RECOVERY OF INTER-ROW AND INTER-COLUMN INFORMATION IN TWO-WAY DESIGNS*

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1. Introduction

Precision of estimates of treatment contrasts in two-way designs (i.e., designs in which heterogeneity is eliminated in two directions:—rows and columns) can be increased by the use of information available from inter-row and inter-column comparisons in addition to the usual "intra" estimates. Let ρ_r denote the ratio of the inter-row variance to the intra-row and column variance and similarly ρ_c denote the ratio of the inter-column variance to the intra-row and column variance. ρ_r and ρ_c play an important role in the combined inter and intra estimates of treatment effects; but these are usually unknown. The usual procedure is then to substitute estimates of these, available from an analysis of variance table for the data. As a result, the final estimate of treatment contrasts are no longer unbiased, in general. The estimates of ρ_r and ρ_c, that are used are also not unbiased. In this paper alternative unbiased estimators of ρ_r and ρ_c are proposed. These estimates have certain desirable properties and in addition, with their use the final estimates of treatment contrasts turn out to be unbiased. However, as estimates of ρ_r and ρ_c are used and not ρ_r, ρ_c themselves, an increase in the variance of the treatment estimates is inevitable. We have considered only a particular class of two-way designs for this, in this paper. If L denotes the row-incidence matrix and M, the column incidence matrix, we consider only those designs for which LL' and MM' have the same eigenvectors. Many of the two-way designs used in practice satisfy this requirement. For example, designs having property A and property B, as defined by Zelen and Federer [7] satisfy this requirement and the results of this paper are valid for them. Most of the results in this paper are extensions of similar results by J. Roy and K. R. Shah [5], in the case of one-way designs or incomplete block designs.

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2. Two-way designs

We consider a two-way design in which $n=uu'$ plots are arranged in u rows and u' columns and v treatments are assigned to them in such a way that each treatment is replicated r times; the ith treatment occurs l_{ij} times ($l_{ij}=0$ or 1) in the jth row and m_{ik} ($m_{ik}=0$ or 1) times in the kth column. The $v\times u$ and $v\times u'$ matrices $L=[l_{ij}]$, $M=[m_{ik}]$ are called the row and column incidence matrices respectively. E_{ab} will denote an $a\times b$ matrix, all the elements of which are unity. It follows that

$$LE_{u1}=rE_{v1}\,, \qquad E_{1v}L=u'E_{1u} \tag{2.1}$$

$$ME_{u'1}=rE_{v1}\,, \qquad E_{1v}M=uE_{1u'} \tag{2.2}$$

and

$$n=uu'=vr\,. \tag{2.3}$$

The model assumed is

$$y_{jk}=(\mu+\alpha_j+\beta_k+t_i+e_{ijk}) \qquad j=1, 2, \cdots, u, \; k=1, 2, \cdots, u' \tag{2.4}$$

when $l_{ij}=m_{ik}=1$ and, where

(2.5) y_{jk}=yield of the plot in the jth row and kth column,

(2.6) μ=the general mean,

(2.7) α_j=effect of the jth row,

(2.8) β_k=effect of the kth column,

(2.9) t_i=effect of the ith treatment,

(2.10) e_{ijk}=error.

e_{ijk} are assumed to be normally and independently distributed with zero means and a common variance σ^2. We shall express this by writing e_{ijk} are $NI(0, \sigma^2)$. When, however, inter-row and inter-column information is to be recovered, we make a further assumption that the α_j are $NI(0, \sigma_r^2)$ and the β_k are $NI(0, \sigma_c^2)$ and that α_j, β_k, e_{ijk} are all independently distributed. For $i=1,\cdots,v$; $j=1,\cdots,u$; $k=1,\cdots,u'$, we shall use the following symbols also:

(2.11) $R_j=\sum_k y_{jk}$=total of the jth row,

(2.12) $C_k=\sum_j y_{jk}$=total of the kth column, (2.13) T_i = total of the plots receiving the ith treatment

(2.14) $g=\sum_j R_j=\sum_k C_k=\sum_j\sum_k y_{jk}$ = the grand total,

We need the following vectors:

$$t'=[t_1,\cdots,t_v]\ ,\qquad \alpha'=[\alpha_1,\cdots,\alpha_u]\ ,\qquad \beta'=[\beta_1,\cdots,\beta_{u'}]$$

$$R'=[R_1,\cdots,R_u]\ ,\qquad C'=[C_1,\cdots,C_{u'}]\ ,\qquad T'=[T_1,\cdots,T_v]\ .$$

Only treatment contrasts i.e., functions of the type $\xi' t$ where $E_{iv}\xi=0$ are estimable. The intra-row and column estimate of $\xi' t$ will be denoted by $\xi'\hat{t}$, the inter-row estimate alone (when it exists) by $\xi'\hat{t}_r$, the inter-column estimate alone (when it exists) by $\xi'\hat{t}_c$ and the combined intra and inter-row and column estimate by $\xi'\bar{t}$. The reduced normal equations for determining $\hat{t}$, $\hat{t}_r$, $\hat{t}_c$ or $\bar{t}$ are given below:

