures based on rank-scores have been offered by Lehmann [10] and recently by Puri and Puri [12].

The procedure based on U-Statistics offered now is:

$$\text{Select } \pi_s \qquad \text{if } U_s = \max_i U_i . \tag{3.5}$$

The probability of correct selection of the 'best' population (i.e. with the smallest θ_i among $\theta_1, \theta_2, \cdots, \theta_C$) under the condition (3.4) is given by

$$P[U_s > U_j \quad \forall j \neq s / \theta_s \leqq \theta_j - \Delta \quad \forall j \neq s] . \tag{3.6}$$

We now use the following result from [3];

LEMMA 3.1. *Assume the sequence of distributions*

$$F_i^{(n)}(x) = F(x - n^{-1/2}\theta_i)$$

of independent random variables x_{ij}, $j=1, 2, \cdots, n_i$ *for each index* $n = 1, 2, \cdots$ *where* $n_i = nr_i$ *with* r_i *a positive integer,* $i = 1, 2, \cdots, C$. *Suppose further that* F *possesses a continuous derivative* f *and there exists a function* g *such that*

$$|[f(y+h) - f(y)]/h| \leqq g(y)$$

and

$$\int_{-\infty}^{\infty} g(y)f(y)dy<\infty .$$

If $N=\sum n_i$ *and* $U'=[U_1, U_2, \cdots, U_C]$, $L'=[1]_{1\times C}$, *then, as* $n\to\infty$, *the limiting distribution of* $\sqrt{N}(U-C^{-1}L)$ *is C-variate normal with mean* $\mu(\sum r_j)^{1/2}$ *and covariance matrix* $\sum=[\sigma_{ij}]=\lambda$ *given by*

$$\mu_i=-\lambda_C\left(C\theta_i-\sum_1^C \theta_j\right)$$

where

$$\lambda_C=\int_{-\infty}^{\infty}[1-F(y)]^{C-2}f^2(y)dy ,$$

and

$$\sigma_{ii}=\frac{\sum r_k}{C^2(2C-1)}\left[\frac{(C-1)^2}{r_i}+\sum_{j\neq i}\frac{1}{r_j}\right] ,$$

$$\sigma_{ij}=\frac{\sum r_k}{C^2(2C-1)}\left[\sum_k \frac{1}{r_k}-\frac{C}{r_i}-\frac{C}{r_j}\right] , \qquad i\neq j .$$

For the immediate application we take $r_i=1$ for all i.

Now the probability of correct selection given by (3.6) is

$$=P[U_s-U_j>0 \quad \forall j\neq s/\theta_s\leqq\theta_j-\Delta \quad \forall j\neq s]$$

$$=P\left[Z_j<\lambda_C C\sqrt{\frac{2C-1}{2}}(\theta_j-\theta_s) \quad \forall j\neq s/\theta_j-\theta_s\geqq\Delta \quad \forall j\neq s\right]$$

where the $(C-1)$ dimensional vector with components

$$Z_j=\frac{\sqrt{nC}(U_j-U_s)+\lambda_C C^{3/2}(\theta_j-\theta_s)}{\sqrt{2C/2C-1}} , \qquad j\neq s$$

is, in the limit, distributed as a $(C-1)$-variate normal variate, say $\boldsymbol{Y}$ with zero mean vector and covariance matrix $\boldsymbol{B}=[b_{ij}]$ with $b_{ii}=1$, $b_{ij}=1/2$, $i\neq j$.

