ON THE CONVERGENCE OF OPTIMUM STRATIFICATIONS FOR EMPIRIC DISTRIBUTION FUNCTION IN UNIVARIATE CASE

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Summary

Let $\{X_n\}$, $n=1, 2, \cdots$, be a sequence of independent random variables distributed according to a distribution function $F(x)$ with finite variance, $F_n(x)$ be the empiric distribution function of $X_1, \cdots, X_n$ for each n, and $\phi^*_{(n)}$ and ϕ^* be optimum stratifications corresponding to $F_n(x)$ and $F(x)$ respectively.

It is shown in this paper that $\phi^*_{(n)}$ tends almost surely to ϕ^* under a suitable criterion.

1. Introduction

Let X be an unvariate stratification variable with a marginal distribution function $F(x)$, and Y be an unvariate objective variable with finite mean μ_0 and variance σ_0^2, and $\eta(x)$ be the regression function of Y on X.

Let us suppose that stratification operation should be made only using the stratification variable X for estimating the mean μ_0 of Y, and that the number l of strata, the total sample size m and the sample allocation $\{m_i, 1\leqq i\leqq l\}$ are preassigned.

Such a stratification may be expressed by a decomposition $\{F_i, 1\leqq i\leqq l\}$ of the marginal distribution function $F(x)$ of X, i.e.

$$\sum_{i=1}^{l} F_i(x)=F(x) \qquad \text{for all } x ,$$

where $F_i(x)$ is non-negative and non-decreasing in x.

Since each measure F_i corresponding to the function $F_i(x)$ is absolutely continuous with respect to the measure F corresponding to $F(x)$, there exists a vector-valued measurable function $\phi(x)=(\phi_1(x), \cdots, \phi_l(x))$ for each decomposition $\{F_i\}$ of F such that

$$\sum_{i=1}^{l} \phi_i(x)=1 \qquad \text{a.e. } (F), \quad \phi_i(x)\geqq 0 \ (1\leqq i\leqq l) ,$$

and the correspondence between $\{F_i\}$ and ϕ may be regarded as one to one (see the section 3 in [1]).

In case of proportionate allocation ($m_i=w_i m$) the variance of an unbiased estimator $\bar{Y}=\sum_{i=1}^{l} w_i\bar{Y}_i$ of μ_0, based on the random sample under a stratification ϕ, may be expressed as

$$(1.1)\qquad V(\bar{Y}\,|\,\phi)=\frac{1}{m}\left\{\sigma_0^2(1-\rho^2)+\sum_{i=1}^{l}\int_{-\infty}^{\infty}[\eta(x)-\mu_{0i}]^2\phi_i(x)dF(x)\right\},$$

where $w_i=F_i(+\infty)$ is the weight of the ith stratum, σ_0^2 is the total variance of Y, μ_{0i} is the mean of Y in the ith stratum and ρ is the correlation ratio of Y on X (see the section 4 in [1]).

It is easily seen from (1.1) that $V(\bar{Y}\,|\,\phi)$ takes the minimum value at $\phi=\phi^*$ if and only if ϕ^* attains the infinum of the second term in the bracket of the right-hand side of (1.1) which is rewritten as

$$(1.2)\qquad \sum_{i=1}^{l}\int_{-\infty}^{\infty}[\eta(x)-\mu_{0i}]^2\phi_i(x)dF(x)=\int_{-\infty}^{\infty}\eta^2(x)dF(x)-\sum_{i=1}^{l}w_i\mu_{0i}^2\,.$$

Further if the regression function is linear, i.e. $\eta(x)=\beta_0+\beta_1 x$, then (1.2) reduces to

$$(1.3)\qquad \sum_{i=1}^{l}\int_{-\infty}^{\infty}[\eta(x)-\mu_{0i}]^2\phi_i(x)dF(x)=\beta_1^2\left\{\int_{-\infty}^{\infty}x^2dF(x)-\sum_{i=1}^{l}w_i\mu_i^2\right\},$$

where $\mu_i=\frac{1}{w_i}\int_{-\infty}^{\infty}x\phi_i(x)dF(x)$ is the mean of X in the ith stratum.