(a) Only intra-row and column estimates

$$Q=F\hat{t} \tag{2.15}$$

(b) Inter-row estimates only

$$Q_r=\frac{1}{u'}LL'\hat{t}_r \tag{2.16}$$

(c) Inter-column estimates only

$$Q_c=\frac{1}{u}MM'\hat{t}_c \tag{2.17}$$

(d) Combined intra and inter estimate

$$P=\left(WF+\frac{W_r}{u'}LL'+\frac{W_c}{u}MM'\right)\bar{t}\ . \tag{2.18}$$

Here

$$Q=T-\frac{1}{u'}LR-\frac{1}{u}MC'+\frac{rg}{n}E_{v1}\ , \tag{2.19}$$

$$Q_r=\frac{1}{u'}LR-\frac{rg}{n}E_{v1}\ , \tag{2.20}$$

$$Q_c=\frac{1}{u}MC-\frac{rg}{n}E_{v1}\ , \tag{2.21}$$

$$F=rI-\frac{1}{u'}LL'-\frac{1}{u}MM'+\frac{r^2}{n}E_{vv}\ , \tag{2.22}$$

$$P = WQ + W_r Q_r + W_c Q_c \,, \tag{2.23}$$

$$W = \frac{1}{\sigma^2} \,, \tag{2.24}$$

$$\left. \begin{aligned} W_r &= \frac{1}{\sigma^2 + u'\sigma_r^2} \\ W_c &= \frac{1}{\sigma^2 + u\sigma_c^2} \end{aligned} \right\} . \tag{2.25}$$

The quantities ρ_r and ρ_c mentioned in Section 1 are respectively

$$\rho_r = \frac{W}{W_r} \quad \text{and} \quad \rho_c = \frac{W}{W_c} \,. \tag{2.26}$$

One has to solve (a), (b), (c) and (d), in conjunction with some suitable additional equation like $E_{1v}t=0$, to obtain any solutions $\hat{t}$, $\hat{t}_r$, $\hat{t}_c$, $\bar{t}$ of these 4 sets. It can be readily seen that the variance-covariance matrices of Q, Q_r, Q_c are respectively $(1/W)F$, $(1/W_r)(LL'/u' - r^2E_{vv}/n)$ and $(1/W_c)(MM'/u - r^2E_{vv}/n)$. The covariance matrix of any two of them is null.

We assume that the rank of LL' is q_r+1, of MM' is q_c+1 and that LL' and MM' have the same eigenvectors. Martin and Zyskind [3] have observed that this condition is sufficient for best combinability of inter and intra information. Note that $(1/\sqrt{v})E_{v1}$ is an eigenvector of both LL', MM', the corresponding eigenvalues being $u'r$ and ur respectively. Let the other eigenvectors of LL' and MM' be ξ_s $(s=1, 2, \cdots, v-1)$ and we shall choose them to be all unit and mutually orthogonal (orthogonal to $(1/\sqrt{v})E_{v1}$ also). Let the corresponding eigenvalues for LL' be e_s and for MM' be g_s. Of course, $e_s=0$ for $s>q_r$ and $g_s=0$ for $s>q_c$. As a result of these assumptions, F also has the same eigenvectors viz

$$\frac{1}{\sqrt{v}}E_{v1}, \xi_1, \cdots, \xi_{v-1} \,,$$

the corresponding eigenvalues being

$$\phi_0 = 0, \phi_1, \cdots, \phi_{v-1}$$

where

$$\phi_s = r - \frac{1}{u'}e_s - \frac{1}{u}g_s \qquad s=1, 2, \cdots, v-1 \,. \tag{2.27}$$

All treatment contrasts are estimable (see Chakrabarti, [1]) if and only if rank $F=v-1$ and we assume so. ϕ_s $(s=1, \cdots, v-1)$ are then all non-null. From (2.18), the combined inter and intra estimator of $\xi_s't$ is ($s=$

1, 2, $\cdots$, $v-1$)

(2.28)
$$\xi_s'\bar{t}=\frac{W(\xi_s'Q)+W_r(\xi_s'Q_r)+W_c(\xi_s'Q_c)}{W\phi_s+(W_r/u')e_s+(W_c/u)g_s}=\frac{\xi_s'Q+(1/\rho_r)\xi_s'Q_r+(1/\rho_c)\xi_s'Q_c}{\phi_s+(1/u'\rho_r)e_s+(1/u\rho_c)g_s}\ .$$

It can be easily proved that this is unbiased for $\xi_s't$, its variance is

(2.29)
$$\frac{1}{W\phi_s+(W_r/u')e_s+(W_c/u)g_s}$$

and that

(2.30)
$$\mathrm{Cov}\,(\xi_s'\bar{t},\ \xi_l'\bar{t})=0 \qquad s\neq l\ .$$

However ρ_r and ρ_c are not known and we use some estimates P_r and P_c of them in (2.28). In that case, the difference between this latter expression and (2.28) is easily seen to be (we use ρ_r, ρ_c or P_r, P_c in the paranthesis of (2.28), to indicate whether we are referring to the true values or estimates),

(2.31)
$$\xi_s'\bar{t}(P_r,\ P_c)-\xi_s'\bar{t}(\rho_r,\ \rho_c)=Z_s\ , \qquad \text{say } s=1, 2, \cdots, v-1$$

where

(2.32)
$$Z_s=\frac{\phi_s e_s}{u'}\left(\frac{1}{\rho_r}-\frac{1}{P_r}\right)w_s+\frac{e_s g_s}{uu'\rho_c}\left(\frac{1}{\rho_r}-\frac{1}{P_r}\right)(w_s-x_s)$$
$$+\frac{\phi_s g_s}{u}\left(\frac{1}{\rho_c}-\frac{1}{P_c}\right)x_s-\frac{e_s g_s}{uu'\rho_r}\left(\frac{1}{\rho_c}-\frac{1}{P_c}\right)(w_s-x_s)\ ,$$