Thus, for large n, the probability (3.6) is approximated by

$$P\left[Y_j<\lambda_C C\sqrt{\frac{2C-1}{2}}\Delta_j \quad \forall j=1, 2, \cdots, C-1\right] \tag{3.7}$$

where

$$\Delta_j=\theta_{[j+1]}-\theta_{[1]}\geqq\Delta .$$

Since the lower bound of (3.7) under (3.4) is attained at the 'least favourable configuration'

$$\theta_{[j]}=\theta_{[1]}+\Delta , \qquad j=2, 3, \cdots, C ,$$

it follows that in order to satisfy (3.3) we must have

$$\lambda_C C\sqrt{(2C-1)/2}\,\Delta=\delta\ , \tag{3.8}$$

where

$$P[Y_j<\delta \quad \forall\, j=1, 2, \cdots, C-1]=\gamma\ .$$

Thus an asymptotic expression for the smallest 'n' necessary is provided by

$$n=\frac{2\delta^2}{(2C-1)\lambda_C^2 C^2\mu_n^2} \tag{3.9}$$

writing

$$\mu_n=\theta_{[2]}^{(n)}-\theta_{[1]}^{(n)} \qquad \text{for } n^{-1/2}\Delta$$

in view of (3.2) and (3.8). It is to be noted here that (3.9) provides only a large-sample approximation so that (3.3) holds; as Rizvi and Woodworth [13] have shown, we cannot claim

$$P\ [\text{Correct Selection}]\geqq\gamma\ ,$$

when $\mu_n\geqq n^{-1/2}\Delta$, necessarily for any finite n. A similar argument for the Bechhofer procedure using samples of size m leads to the asymptotic expression (for smallest m)

$$m=\frac{2\sigma^2\delta^2}{\mu_m^{*2}} \tag{3.10}$$

where σ^2 is the variance of distribution F in (3.2) and

$$\mu_m^*=\theta_{[2]}^{*(m)}-\theta_{[1]}^{*(m)}\ .$$

With the same configuration of θ-values, i.e. with $\theta_i^{(n)}$ and $\theta_i^{*(m)}$ for the two respective procedures equal, we set $\mu_n=\mu_m^*$; therefore the asymptotic expression for the ratio of smallest sample sizes necessary in order to satisfy (3.3) is given by

$$\frac{m}{n}=\sigma^2(2C-1)C^2\lambda_C^2\ . \tag{3.11}$$

Thus the relative asymptotic efficiency of the proposed procedure with respect to the standard Bechhofer procedure given by (3.11) is precisely the same as that of the V-Statistic based on U_j's, with respect to the standard F-Statistic, based on $\bar{X}_i$'s, for testing the hypothesis $H_0: F_1=F_2=\cdots=F_C$ aginst location alternatives (2.1). Since in many situations (for example see [3]) the V-test is asymptotically more efficient than the standard F-test, or some other nonparametric competitors, the same applies to the corresponding selection procedures.

Along the same lines it can be shown that the selection procedures

$$\text{(3.12)} \qquad \text{Select } \pi_s \quad \text{if } U_s = \min_i U_i$$

using U-Statistics corresponding to functions $\phi^{(1)}$ and $\phi^{(3)}$ in Section 2 have asymptotic efficiencies (relative to the Bechhofer procedure) the same as those of the L and W-Statistics, respectively, with respect to the F-Statistic, viz.

$$\text{(3.13)} \qquad \frac{\sigma^2(2C-1)(C-1)^2\binom{2C-2}{C-1}}{2\left\{\binom{2C-2}{C-1}-1\right\}}(\lambda_C+\lambda_C^*)^2$$

and

$$\text{(3.14)} \qquad 12\sigma^2\lambda_2^2 .$$

Here

$$\text{(3.14)} \qquad \lambda_C^* = \int_{-\infty}^{\infty} f^2(y)\{F(y)\}^{C-2}dy .$$

Incidentally, (3.14) is also the efficiency of the procedure using mean ranks corresponding to the Kruskal-Wallis [8] statistic.

4. Location problem, unequal samples

The assumption of equal sample sizes does not seem to be necessary for comparing performance of two selection procedures. Suppose 'n_i' is the size of the ith sample and we use the procedure (3.5). In order to get an asymptotic solution we require the same guarantee (3.3) under (3.2), but 'n' here denotes the index $n=1, 2, \cdots$ and we assume $n_i = nr_i$, r_i being a fixed positive integer.