Therefore the optimum stratification ϕ^* attains the infimum of the function

$$(1.4)\qquad G_1(w(\phi),\,u(\phi))=-\sum_{i=1}^{l}\frac{[u_i(\phi)]^2}{w_i(\phi)}\,,$$

where $w_i(\phi)=\int_{-\infty}^{\infty}\phi_i(x)dF(x)$ and $u_i(\phi)=\int_{-\infty}^{\infty}x\phi_i(x)dF(x)$. (see example i in [1], p. 123).

This fact shows that ϕ^* is identical to an optimum stratification for the marginal distribution function $F(x)$ of the stratification variable X in the case of proportionate allocation and linear regression.

If $F(x)$ is unknown but a simple random sample of size n $(X_1,\cdots,X_n)$ is given under the above situation in advance of stratification operation, it is necessary to make an estimate $\phi^*_{(n)}$ of ϕ^* using the given sample $(X_1,\cdots,X_n)$ and investigate its probabilistic behavior.

One way of doing this is to obtain $\phi^*_{(n)}$ as an optimum stratification for the empiric distribution function $F_n(x)$ of $(X_1,\cdots,X_n)$, and to examine whether $G_1(w(\phi^*_{(n)}),\,u(\phi^*_{(n)}))$ converges almost surely to $G_1(w(\phi^*),\,u(\phi^*))$

or not as $n \to \infty$.

It is noted here that there exists an optimum ϕ^* for any $F(x)$ (discrete or continuous) such that

$$\phi_i^*(x) = \begin{cases} 1 & \text{if } x_{i-1}^* \leqq x < x_i^* \quad (1 \leqq i \leqq l) \\ 0 & \text{otherwise}, \end{cases}$$

where $w_i^* = \int_{-\infty}^{\infty} \phi_i^*(x) dF(x)$, $\mu_i^* = \int_{-\infty}^{\infty} x \phi_i^*(x) dF(x)$, $(1 \leqq i \leqq l)$ $\mu_1^* < \mu_2^* < \cdots < \mu_l^*$, $x_i^* = (\mu_i^* + \mu_{i+1}^*)/2$ with no probability mass on it $(1 \leqq i \leqq 1)$, $x_0^* = -\infty$ and $x_l^* = +\infty$, if the support of $F(x)$ contains at least l points (see also [1]).

In case of Neyman allocation $\left(n_i = \frac{w_i \sigma_i}{\sum_j w_j \sigma_j} n \right)$ there exists an optimum stratification ϕ^{**} for $F(x)$ which minimizes the function

$$G_2(w(\phi), u(\phi), v(\phi)) = \sum_{i=1}^{l} [w_i(\phi) v_i(\phi) - u_i^2(\phi)]^{1/2}, \tag{1.5}$$

where $w(\phi) = \int_{-\infty}^{\infty} \phi(x) dF(x)$, $u(\phi) = \int_{-\infty}^{\infty} x \phi(x) dF(x)$ and $v(\phi) = \int_{-\infty}^{\infty} x^2 \phi(x) \cdot dF(x)$ (see [2]).

It is also of interest to investigate the probabilistic behavior of an optimum stratification $\phi_{(n)}^{**}$ for $F_n(x)$, i.e. to examine whether $G_2(w(\phi_{(n)}^{**}), u(\phi_{(n)}^{**}), v(\phi_{(n)}^{**}))$ converges almost surely to $G_2(w(\phi^{**}), u(\phi^{**}), v(\phi^{**}))$ or not as $n \to \infty$.

2. Preparatory lemmas

In this section we state two lemmas useful for deriving main theorems in the following sections.

LEMMA 1. *Let $F(x)$ be a distribution function having finite mean μ and variance σ^2 and $F_n(x)$ be the empiric distribution function of random variables distributed independently according to $F(x)$. If $g(x)$ is a continuous and integrable function with respect to F, then*

$$P\left\{ \lim_{n\to\infty} \sup_{x \in \mathcal{I}} \left| \int_I g(x) dF_n - \int_I g(x) dF \right| = 0 \right\} = 1 \tag{2.1}$$

holds, where $\mathcal{I}$ is the family of all intervals on the real line.