(2.33)
$$w_s=\frac{\xi_s'Q}{\phi_s}-\frac{u'\xi_s'Q_r}{e_s}\ ,$$

(2.34)
$$x_s=\frac{\xi_s'Q}{\phi_s}-\frac{u\xi_s'Q_c}{g_s}\ .$$

In the next section, we consider the classical estimates of ρ_r and ρ_c, suggest some alternative estimates of them, having some desirable properties and examine whether the expected value of Z_s, above reduces to zero, for these estimates, so that even $\xi_s'\bar{t}(P_r,\ P_c)$ is unbiased for $\xi_s't$.

3. Structure of the analysis of variance

The adjusted treatment sum of squares (s.s.) in the analysis of variance for such a design is $Q'\hat{t}$, where $\hat{t}$ is any solution of (2.15) and can be easily seen to be equal to $\sum_{s=1}^{v-1}(\xi_s'Q)^2/\phi_s$, (d.f. $v-1$), as ξ_s are eigenvectors of F, with ϕ_s as the corresponding eigenvalues. Also the error s.s.

(Intra-row and column) viz

$$\underset{\mu,\,\boldsymbol{\alpha},\,\boldsymbol{\beta},\,\boldsymbol{t}}{\text{Min}} \; \{y_{jk}-E(y_{jk}\,|\,\boldsymbol{\alpha}_1\boldsymbol{\beta})\}^2$$

reduces to

$$(3.1)\quad E_i=\left(\sum_j \sum_k y_{jk}^2-\frac{g^2}{n}\right)-\sum_{s=1}^{v-1} (\boldsymbol{\xi}_s'\boldsymbol{Q})^2/\phi_s-(\boldsymbol{R}'\boldsymbol{R}/u'-g^2/n)-\left(\frac{\boldsymbol{c}'\boldsymbol{c}}{u}-\frac{g^2}{n}\right) .$$

In other words,

$$E_i=\text{total s.s.}-\text{treatment s.s. (adj.)}$$
$$-\text{row s.s. (unadj.)}-\text{column s.s. (unadj.)}.$$

From the least squares theory, it is well-known that E_i has the $x^2\sigma^2$ distribution and is independently distributed of $\boldsymbol{Q}$, any row contrast or any column contrast. E_i has $(n-1)-(v-1)-(u-1)-(u'-1)=\nu$ degrees of freedom (d.f.). It can also be shown that, in the analysis without recovery of inter-row and inter-column information i.e., when $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are fixed effects, the adjusted row s.s. for testing the significance of row effects viz

$$\underset{\mu,\,\boldsymbol{\beta},\,\boldsymbol{t}}{\text{Min}} \sum_j \sum_k (y_{jk}-\mu-\beta_k-t_i)^2-\underset{\mu,\,\boldsymbol{\alpha},\,\boldsymbol{\beta},\,\boldsymbol{t}}{\text{Min}} \sum\sum (y_{jk}-\mu-\alpha_j-\beta_k-t_i)^2$$

comes out to be

$$(3.2)\qquad R_a=(\boldsymbol{R}'\boldsymbol{R}/u'-g^2/n)+\sum_{s=1}^{v-1} (\boldsymbol{\xi}_s'\boldsymbol{Q})^2/\phi_s$$
$$-\left(\boldsymbol{T}-\frac{1}{u}\boldsymbol{MC}\right)\left(r\boldsymbol{I}-\frac{1}{u}\boldsymbol{MM}'\right)^*\left(\boldsymbol{T}-\frac{1}{u}\boldsymbol{MC}\right) .$$

Here $(r\boldsymbol{I}-(1/u)\boldsymbol{MM}')^*$ denotes a generalized inverse of $r\boldsymbol{I}-(1/u)\boldsymbol{MM}'$ (Rao [4]). We can take

$$(3.3)\qquad \left(r\boldsymbol{I}-\frac{1}{u}\boldsymbol{MM}'\right)^*=\sum_{s=1}^{v-1}\left(r-\frac{1}{u}g_s\right)^{-1}\boldsymbol{\xi}_s\boldsymbol{\xi}_s'$$

as $\boldsymbol{\xi}_s$ are eigenvectors of $r\boldsymbol{I}-(1/u)\boldsymbol{MM}'$ and $r-(1/u)g_s$ are the corresponding non-zero eigenvalues. Using (2.19), (2.20), (2.27) and (3.3) in (3.2), we find

$$(3.4)\quad R_a=(\boldsymbol{R}'\boldsymbol{R}/u'-g^2/n)+\sum_{s=1}^{v-1} (\boldsymbol{\xi}_s'\boldsymbol{Q})^2/\phi_s-u'\sum_{s=1}^{v-1} (\boldsymbol{\xi}_s'\boldsymbol{Q}+\boldsymbol{\xi}_s'\boldsymbol{Q}_r)^2/(u'\phi_s+e_s) .$$

It has $u-1$ d.f.