The probability of correct selection (3.6) is

$$P\left[\sqrt{N}(U_j - U_s) + \lambda_C C\left(\sum_k r_k\right)^{1/2}(\theta_j-\theta_s) < \lambda_C C\left(\sum_k r_k\right)^{1/2}(\theta_j-\theta_s) \quad \forall j \neq s / \theta_s \leqq \theta_j - \Delta, \quad \forall j \neq s\right]$$

and this is approximated for large 'n' by

$$\text{(4.1)} \qquad P\left[W_j^{(s)} < C\lambda_C \Delta_j \sqrt{\frac{(2C-1)r_j r_s}{r_j+r_s}}, \ \forall j \neq s\right]$$

where $\Delta_j=\theta_j-\theta_s\geqq\Delta$, and the $(C-1)$ vector $\boldsymbol{W}^{(s)}$ has normal distribution with zero mean-vector, unit variances and correlation matrix

$$\boldsymbol{\rho}^{(s)}=[\rho_{ij}^{(s)}]$$

with

$$\rho_{ij}^{(s)}=\left[\frac{r_i r_j}{(r_i+r_s)(r_j+r_s)}\right]^{1/2}, \qquad s\neq i\neq j\neq s ,$$

in view of Lemma 3.1. For each $s=1, 2, \cdots, C$, the lower bound of (4.1) is attained at $\Delta_j=\Delta$ for all $j\neq s$. We thus have

$$\min_{s=1,\cdots,C} P\left[W_j^{(s)}<\sqrt{\frac{(2C-1)r_j r_s}{r_j+r_s}}\,C\lambda_C\Delta \quad \forall\, j\neq s\right]=\gamma . \tag{4.2}$$

It may be noted here that the left-hand side is a continuous non-decreasing function of Δ. Suppose, therefore, (4.2) is satisfied for Δ equal to β with the minimum attained for s; then we have β given by

$$P\left[W_j^{(s)}<\sqrt{\frac{(2C-1)r_j r_s}{(r_j+r_s)}}\,C\lambda_C\beta, \quad \forall\, j\neq s\right]=\gamma . \tag{4.3}$$

A large sample solution for the smallest 'n' necessary is provided by

$$n=\beta^2/\mu_n^2 \tag{4.4}$$

writing, as in (3.9), $\Delta=\sqrt{n}\,\mu_n$ with $\mu_n=\theta_{[2]}^{(n)}-\theta_{[1]}^{(n)}$.

A parallel argument for the Bechhofer procedure, using index m, leads to the equation

$$\min_{s=1,2,\cdots,C} P\left[W_j^{(s)}<\sqrt{\frac{r_s r_j}{r_s+r_j}}\,\frac{\Delta}{\sigma} \quad \forall\, j\neq s\right]=\gamma . \tag{4.5}$$

Then (4.5) would be satisfied for Δ equal to α given by

$$P\left[W_j^{(s)}<\sqrt{\frac{r_s r_j}{r_j+r_s}}\,\frac{\alpha}{\sigma}, \quad \forall\, j\neq s\right]=\gamma$$

with the same s as in (4.3). Therefore

$$\alpha=\sqrt{2C-1}\,C\lambda_C\beta\sigma . \tag{4.6}$$

The large sample solution for the smallest 'm' necessary is

$$m=\alpha^2/\mu_m'^2 \tag{4.7}$$

writing, as in (3.10), $\Delta=\sqrt{m}\,\mu_m'$. The large sample approximation for the ratio m/n is then

$$\frac{m}{n}=\frac{\mu_n^2\alpha^2}{\beta^2\mu_m'^2}\,.$$

Setting $\mu_n=\mu_m'$, as in the case of equal samples, we get the asymptotic efficiency of procedure (3.5), relative to the Bechhofer procedure, in the form

$$(2C-1)C^2\lambda_C^2\sigma^2\,.$$

This is precisely the same expression obtained in the case of equal samples.