PROOF. Since g is integrable with respect to F, for any positive ε there exists a positive constant K_ε such that

$$\int_{|x|>K_\varepsilon} |g(x)|dF(x)<\varepsilon$$

holds.

By the strong law of large numbers there exists a positive integer $n_1(\varepsilon)$ depending only on ε such that

$$(2.2)\qquad \int_{|x|>K_\varepsilon} |g(x)|dF_n(x)<\int_{|x|>K_\varepsilon} |g(x)|dF(x)+\varepsilon<2\varepsilon$$

holds with probability one for all $n>n_1(\varepsilon)$.

Let us divide the interval $[-K_\varepsilon, K_\varepsilon]$ into ν_ε intervals $\{I_j\}$ with equal length such that

$$|g(x)-g(x)|<\varepsilon \qquad \text{for } |x-x'|<\delta_\varepsilon,\ x \text{ and } x'\in[-K_\varepsilon, K_\varepsilon]$$

and

$$\nu_\varepsilon=\left[\frac{2K_\varepsilon}{\delta_\varepsilon}\right]+1\ ,\quad I_j=\left(-K_\varepsilon+(j-1)\frac{2K_\varepsilon}{\nu_\alpha},\ -K_\varepsilon+j\frac{2K_\varepsilon}{\nu_\varepsilon}\right],\qquad (2\leqq j\leqq\nu_\varepsilon)\ ,$$

and

$$I_1=\left[-K_\varepsilon,\ -K_\varepsilon+\frac{2K_\varepsilon}{\nu_\varepsilon}\right]\ .$$

Let us take any fixed point x_j in each I_j $(1\leqq j\leqq\nu_\varepsilon)$. Then the following relations hold:

$$(2.3)\qquad \left|\int_{I\cap(-K_\varepsilon,K_\varepsilon)} g(x)dF(x)-\sum_{j=1}^{\nu_\varepsilon} g(x_j)F(I\cap I_j)\right|<\varepsilon\ ,$$

$$(2.4)\qquad \left|\int_{I\cap(-K_\varepsilon,K_\varepsilon)} g(x)dF_n(x)-\sum_{j=1}^{\nu_\varepsilon} g(x_j)F_n(I\cap I_j)\right|<\varepsilon\ ,$$

and then for any $\eta>0$

$$(2.5)\qquad \left|\int_{I\cap(-K_\varepsilon,K_\varepsilon)} g(x)dF_n(x)-\int_{I\cap(-K_\varepsilon,K_\varepsilon)} g(x)dF(x)\right|$$
$$<2\varepsilon+M_\varepsilon\sum_{j=1}^{\nu_\varepsilon}|F_n(I\cap I_j)-F(I\cap I_j)|<2\varepsilon+M_\varepsilon\nu_\varepsilon\eta$$

uniformly in I with probability one for all $n>n_2(\eta)$ by the theorem of Glivenko-Cantelli where $M_\varepsilon=\sup_{|x|\leqq K_\varepsilon}|g(x)|$. If we take η for η_ε such that $M_\varepsilon\nu_\varepsilon\eta_\varepsilon<\varepsilon$, then from (2.5)

$$(2.6)\qquad \left|\int_{I\cap[-K_\varepsilon,K_\varepsilon]} g(x)dF(x)-\int_{I\cap[-K_\varepsilon,K_\varepsilon]} g(x)dF(x)\right|<3\varepsilon$$

holds uniformly in I with probability one for all $n>n_2(\eta_\varepsilon)$.

Therefore from (2.2) and (2.6) we can find a positive integer $n_0(\varepsilon)=$

$\max(n_1(\varepsilon), n_2(\eta_\varepsilon))$ such that the inequality

$$\left|\int_I g(x)dF_n(x)-\int_I g(x)dF(x)\right|<6\varepsilon \tag{2.7}$$

holds uniformly in I with probability 1 for all $n>n_0(\varepsilon)$. This result shows that (2.1) holds. Thus the proof is completed.

COROLLARY. *If $F(x)$ has the finite rth moment, then the relation*

$$P\left\{\lim_{n\to\infty}\sup_{I\in\mathcal{I}}\left|\int_I x^k dF_n(x)-\int_I x^k dF(x)\right|=0\right\}=1 \tag{2.8}$$

holds for $0\leqq k\leqq r$.