When we recover inter-row information, we minimize

$$\frac{1}{u'}\sum_{j=1}^{u} \{R_j-E(R_j)\}^2$$

with respect to μ and $\boldsymbol{t}$, leading to (2.16). The minimum value is called the inter-row error s.s. We shall denote it by E_r and comes out to be

$$E_r=(\boldsymbol{R}'\boldsymbol{R}/u'-g^2/n)-\boldsymbol{Q}_r'\hat{\boldsymbol{t}}_r \tag{3.5}$$
$$=(\boldsymbol{R}'\boldsymbol{R}/u'-g^2/n)-u'\sum_{s=1}^{q_r}(\boldsymbol{\xi}_s'\boldsymbol{Q}_r)^2/e_s$$

and has $u-1-q_r$ d.f. It is independently distributed of $\boldsymbol{Q}_r$ and by least squares theory, has the $\chi^2(\sigma^2+u'\sigma_r^2)$ distribution as $V(R_j)=u'(\sigma^2+u'\sigma_r^2)$. Obviously E_r is the sum of squares due to those row-contrasts, which are uncorrelated with $\boldsymbol{Q}_r$. This E_r is a part of the adjusted row s.s. R_a also and we can show, by a little algebra, that

$$R_a=E_r+\sum_{s=1}^{q_r}\frac{e_s\phi_s}{u'\phi_s+e_s}w_s^2 \tag{3.6}$$

where

$$w_s=\frac{\boldsymbol{\xi}_s'\boldsymbol{Q}}{\phi_s}-\frac{u'\boldsymbol{\xi}_s'\boldsymbol{Q}_r}{e_s} \qquad s=1, ,\cdots, q_r\ . \tag{3.7}$$

Observe that w_s are normal variables with

$$E(w_s)=0 \tag{3.8}$$

$$V(w_s)=\frac{1}{W}\left(\frac{1}{\phi_s}+\frac{u'\rho_r}{e_s}\right) \tag{3.9}$$

and

$$\operatorname{Cov}(w_s, w_l)=0\ , \qquad s\neq l\ . \tag{3.10}$$

In exactly a similar manner, the adjusted column s.s. C_a is (d.f. $u'-1$)

$$(\boldsymbol{C}'\boldsymbol{C}/u-g^2/n)+\sum_{s=1}^{v-1}(\boldsymbol{\xi}_s'\boldsymbol{Q})^2/\phi_s)-u\sum_{s=1}^{v-1}(\boldsymbol{\xi}_s'\boldsymbol{Q}+\boldsymbol{\xi}_s'\boldsymbol{Q}_c)^2/(u\phi_s+g_s)\ . \tag{3.11}$$

The inter-column error s.s. E_c (with d.f. $u'-1-q_c$) is

$$E_c=(\boldsymbol{C}'\boldsymbol{C}/u-g^2/n)-u\sum_{s=1}^{q_c}(\boldsymbol{\xi}_s'\boldsymbol{Q}_c)^2/g_s\ . \tag{3.12}$$

It is independently distributed of $\boldsymbol{Q}_c$ and has the $\chi^2(\sigma^2+u\sigma_c^2)$ distribution. It is the s.s. due to those column contrasts which are uncorrelated with $\boldsymbol{Q}_c$. Also it is a part of the adjusted column s.s. C_a and

$$C_a=E_c+\sum_{s=1}^{q_c}\frac{g_s\phi_s}{u\phi_s+g_s}x_s^2 \tag{3.13}$$

where

$$x_s = \frac{\boldsymbol{\xi}_s' \boldsymbol{Q}}{\phi_s} - \frac{u \boldsymbol{\xi}_s' \boldsymbol{Q}_c}{g_s} \qquad s=1, 2, \cdots, q_c \tag{3.14}$$

are normal variables with

$$E(x_s)=0 \tag{3.15}$$

$$V(x_s)=\frac{1}{W}\left(\frac{1}{\phi_s}+\frac{u\rho_c}{g_s}\right) \tag{3.16}$$

$$\text{Cov}\,(x_s, x_l)=0\ , \qquad s\neq l \tag{3.17}$$

and

$$\text{Cov}\,(w_s, x_l)=\begin{cases} \sigma^2/\phi_s\ , & s=l \\ 0, & s\neq l\ , \end{cases} \tag{3.18}$$

Consider any row contrast $\boldsymbol{a}'\boldsymbol{R}$ (where $a'E_{u1}=0$), which is uncorrelated with $\boldsymbol{Q}_r$. Then it is easy to observe that $\boldsymbol{a}'\boldsymbol{R}$ is uncorrelated with any w_s. Since E_r is the s.s. of contrasts like this $\boldsymbol{a}'\boldsymbol{R}$, it is obvious that E_r and w_s are independently distributed. Further, as

$$\text{Cov}\,(\boldsymbol{R}, \boldsymbol{C})=(\sigma^2+\sigma_r^2+\sigma_c^2)E_{uu'}$$

and row contrast is uncorrelated with $\boldsymbol{C}$ and hence

$$\text{Cov}\,(\boldsymbol{a}'\boldsymbol{R}, x_s)=0\ .$$

Thus E_r is independently distributed of x_s $(s=1, 2, \cdots, q_c)$. By a similar reasoning, E_c is independently distributed of w_s $(s=1, 2, \cdots, q_r)$ and of x_s $(s=1, 2, \cdots, q_c)$.