Similar results would follow for procedures (3.12) corresponding to functions $\phi^{(1)}$ and $\phi^{(3)}$.

5. Comparison of performance

The above asymptotic relative efficiency is obtained by comparing the smallest sample sizes necessary in order that (3.3) holds and, thus, reflects the relative performance at the least favourable configuration $\theta_{[j]}=\theta_{[1]}+\Delta$, $j=2,\cdots,C$. For any given configuration $\theta_j-\theta_s=\Delta_j$, the respective probabilities of correct selection are approximated by expression (3.15) and the one obtained after replacing in (3.15) $\sqrt{2C-1}\,C\lambda_C$ by $1/\sigma$. These two expressions are equal if

$$\sqrt{2C-1}\,C\lambda_C\sqrt{n}\,(\theta_j^{(n)}-\theta_s^{(n)})=\sqrt{m}\,(\theta_j^{(m)}-\theta_s^{(m)})/\sigma$$

asymptotically, for all $j\neq s$. With the given configuration, taking

$$(\theta_j^{(n)}-\theta_s^{(n)})=(\theta_j^{(m)}-\theta_s^{(m)})\,,$$

we get the same expression for the asymptotic relative efficiency for any configuration.

6. Scale problem

For the scale problem we now assume (2.2). Suppose 'μ' is in the nature of parameter like median, we then offer the selection rule,

$$\text{(6.1)}\qquad \text{Select } \pi_s \quad \text{if } U_s=\max_i U_i\,,$$

corresponding to function ϕ. Using the index $n=1,2,\cdots$, and sample sizes $n_i=nr_i$, as in Section 4, we assume the sequence of distributions

$$F_i^{(n)}(x)=F[(x-\mu)\theta_i^{(n)}]$$

with

(6.2) $$\theta_i^{(n)}=1+n^{-1/2}\theta_i$$

and require the same guarantee (3.3) under condition (3.4). There we need the following result from [6] incorporating the necessary modification in Lemma 3.1.

LEMMA 6.1. *Assume the sequence of distributions*

$$F_i^{(n)}(x)=F[(x-\mu)(1+n^{-1/2}\theta_i)]$$

of independent random variables $[x_{ij},\ j=1, 2, \cdots, n_i]$, *for each index 'n',* $n_i=nr_i$, *with 'r_i' a positive integer,* $i=1, 2, \cdots, C$. *Suppose that F possesses a derivative f and there exists a function g such that*

$$\left|\frac{f(x)-f(x+hx)}{h}\right|\leqq g(x)$$

for sufficiently small 'h' and

$$\int [|x|g(x)]^i f(x)d(x)<\infty\ , \qquad i=1, 2, \cdots, 2C-1\ .$$

We assume further that there exists $A<\infty$ *such that*

$$P_F[|x|f(x)<A]=1\ .$$

Then, as $n\to\infty$, *the limiting distribution of* $\sqrt{N}(U-C^{-1}L)$ *is C-variate normal with mean vector*-$(\sum r_j)^{1/2}\eta$, *where*

$$\eta_i=\xi_C\left(C\theta_i-\sum_j \theta_j\right)\ ,$$

with

$$\xi_C=-\int xf^2(x)[1-F(x)]^{C-2}dx\ ,$$

and the covariance matrix Σ *as in Lemma* 3.1.

The probability of correct selection

$$\begin{aligned}P[U_j<U_s\quad &\forall j\neq s/\theta_s\leqq\theta_j-\Delta\quad \forall j\neq s]\\ &=P\Big[\sqrt{N}(U_j-U_s)+\xi_C C\Big(\sum_k r_k\Big)^{1/2}(\theta_j-\theta_s)\\ &\qquad <\xi_C C\Big(\sum_k r_k\Big)^{1/2}(\theta_j-\theta_s),\quad \forall j\neq s/\theta_s\leqq\theta_j-\Delta\Big]\ ,\end{aligned}$$

and this is approximated by

(6.3) $$P\left[W_j^{(s)}<C\xi_C\Delta_j\sqrt{\frac{(2C-1)r_jr_s}{r_j+r_s}},\quad \forall j\neq s\right]$$

as in (4.1). Proceeding along the same lines as in Section 4, a large sample solution for the smallest 'n' is provided by

(6.4) $$n=\beta'^2/\mu_n^2$$

where β' satisfies equation like (4.3) in place of β with ξ_C replacing λ_C.