Remark. In the case $k=0$ (2.8) is identical to the theorem of Glivenko-Cantelli.

LEMMA 2. *Under the same assumptions in Lemma 1 and the assumption that $0<\tau=\int[g(x)]^4 dF(x)<\infty$,*

$$P\left\{\lim_{n\to\infty}\sup_{I\in\mathcal{I}}\left|\frac{\left[\int_I g(x)dF_n(x)\right]^2}{\int_I dF_n(x)}-\frac{\left[\int_I g(x)aF(x)\right]^2}{\int_I dF(x)}\right|=0\right\}=1\,. \tag{2.9}$$

holds, where both terms in the bracket in the left-hand side of (2.9) should be taken to be zero if

$$\int_I dF_n(x)=0 \quad or \quad \int_I dF(x)=0 \qquad respectively\,.$$

PROOF. Let $\mathcal{I}$ be the family of all intervals on the real line, and let us divide it into two sets $\mathcal{I}_\varepsilon$ and $\mathcal{I}'_\varepsilon$ for any positive ε such that

$$\mathcal{I}_\varepsilon=\left\{I;\ \int_I dF(x)<\frac{\varepsilon^2}{9\tau}\right\} \qquad \text{and} \qquad \mathcal{I}'_\varepsilon=\mathcal{I}-\mathcal{I}_\varepsilon\,.$$

Then it is easily seen by the Schwarz's inequality

$$\frac{\left[\int_I gdF\right]^2}{\int_I dF}\leqq\int_I g^2dF\leqq\sqrt{\int_I g^4dF}\,\sqrt{\int_I dF}<\frac{\varepsilon}{3} \tag{2.10}$$

holds for any $I\in\mathcal{I}_\varepsilon$. In the same way

$$\frac{\left[\int_I gdF_n\right]^2}{\int_I dF_n}\leqq\int_I g^2dF_n \tag{2.11}$$

holds for any possible F_n. By Lemma 1 there exists a positive integer $n_1(\varepsilon)$ such that

$$\int_I g^2 dF_n < \int_I g^2 dF + \frac{\varepsilon}{3} < \frac{2}{3}\varepsilon \tag{2.12}$$

holds with probability 1 for any $I \in \mathfrak{T}_\varepsilon$ for $n > n_1(\varepsilon)$.

Therefore from (2.10) and (2.12)

$$\sup_{I \in \mathfrak{T}_\varepsilon} \left| \frac{\left[\int_I g dF_n\right]^2}{\int_I dF_n} - \frac{\left[\int_I g dF\right]^2}{\int_I dF} \right| < \varepsilon \tag{2.13}$$

holds with probability 1 for any $I \in \mathfrak{T}_\varepsilon$ for $n > n_1(\varepsilon)$.

On the other hand the inequality

$$\int_I dF(x) \geqq \frac{\varepsilon^2}{9\tau}$$

holds for any $I \in \mathfrak{T}'_\varepsilon$. Let us put

$$\eta_{1n} = \int_I g dF_n - \int_I g dF \quad \text{and} \quad \eta_{2n} = \int_I dF_n - \int_I dF .$$

For any $\delta > 0$ there exists a positive integer $n_2(\delta)$ by Lemma 1 such that

$$|\eta_{1n}| < \delta \quad \text{and} \quad |\eta_{2n}| < \delta$$

hold with probability 1 uniformly in $I \in \mathfrak{T}'_\varepsilon$ for all $n > n_2(\delta)$.

Therefore it is easily seen that for $\delta < 9\tau\varepsilon^{-2}$

$$\sup_{I \in \mathfrak{T}'_\varepsilon} \left| \frac{\left[\int_I g dF_n\right]^2}{\int_I dF_n} - \frac{\left[\int_I g dF\right]^2}{\int_I dF} \right| \leqq \frac{\delta}{\varepsilon^2/9\tau - \delta}\left(2\rho + \delta + \frac{9\tau\rho^2}{\varepsilon^2}\delta\right) \tag{2.14}$$

holds with probability 1 if $n > n_2(\delta)$, where $\rho = \int_{-\infty}^{\infty} |g(x)| dF(x)$. The right-hand side of (2.14) may be smaller than ε for sufficiently small $\delta = \delta_\varepsilon$.

