SOME ASYMPTOTIC PROPERTIES OF THE LINEARIZED MAXIMUM LIKELIHOOD ESTIMATE AND BEST LINEAR UNBIASED ESTIMATE*

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(Received Oct. 22, 1969)

1. Introduction

The author [4] studied a strictly unbiased linearized maximum likelihood estimate $\boldsymbol{\theta}^0=(\theta_1^0, \theta_2^0)$ (see also Plackett [9]) of the location and scale parameters θ_1 and θ_2 based on a Type II doubly censored sample from an absolutely continuous distribution satisfying certain general conditions. This estimate is asymptotically normal and efficient (in the sense that $n^{1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta})$, where $\boldsymbol{\theta}=(\theta_1, \theta_2)$, converges in distribution to the bivariate normal distribution $n(\mathbf{0}, \theta_2^2\boldsymbol{I}^{-1})$, where $\theta_2^{-2}\boldsymbol{I}=\theta_2^{-2}\,||\, I_{rs}(p, q)\,||$, $r, s=1, 2$ is the Fisher information matrix whose expression is given in condition C_3 of [4]). An asymptotically unbiased linear estimate $\boldsymbol{\theta}^\dagger=(\theta_1^\dagger, \theta_2^\dagger)$ related to $\boldsymbol{\theta}^0$ with simpler coefficients was also studied. $\boldsymbol{\theta}^\dagger$ was also obtained by Bennett [1], Weiss [11] and Chernoff, et al. [5].

In Section 2 we show that $n^{1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta}^\dagger)$ converges in probability to $\mathbf{0}$. By the Gauss-Markov theorem, one can obtain the best linear unbiased estimate $\boldsymbol{\theta}'=(\theta_1', \theta_2')$ of $\boldsymbol{\theta}$ (c.f. Lloyd [8]). The coefficients of such an estimate involve means and covariances of standardized parameter-free order statistics which in general are difficult to evaluate. We approximate the means and covariances by suitable functions of population quantiles and denote the resulting estimate by $\boldsymbol{\theta}^*=(\theta_1^*, \theta_2^*)$. This estimate will be called the approximately best linear estimate. In Section 3 we show that $\sqrt{n}(\boldsymbol{\theta}^\dagger-\boldsymbol{\theta}^*)$ and $\sqrt{n}(\boldsymbol{\theta}^*-\boldsymbol{\theta}^{*\prime})$ converges in probability to $\mathbf{0}$, where $\boldsymbol{\theta}^{*\prime}=(\theta_1^{*\prime}, \theta_2^{*\prime})$ is a strictly unbiased estimate proposed by Särndal ([10], § 1.4.2). An immediate consequence of this is that $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^{*\prime}$ are asymptotically normal and efficient. In Section 4 we show that the moments of $\boldsymbol{\theta}^0$, $\boldsymbol{\theta}^\dagger$, $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^{*\prime}$ tend to the corresponding moments of their limiting normal distribution. In Section 5 we show that the variances $V(n^{1/2}(\theta_1^0-\theta_1'))$ and $V(n^{1/2}(\theta_2^0-\theta_2'))$ tend to zero as the sample size n tend to ∞. Consequently, the best linear unbiased estimate $\boldsymbol{\theta}'$ is asymptotically normal and efficient.

* Supported by Grants-in-Aid, Department of University Affairs, Ontario, Canada.

The notations used in this paper are almost exactly the same as those in [4]. Let

$$(1)\qquad y_u < y_{u+1} < \cdots < y_v$$

($u=[np]$, the greatest integer $\leqq np$, $v=[nq]$, $0<p<q<1$) be a Type II doubly censored sample of ordered observations corresponding to an ordered complete sample of size n from an absolutely continuous distribution with distribution function $F((y-\theta_1)/\theta_2)$ and probability density function $(1/\theta_2)f((y-\theta_1)/\theta_2)$, where the location parameter θ_1 and scale parameter θ_2 are unknowns to be estimated, and let $x_i=(y_i-\theta_1)/\theta_2$, $i=u, u+1, \cdots, v$, be the standardized order statistics.

We assume that the distribution considered satisfies conditions C_1, C_2, C_3, C_4, C_5 and C_6 given in [4]. To avoid repetition, they are not listed here again. But whenever we say condition C_i is satisfied, it should be understood that we refer to condition C_i of [4].

LEMMA. *If*

$$(2)\qquad f(x)>0 \qquad \text{on } \{x \mid 0<F(x)<1\}$$

and C_5 is satisfied, for every fixed integer k, the left hand term in the following equation eventually exists as n becomes large and

$$n^{k/2}E\left[\prod_{j=1}^{k}(x_{i_j}-\xi_{i_j})\right]=\mu_{i_1,\cdots,i_k}+o(1) \qquad \text{as } n\to\infty$$

uniformly for $np\leqq i_j\leqq nq$, where $\xi_{i_j}=F^{-1}(i_j/(n+1))$ and $\mu_{i_1,\cdots,i_k}$ is the moment of order k of the k-variate normal distribution with mean $(0, 0, \cdots, 0)$ *and dispersion matrix*

$$\| K(i_j/n, i_{j'}/n)/[f(F^{-1}(i_j/n))f(F^{-1}(i_{j'}/n))] \| , \qquad j, j'=1, 2, \cdots, k ,$$

and

$$K(w, z)=K(z, w)=w(1-z) \qquad \text{if } 0\leqq w\leqq z\leqq 1 .$$

It should be noted that i_j can equal $i_{j'}$ for some $j\neq j'$. (2) is implied by C_1. Also by C_2 we see that $f(x)$ is continuous. So

$$(3)\qquad F^{-1}(i/n)=\xi_i+0(1/n) \qquad \text{uniformly on } np\leqq i\leqq nq .$$

Consequently, for many situations about Taylor expansions appearing later, one can replace $F^{-1}(i/n)$ by ξ_i or vice versa without changing the rates of convergence of remainders of the expansions considered.