We shall compare the above procedure with the parametric procedure used *when F is normal*; θ_i then, is the reciprocal of the standard deviation σ_i. This procedure is

(6.5) $$\text{Select } \pi_s \quad \text{if } V_s=\max_i V_i$$

where

$$v_i=\sum_{j=1}^{m_i}(x_{ij}-\bar{x}_i)^2/(m_i-1) ,$$

m_i denoting the size of the ith sample. The probability of correct selection

$$P[V_j<V_s \quad \forall j\neq s/\theta_s\leqq\theta_j-\Delta]$$

is approximated for large 'm' by

(6.6) $$P\left[W_j^{(s)}<\Delta_j\sqrt{\frac{2r_jr_s}{r_j+r_s}}, \quad \forall j\neq s\right]$$

in view of the result (see [9], p. 274) that

$$\sqrt{m}[\log V_j+2\log\theta_j^{(m)}]$$

is asymptotically normal with zero mean and variance equal to $2/r_j$. Proceeding again along similar lines as in Section 4, a large sample approximation for the smallest 'm' is provided by

(6.7) $$m=\alpha'^2/\mu_m'^2$$

where

$$\alpha'=\sqrt{(2C-1)/2}\,C\xi_C\beta' .$$

The argument as in Section 4 leads to

(6.8) $$\frac{2C-1}{2}C^2\xi_C^2 ,$$

as the asymptotic efficiency of the procedure (6.1) relative to the procedure (6.5) if F is normal. This is precisely the asymptotic efficiency of the V-test based on U's relative to the test proposed by Lehmann ([9], p. 274) based on V's against scalar alternatives if F happens to be the normal c.d.f.

It may be pointed out here that if μ in (2.2) is of the type of a 'natural' location parameter like μ in

$$f(x, \mu, \theta)=\theta e^{-(x-\mu)\theta}, \qquad x \geqq \mu ,$$

instead of being of the type of median in the discussion so far in this section, the selection procedure offered is

$$\text{(6.9)} \qquad \text{Select } \pi_s \quad \text{if } U_s=\min_i U_i .$$

It may be verified that the expression (6.8) continues to hold as the asymptotic relative efficiency; we only need substitute $\xi'_C=-\xi_C$ in the preceding discussion.

Along the same lines it can be shown that the selection procedure using the rule

$$\text{Select } \pi_s \quad \text{if } U_s=\max_i U_i$$

using U-statistics corresponding to function $\phi^{(2)}$ in Section 2 has asymptotic efficiency

$$\text{(6.10)} \qquad \frac{C^2(C-1)^2(2C-1)\binom{2C-2}{C-1}}{4\left[C^2+\binom{2C-2}{C-1}(C^2+4C-2)\right]}(\xi_C+\xi_C^*)^2 ,$$

with

$$\xi_C^*=\int xf^2(x)[F(x)]^{C-2}dx$$

relative to the procedure (6.5), this is again precisely the asymptotic efficiency of the D-test [4] relative to Lehmann's test if F is normal.

Before closing we may point out here that the approximation to the smallest sample sizes of the type (3.9), (4.4), (6.4) etc. hold for any F satisfying conditions in Lemma 3.1 or 6.1 while that viz. (3.10) or (4.7) for the Bechhofer procedure holds only if F has finite variance; such an approximation for the scale procedure (6.5) is available only if F is normal. In this sense the procedures offered here, based on U-Statistics and the Bechhofer procedure may be called non-parametric while the one (6.5) related to Lehmann's test is essentially parametric.

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