Hence from (2.13) and (2.14)

$$\sup_{I \in \mathfrak{T}} \left| \frac{\left[\int_I g dF_n\right]^2}{\int_I dF_n} - \frac{\left[\int_I g dF\right]^2}{\int_I dF} \right| < \varepsilon \tag{2.15}$$

holds with probability 1 if $n > n_0(\varepsilon) = \max(n_1(\varepsilon), n_2(\delta_\varepsilon))$, and the proof is completed.

3. Main theorems

In this section we shall state two main theorems which shows the optimum stratifications $\phi^*_{(n)}$ and $\phi^{**}_{(n)}$ for the empiric distribution function F_n in cases of proportionate and Neyman allocations converge almost surely in the following sense to optimum ones ϕ^* and ϕ^{**} for the true distribution function F respectively.

As stated in Section 1, the optimum stratifications ϕ^* and ϕ^{**} for the true F attain the minimum values of the functions

$$G_1(y(\phi)) = -\sum_{i=1}^{l} \frac{[u_i(\phi)]^2}{w_i(\phi)} \tag{3.1}$$

and

$$G_2(y(\phi)) = \sum_{i=1}^{l} \sqrt{w_i(\phi)v_i(\phi) - [u_i(\phi)]^2} \tag{3.2}$$

in cases of proportionate and Neyman allocations respectively.

Let $\phi^*_{(n)}$ and $\phi^{**}_{(n)}$ be optimum stratifications, corresponding to ϕ^* and ϕ^{**}, for the empiric distribution function $F_n(x)$ based on the simple random sample of size n which is given to us in advance of stratification operation.

Further let us define

$$y^{(n)} = (w^{(n)}(\phi), u^{(n)}(\phi), v^{(n)}(\phi)) , \tag{3.3}$$

$$w^{(n)}(\phi) = \int \phi dF_n , \qquad u^{(n)}(\phi) = \int x\phi dF_n$$

$$\text{and} \qquad v^{(n)}(\phi) = \int x^2 \phi dF_n .$$

Then $\phi^*_{(n)} = (\phi^*_{(n)1}, \cdots, \phi^*_{(n)l})$ may be represented by a partition of the real line consisting of l disjoint intervals each of which has no sample point on its end points as shown in Section 1.

THEOREM 1. *Let $F(x)$ be a distribution function which has a finite fourth moment and contains at least l points in its support. Then*

$$P\{\lim_{n\to\infty} G_1(y(\phi^*_{(n)})) = G_1(y(\phi^*))\} = 1 \tag{3.4}$$

holds, where G_1 is defined by (3.1) or (1.4), ϕ^* and $\phi^*_{(n)}$ are optimum stratifications in case of proportionate allocation for F and F_n respectively and $y(\phi) = (w(\phi), u(\phi))$. That is, $\phi^*_{(n)}$ converges almost surely to ϕ^* in the above sense.

PROOF. Let T be the set of all possible partitions of the real line consisting of l disjoint interval, i.e.

$$T=\{J=(J_1,\cdots,J_l)\}\ ,\qquad \bigcup_{i=1}^{l} J_i=(-\infty,\infty)$$

and

$$J_i\cap J_k=0\ \text{(empty)}\qquad \text{for } i\neq k\ .$$

Then it is easily seen by Lemma 2 that

$$\text{(3.5)}\qquad P\left\{\lim_{n\to\infty}\ \sup_{J\in T}\left|\sum_{i=1}^{l}\frac{\left[\int_{J_i} x dF_n(x)\right]^2}{\int_{J_i} dF_n(x)}-\sum_{i=1}^{l}\frac{\left[\int_{J_i} x dF(x)\right]^2}{\int_{J_i} dF(x)}\right|=0\right\}=1$$

holds. This means that for any positive ε there exists a positive integer $n_0(\varepsilon)$ such that

$$\text{(3.6)}\qquad |G_1[y^{(n)}(\phi_J)]-G_1[y(\phi_J)]|<\varepsilon$$

holds almost surely for any $J\in T$ and any $n>n_0(\varepsilon)$, where the ith component $\phi_{J_i}(x)$ of $\phi_J(x)$ donotes the indicator function of the interval J_i.