4. Estimation of ρ_r and ρ_c

By least squares theory, $E(E_i\,|\,\boldsymbol{\alpha}, \boldsymbol{\beta})=\nu\sigma^2$ and hence, even when $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are random, $E(E_i)=\nu\sigma^2$ and E_i/ν provides an estimate of σ^2. Now from (3.4),

$$E(R_a)=(u-1)\sigma^2+u'(u-1-\gamma_r)\sigma_r^2 \tag{4.1}$$

where

$$\gamma_r=\sum_{s=1}^{q_r}\frac{e_s}{u'\phi_s+e_s}\ . \tag{4.2}$$

Similarly

$$E(C_a)=(u'-1)\sigma^2+u(u'-1-\gamma_c)\sigma_c^2 \tag{4.3}$$

(4.4) $$\gamma_c=\sum_{s=1}^{q_c}\frac{g_s}{u\phi_s+g_s}\ .$$

Hence, the classical estimates of σ_r^2 and σ_c^2 are respectively

(4.5) $$\hat{\sigma}_r^2=\frac{R_a-(u-1)E_i/\nu}{u'(u-1-\gamma_r)}$$

(4.6) $$=\frac{E_r+\sum_{s=1}^{q_r}\frac{e_s\phi_s}{u'\phi_s+e_s}w_s^2-(u-1)E_i/\nu}{u'(u-1-\gamma_r)}\qquad\text{(by 3.6)}$$

and

(4.7) $$\hat{\sigma}_c^2=\frac{C_a-(u'-1)E_i/\nu}{u(u'-1-\gamma_c)}$$

(4.8) $$=\frac{E_c+\sum_{s=1}^{q_c}\frac{g_s\phi_s}{u\phi_s+g_s}x_s^2-(u'-1)E_i/\nu}{u(u'-1-\gamma_c)}\qquad\text{(by 3.13).}$$

These however, could be negative. From these estimates, the classical estimates of ρ_r and ρ_c are obtained as

(4.9) $$\hat{\rho}_r=(\text{Estimate of }\sigma^2+u'\sigma_r^2)/\text{Estimate of }\sigma^2$$
$$=\nu R_a/(u-1-\gamma_r)E_i-\gamma_r/(u-1-\gamma_r)$$

(4.10) $$=\frac{\nu}{(u-1-\gamma_r)E_i}\left\{E_r+\sum_{s=1}^{q_r}\frac{e_s\phi_s}{u'\phi_s+e_s}w_s^2\right\}-\frac{\gamma_r}{u-1-\gamma_r}$$

and similarly

(4.11) $$\hat{\rho}_c=\frac{\nu C_a}{(u'-1-\gamma_c)}-\frac{\gamma_c}{u'-1-\gamma_c}$$

(4.12) $$=\frac{\nu}{(u'-1-\gamma_c)E_i}\left\{E_c+\sum_{s=1}^{q_c}\frac{g_s\phi_s}{u\phi_s+g_s}x_s^2\right\}-\frac{\gamma_c}{u'-1-\gamma_c}$$

$\hat{\rho}_r$ and $\hat{\rho}_c$ are not unbiased and they could be less than 1 also, even if ρ_r and ρ_c cannot be. The bias can be removed easily. From the distribution of E_i, E_r, E_c, x_s, ω_s (which have been already stated in the last section), one can show that

(4.13) $$E(\hat{\rho}_r)=\frac{(u-1)\nu}{(u-1-\gamma_r)(\nu-2)}\rho_r-\frac{\gamma_r}{u-1-\gamma_r}$$

and hence

(4.14)
$$\hat{\hat{\rho}}_r=\frac{(u-1-\gamma_r)(\nu-2)}{(u-1)\nu}\left\{\hat{\rho}_r+\frac{\gamma_r}{u-1-\gamma_r}\right\}$$
$$=\frac{R_a(u-1)}{E_i/\nu}\cdot\frac{\nu-2}{\nu}$$

is unbiased for ρ_r and similarly

(4.15)
$$\hat{\hat{\rho}}_c=\frac{C_a/(u'-1)}{E_i/\nu}\cdot\frac{\nu-2}{\nu}$$

is unbiased for ρ_c.

Following J. Roy and K. R. Shah [5], we consider a more general form than (4.10) viz

(4.16)
$$P_r=\frac{aE_r+\sum_{s=1}^{q_r} b_s w_s^2}{E_i}+c$$

where a, b_s, c are arbitrary constants and are so determined that

(i) P_r is unbiased for ρ_r,

(ii) the dominant term viz the coefficient of ρ_r^2 in the variance of ρ_r, is minimum. From the distributions of E_r, w_s and E_i, we can find $E(P_r)$ and $V(P_r)$ and this, after a considerable algebra, leads to (or we can use J. Roy and K. R. Shah's results for one-way designs with appropriate changes to suit this situation)

(4.17)
$$a=\frac{3(\nu-2)}{3(u-1-q_r)+(u+1+q_r)q_r}$$

(4.18)
$$b_s=(u+1+q_r)ae_s/3u'$$

and

(4.19)
$$c=\frac{-1}{\nu-2}\sum_{s=1}^{q_r}\frac{b_s}{\phi_s}\,.$$

By changing E_r to E_c, q_r to q_c, w_s to x_s in P_r, we shall get a similar estimate P_c of ρ_c and the values of a, b_s, c for that can be easily obtained from (4.17), (4.18) and (4.19) by making these changes there and in addition changing u to u'.

This estimate P_r (or P_c) of ρ_r (or ρ_c) is better than the classical estimate, as it is optimum in a certain sense viz the coefficient of ρ_r^2 (or ρ_c^2) in its variance is minimum.

Following Roy and Shah we also consider a quadratic form of the type

(4.20)
$$b_0E_i+b_1E_r+\sum_{s=1}^{q_r} a_s\phi_s w_s^2$$

to estimate $\sigma^2+u'\sigma_r^2$. The constants b_0, b_1, a_s are so chosen as to make this an unbiased estimate and minimize the dominant term in its variance viz the coefficient of ρ_r^2. This yields

$$v_r = -\frac{E_i}{\nu(u-1)}\sum_{s=1}^{q_r}\frac{e_s}{u'\phi_s} - \frac{1}{u-1}\left\{E_r + \sum_{s=1}^{q_r}\frac{e_s w_s^2}{u'}\right\} \tag{4.21}$$

as an optimum estimate of $\sigma^2+u'\sigma_r^2$. If we employ this method for estimating $\sigma^2+u\sigma_c^2$ or σ^2 alone, we get

$$v_c = -\frac{E_i}{\nu(u'-1)}\sum_{s=1}^{q_c}\frac{g_s}{u\phi_s} + \frac{1}{u'-1}\left\{E_c + \sum_{s=1}^{q_r}\frac{g_s x_s^2}{u}\right\} \tag{4.22}$$

for estimating $\sigma^2+u'\sigma_c^2$ and

$$v_0 = E_i/\nu \tag{4.23}$$

for σ^2. Using these estimates, we find again that an unbiased estimate of ρ_r is provided by

$$\left(1-\frac{2}{\nu}\right)\frac{v_r}{v_0} - \frac{2}{u-1}\,\frac{1}{\nu}\sum_{s=1}^{q_r}\frac{e_s}{u'\phi_s} \tag{4.24}$$

and of ρ_c by

$$\left(1-\frac{2}{\nu}\right)\frac{v_c}{v_0} - \frac{2}{u'-1}\,\frac{1}{\nu}\sum_{s=1}^{q_c}\frac{g_s}{u\phi_s}\,. \tag{4.25}$$

Let

$$\boldsymbol{R}(\hat{\boldsymbol{t}}) = \boldsymbol{R} - \boldsymbol{L}'\hat{\boldsymbol{t}}\,. \tag{4.26}$$

Then

$$\begin{aligned} \frac{1}{u'}\boldsymbol{R}'(\hat{\boldsymbol{t}})\boldsymbol{R}(\hat{\boldsymbol{t}}) - \frac{g^2}{n} &= (\boldsymbol{R}'\boldsymbol{R}/u' - g^2/n) - 2\hat{\boldsymbol{t}}'\boldsymbol{L}\boldsymbol{R}/u' + \hat{\boldsymbol{t}}'\boldsymbol{L}\boldsymbol{L}'\hat{\boldsymbol{t}}/u' \\ &= (\boldsymbol{R}'\boldsymbol{R}/u' - g^2/n) - 2\sum_{s=1}^{q_r}\frac{1}{\phi_s}(\boldsymbol{\xi}_s'\boldsymbol{Q})(\boldsymbol{\xi}_s'\boldsymbol{Q}_r) + \frac{1}{u'}\sum_{s=1}^{q_r}\frac{e_s(\boldsymbol{\xi}_s'\boldsymbol{Q})^2}{\phi_s^2} \\ &= E_r + \sum_{s=1}^{q_r}\frac{e_s w_s^2}{u'}\,, \end{aligned} \tag{4.27}$$

from (3.6) and (3.7). Similarly, if

$$\boldsymbol{C}(\hat{\boldsymbol{t}}) = \boldsymbol{C} - \boldsymbol{M}'\hat{\boldsymbol{t}}\,, \tag{4.28}$$

$$\frac{1}{u}\boldsymbol{C}'(\hat{\boldsymbol{t}})\boldsymbol{C}(\hat{\boldsymbol{t}}) - \frac{g^2}{n} = E_c + \sum_{s=1}^{q_c}\frac{g_s x_s^2}{u}\,. \tag{4.29}$$

This shows that in actual computation of v_r or v_c, it is easier to use $(1/u')\boldsymbol{R}'(\hat{\boldsymbol{t}})R(\hat{\boldsymbol{t}})-g^2/n$ and $(1/u)\boldsymbol{C}'(\hat{\boldsymbol{t}})\boldsymbol{C}(\hat{\boldsymbol{t}})-g^2/n$ rather than E_r, E_c, e_s, g_s, w_s, x_s.

It may be added here that, as E_r and E_c are respectively $\chi^2(\sigma^2+u'\sigma_r^2)$ and $\chi^2(\sigma^2+u\sigma_c^2)$, with $u-1-q_r$ and $u'-1-q_c$ d.f., we can even use $E_r/(u-1-q_r)$ and $E_c/(u'-1-q_c)$ for estimating $\sigma^2+u'\sigma_r^2$ and $\sigma^2+u\sigma_c^2$ respectively. Further these estimates are always positive.