Since the optimum $\phi^*_{(n)}$ for F_n may be represented as $\phi_{J^{(n)}}$ using a partition $J^{(n)}$ in T, then the inequality (3.6) holds almost surely for $\phi_{J^{(n)}}$ if $n>n_0(\varepsilon)$. Therefore

$$\text{(3.7)}\qquad G_1[y^{(n)}(\phi_{J^{(n)}})]<G_1[y(\phi_{J^{(n)}})]+\varepsilon$$

holds almost surely for any $n>n_0(\varepsilon)$.

On the other hand

$$\text{(3.8)}\qquad G_1[y(\phi^*)]-\varepsilon<G_1[y^{(n)}(\phi^*)]\leqq G_1[y^{(n)}(\phi_{J^{(n)}})]$$

holds almost surely for any $n>n_0(\varepsilon)$ by (3.6) and the optimality of $\phi_{J^{(n)}}$ for F_n.

From (3.7) and (3.8) it is easily seen that

$$\text{(3.9)}\qquad G_1[y(\phi^*)]-\varepsilon<G_1[y(\phi_{J^{(n)}})]+\varepsilon\leqq G_1[y(\phi^*)]+\varepsilon$$

hold almost surely for any $n>n_0(\varepsilon)$. This is equivalent to (3.4), and the proof is completed.

THEOREM 2. *Let $F(x)$ be a distribution function which has a finite second moment and contains at least l points in its support. Then*

$$\text{(3.10)}\qquad P\{\lim_{n\leftarrow\infty} G_2(y^{**}_{(n)})]=G_2[y(\phi^{**})]\}=1$$

where G_2 is defined in (3.2) or (1.5), ϕ^{**} and $\phi^{**}_{(n)}$ are optimum stratifications in case of Neyman allocation for F and F_n respectively, and $y(\phi)$ $=(w(\phi), u(\phi), v(\phi))$.

PROOF. Let us first note that the image S and $S^{(n)}$ of the Φ of all stratifications by the linear mapping $y(\phi)$ and $y^{(n)}(\phi)$ are compact sets in R^{3l}. Further it is easily seen by the strong law of large numbers that for any positive ε

$$|u_i^{(n)}(\phi)| \leqq \int_{-\infty}^{\infty} |x| dF_n(x) < \int_{-\infty}^{\infty} |x| dF(x) + \varepsilon$$

and

$$|v_i^{(n)}(\phi)| \leqq \int_{-\infty}^{\infty} x^2 dF_n(x) < \int_{-\infty}^{\infty} x^2 dF(x) + \varepsilon$$

hold almost surely for $n > n_0(\varepsilon)$. This means the set $S^{(n)}$ is included in $S_\varepsilon = \{y'; y' = y \pm \varepsilon, y \in S\}$ for $n > n_0(\varepsilon)$.

Since $G_2(\cdot)$ is continuous on the compact set S_ε, it is uniformly continuous on S_ε. Besides it is easily seen by Lemma 1 that $y^{(n)}(\phi_J) = (w^{(n)}(\phi_J), u^{(n)}(\phi_J), v^{(n)}(\phi_J))$ converges almost surely to $y(\phi_J)$ uniformly in $J \in T$.

Therefore

$$P\{\lim_{n\to\infty} \sup_{J \in T} |G_2[y^{(n)}(\phi_J)] - G_2[y(\phi_J)]| = 0\} = 1 \tag{3.11}$$

holds, and hence the rest of the proof goes on in the same way as shown in Theorem 1.

4. Conclusion

We have shown in this paper that the optimum stratifications $\phi^*_{(n)}$ and $\phi^{**}_{(n)}$ for the empiric distribution function $F_n(x)$ converge to the optimum stratifications ϕ^* and ϕ^{**} for a univariate distribution function $F(x)$ respectively, both in cases of proportionate and Neyman allocation.

There still remain the following problems:

1) It is possible to extend these results to the multivariate distribution function $F(x)$?
2) May we find some useful algorithm to find out $\phi^*_{(n)}$ and $\phi^{**}_{(n)}$ for a finite n?

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