5. Effect of using estimates of ρ_r and ρ_c on $\boldsymbol{\xi}_s'\boldsymbol{t}$

We shall assume that the estimates of ρ_r and ρ_c used, are of the general form (4.16). The classical estimates are also of that form. We have already observed that E_r, E_c, E_i, w_s $(s=1, 2, \cdots, q_r)$ are all independently distributed, all x_s $(s=1, 2, \cdots, q_c)$ are also independent among themselves and independent of E_r, E_c, E_i, w_l $(l\neq s)$ but each x_s is correlated only with the corresponding w_s. Hence

$$(5.1)\qquad E\left(\frac{1}{P_r}w_s\right)=E\left\{\frac{E_i}{aE_r+\sum_1^{q_r}b_s w_s^2+cE_i}w_s\right\}$$

$$=E\{\text{conditional expectation of } (1/P_r)w_s \text{ when } E_i,\ E_r,\ w_l,\ l\neq s \text{ are all fixed}\}$$

$$=E\{\text{conditional expectation of an odd function of } w_s\}$$

$$=0 \text{ as } w_s \text{ has a normal distribution with zero mean}$$

Similarly,

$$(5.2)\qquad E\left(\frac{1}{P_r}\right)x_s=E(\text{conditional expectation of } (1/P_r)x_s, \text{ when } E_i,\ E_r,\ w_s \text{ are all fixed})$$

$$=E\left\{\frac{E(x_s\mid w_s)}{P_r}\right\}$$

$$=E\left\{\frac{1}{P_r}\,\frac{\operatorname{cov}(x_s, w_s)}{V(w_s)}w_s\right\}$$

$$=\text{constant } E\left(\frac{1}{P_r}w_s\right)$$

$$=0 \qquad \text{by (5.1).}$$

Similarly $E((1/P_r)w_s)=E((1/P_c)x_s)=0$ and hence, from (2.31),

$$(5.3)\qquad E\{\boldsymbol{\xi}_s'\boldsymbol{t}(P_r, P_c)-\boldsymbol{\xi}_s'\boldsymbol{t}(\rho_r, \rho_c)\}$$

$$
\begin{aligned}
&= E(Z_s) \\
&= \frac{\phi_s e_s}{u'} E\left[\left(\frac{1}{\rho_r}-\frac{1}{P_r}\right)w_s\right] + \frac{e_s g_s}{uu'\rho_c} E\left[\left(\frac{1}{\rho_r}-\frac{1}{P_r}\right)(w_s - x_s)\right] \\
&\quad + \frac{\phi_s g_s}{u} E\left[\left(\frac{1}{\rho_c}-\frac{1}{P_c}\right)x_s\right] - \frac{e_s g_s}{uu'\rho_r} E\left[\left(\frac{1}{\rho_c}-\frac{1}{P_c}\right)(w_s - x_s)\right] \\
&= 0 .
\end{aligned}
$$

Thus

$$
\text{(5.4)} \qquad \begin{aligned} E[\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(P_r, P_c)] &= E[\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(\rho_r, \rho_c)] \\ &= \boldsymbol{\xi}_s'\bar{\boldsymbol{t}} \end{aligned}
$$

and $\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(P_r, P_c)$ is an unbiased estimate of the treatment contrast $\boldsymbol{\xi}_s'\boldsymbol{t}$, even if we substitute P_r, P_c, for ρ_r and ρ_c. Now $\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(\rho_r, \rho_c)$ is the unbiased minimum variance estimator of $\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(\rho_r, \rho_c)$ and z_s is a zero function and so, by Stein's theorem [6]. $\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(\rho_r, \rho_c)$ is uncorrelated with Z_s and hence

$$
\text{(5.5)} \qquad V\{\boldsymbol{\xi}_s'\boldsymbol{t}(P_r, P_c)\} = V\{\boldsymbol{\xi}_s'\boldsymbol{t}(\rho_r, \rho_c)\} + V(Z_s) .
$$

The effect of substituting estimates of ρ_r and ρ_c is therefore to increase the variance by $V(Z_s)$.

By an argument similar to the one used in (5.1) and (5.2), it can be shown that

$$
\text{(5.6)} \qquad \operatorname{cov}(Z_s, Z_l) = 0 , \qquad s \neq l
$$

and hence, as $\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}$ are independently distributed (see (2.30)), $\boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(P_r, P_c)$
(5.7) and $\boldsymbol{\xi}_l'\bar{\boldsymbol{t}}(P_r, P_c)$ are uncorrelated for $s \neq l$

Now any treatment contrast $\boldsymbol{h}'\boldsymbol{t}$ can be expressed as a linear combination of the contrasts $\boldsymbol{\xi}_s'\boldsymbol{t}$. Say

$$
\tau = \boldsymbol{h}'\boldsymbol{t} = \sum_{s=1}^{v-1} k_s \boldsymbol{\xi}_s'\boldsymbol{t} .
$$

The minimum variance unbiased estimator of $h't$, when ρ_r, ρ_c are known is therefore $\sum_{s=1}^{v-1} k_s \boldsymbol{\xi}_s'\bar{\boldsymbol{t}}(\rho_r, \rho_c) = \bar{\tau}(\rho_r, \rho_c)$. When ρ_r, ρ_c are not known, and we substitute their estimates P_r, P_c (of the suitable form), we shall obtain

$$
\bar{\tau}(P_r, P_c)
$$

as an estimate of τ; it will be unbiased for τ, but

$$
V[\bar{\tau}(P_r, P_c)] = V[\bar{\tau}(\rho_r, \rho_c)] + \sum k_s^2 V(Z_s) .
$$

The second term in the right hand side represents the increase in the variance due to the use of P_r, P_c instead of ρ_r and ρ_c.

6. Designs with property (A) and (B)

Zelen and Federer [7] introduced certain structural properties of two-way designs. These are related to the incidence matrices L and M corresponding to the rows and columns. Let $v=\prod_{i=1}^{m} a_i$ and denote the $a_i \times a_i$ identity matrix by I_i and $E_{a_i a_i}$ by J_i. We then define

$$D_i^{\delta_i}=\begin{cases} I_i & \text{if } \delta_i=0 \\ J_i & \text{if } \delta_i=1 . \end{cases} \tag{6.1}$$

Then a two-way design is said to have property (A) if

$$LL'=\sum_{s=0}^{m}\left\{\sum_{\delta_1+\cdots+\delta_m=s} h_r(\delta_1,\delta_2,\cdots,\delta_m)[D_1^{\delta_1}\times D_2^{\delta_2}\times\cdots\times D_m^{\delta_m}]\right\} \tag{6.2}$$

whe $\times$ denotes Kronecker product and $h_r(\delta_1,\cdots,\delta_m)$ are constants. This can be written, alternatively, in short, as

$$LL'=\sum_{\boldsymbol{\delta}} h_r(\boldsymbol{\delta})\prod_{i=1}^{m}\times D_i^{\delta_i} \tag{6.3}$$

where $\boldsymbol{\delta}=(\delta_1,\delta_2,\cdots,\delta_m)$, each $\delta_i=0$ or 1 and the summation in (6.3) is over all the 2^m binary numbers $\boldsymbol{\delta}$. Similarly the design is said to have property (B) if

$$MM'=\sum_{\boldsymbol{\delta}} h_c(\boldsymbol{\delta})\prod_{i=1}^{m}\times D_i^{\delta_i} \tag{6.4}$$

where $h_c(\boldsymbol{\delta})$ are some other constants. If the design has both the properties (A) and (B), LL' and MM' have the same eigenvectors. This can be shown as below. Let $\chi=(\chi_1,\cdots,\chi_m)$ where each $\chi_i=0$ or 1 only. Define

$$B_i^{\chi_i}=\begin{cases} \frac{1}{a_i}J_i & \text{if } \chi_i=0 \\ I_i-\frac{1}{a_i}J_i & \text{if } \chi_i=1 \end{cases} \tag{6.5}$$

$$B^{\chi}=\prod_{i=1}^{m}\times B_i^{\chi_i} . \tag{6.6}$$

It is easy to show that B^{χ} independent, and

$$B^{\chi}B^{y}=0 \quad \text{if } \chi\neq y \tag{6.7}$$

i.e. columns of B^χ are orthogonal to those of B^y, if $\chi \neq y$.

From (6.3), it can be shown that

$$LL'B^\chi = \sum_\delta h_r(\delta) \prod_{i=1}^{m} \times D_i^{\delta_i} \cdot \prod_{i=1}^{m} \times B_i^{\chi_i} \tag{6.8}$$
$$= E_r(\chi) B^\chi$$

where $E_r(\chi) = \sum_\delta h_r(\delta) \prod_{i=1}^{m} \alpha_i(\chi_i, \delta_i)$,

$$\alpha_i(\chi_i, \delta_i) = \begin{cases} 1 & \chi_i = 0, 1, \quad \delta_i = 0 \\ 0 & \chi_i = 1, \quad \delta_i = 1 \\ a_i & \chi_i = 0, \quad \delta_i = 1 \end{cases} \tag{6.9}$$

(6.8) (along with (6.7)), shows that, the columns from B^χ are eigenvectors of LL', and the corresponding eigenvalue is $E_r(\chi)$ (repeated as many times, as the rank of B^χ). This is true for every binary number χ.

This incidentally shows that

$$MM'B^\chi \text{ is also} = E_c(\chi) B^\chi \tag{6.10}$$

where

$$E_c(\chi) = \sum_\delta h_c(\delta) \prod_{i=1}^{m} \alpha_i(\chi_i, \delta_i) . \tag{6.11}$$

This MM' has also the same eigenvectors, viz columns of B^χ. This, therefore, shows that for these designs, the results of this paper can be applied. Most of the two-way designs occurring in practice, do have properties (A) and (B) and as such satisfy the requirements of this paper.

Of course, columns of B^χ are not unit and are not mutually orthogonal but this can always be achieved by a process of orthogonalization.

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REFERENCES

[1] Chakrabarti, M. C. (1962). *Mathematics of Design and Analysis of Experiments*, Asia Publishing House, Bombay.

[2] Kshirsager, A. M. and Raugauathan, S. (1968). Analysis of a class of two-way designs with recovery of inter-row and inter-column information, *Cal. Stat. Ass. Bull.*, 17, 15-24.

[3] Martin, F. B. and Zyskind, G. (1966). On combinability of information from uncorrelated linear models by simple weighting, *Ann. Math. Statist.*, **37**, 1338-1347.

[4] Rao, C. Radhakrishna (1966). Generalized inverse for matrices and its applications in mathematical statistics, *Research Papers in Statistics*, Festschr; ft for J. Neyman.

[5] Roy, J. and Shah, K. R. (1962). Recovery of inter-block information, *Sankhya*, Series A, **24**, 269-280.

[6] Stein, C. (1950). Unbiased estimates with minimum variance, *Ann. Math. Statist.*, **21**, 406-415.

[7] Zelen, M. and Federer, W. (1964). Applications of the calculus for factorial arrangements II: Two-way elimination of heterogeneity, *Ann. Math. Statist.*, **35**, 658-672.