2. The asymptotic equivalence of θ^0 and θ^*

Let t_i be the expected value Ex_i, L_n be the likelihood function of (1) and $(n^{1/2}\theta_2)^{-1}L_{rn}^0$, $r=1, 2$, be the terms linear in $x_u, \cdots, x_v$ in the

Taylor expansions of $n^{-1/2}\partial \log L_n/\partial\theta_r$, $r=1, 2$ about $x_i=t_i$, $i=u, u+1, \cdots, v$ (c.f. [4], (5), (6)) when these two partial derivatives are considered as functions the x_i's. Then after replacing x_i by $(y_i-\theta_1)/\theta_2$,

$$(4)\qquad \theta_2\left\| \begin{matrix} L_{1n}^0 \\ L_{2n}^0 \end{matrix} \right\| = \left\| \begin{matrix} \Sigma S_{1i}(t_i)y_i \\ \Sigma S_{2i}(t_i)y_i \end{matrix} \right\| - n\boldsymbol{J}^0\boldsymbol{\theta}^T$$

where

$$\Sigma=\sum_{i=u}^{v} \qquad \text{and} \qquad n\boldsymbol{J}^0=n\,\|\,J_{rs}^0(n)\,\|=\left\| \begin{matrix} \Sigma S_{1i}(t_i) & \Sigma t_i S_{1i}(t_i) \\ \Sigma S_{2i}(t_i) & \Sigma t_i S_{2i}(t_i) \end{matrix} \right\| ,$$

$$S_{1i}(t_i)=S_1(t_i)\,, \quad S_{2i}(t_i)=S_2(t_i) \qquad \text{for } i=u+1, u+2, \cdots, v-1\,,$$

$$S_{1u}(t_u)=-(u-1)[f_u'/F_u-f_u^2/F_u^2]+S_1(t_u)\,,$$

$$S_{1v}(t_v)=(n-v)[f_v'/(1-F_v)+f_v^2/(1-F_v)^2]+S_1(t_v)\,,$$

$$S_{2u}(t_u)=-(u-1)[f_u/F_u+t_uf_u'/F_u-t_uf_u^2/F_u^2]+S_2(t_u)\,,$$

$$S_{2v}(t_v)=(n-v)[f_v/(1-F_v)+t_vf_v'/(1-F_v)+t_vf_v^2/(1-F_v)^2]+S_2(t_v)$$

with $f_i=f(t_i)$, $F_i=F(t_i)$, $S_1(t)=-d^2\{\log f(t)\}/dt^2$, $S_2(t)=d\{-1-t(d\log f(t)/dt)\}/dt$. Let L_{rn}^*, $r=1, 2$, be obtained from L_{rn}^0 by replacing $\boldsymbol{J}^0$ by $\boldsymbol{I}$ and the t_i's in the coefficients of y_i by the ξ_i's.

The linearized maximum likelihood estimate $\boldsymbol{\theta}^0$ and the related estimate $\boldsymbol{\theta}^*$ are, respectively, obtained from solving the linearized equations and the related equations

$$L_{rn}^0=0\,, \quad r=1, 2 \qquad \text{and} \qquad L_{rn}^*=0\,, \quad r=1, 2\,.$$

It can be derived that

$$(5)\qquad \boldsymbol{\theta}^0=(n\boldsymbol{J}^0)^{-1}\left\| \begin{matrix} \Sigma S_{1i}(t_i)y_i \\ \Sigma S_{2i}(t_i)y_i \end{matrix} \right\| , \qquad \boldsymbol{\theta}^*=(n\boldsymbol{I})^{-1}\left\| \begin{matrix} \Sigma S_{1i}(\xi_i)y_i \\ \Sigma S_{2i}(\xi_i)y_i \end{matrix} \right\| .$$

In [4] we showed that, if C_1 to C_6 are satisfied, both $n^{1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta})$ and $n^{1/2}(\boldsymbol{\theta}^*-\boldsymbol{\theta})$ converge in distribution to the bivariate normal $N(\boldsymbol{0}, \theta_2^2\boldsymbol{I}^{-1})$ and hence are asymptotically normal and efficient. The following theorem gives the relation between $\boldsymbol{\theta}^0$ and $\boldsymbol{\theta}^*$.

THEOREM 1. *If C_1 to C_6 are satisfied, $n^{1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta}^*)$ converges in probability to* $\boldsymbol{0}$.

PROOF. From [4] (p. 1880, lines 13–18) we note that as $n\to\infty$.

$$(6)\qquad S_{ri}(t_i)=S_{ri}(\xi_i)+o(n^{-1/2})\,, \qquad r=1, 2$$

uniformly for $u\leqq i\leqq v$. Replace each t_i in $\boldsymbol{J}^0$ by ξ_i and denote it by $\boldsymbol{J}^*$. Then by (6)

(7) $$\lim_{n\to\infty} n^{1/2}(\boldsymbol{J}^0-\boldsymbol{J}^*)=\boldsymbol{0} ,$$

where $\boldsymbol{0}$ is a 2 by 2 zero matrix. But

(8) $$\lim_{n\to\infty} n^{1/2}(\boldsymbol{J}^*-\boldsymbol{I})=\boldsymbol{0}$$

(c.f. (12) of [4]). So

(9) $$\lim_{n\to\infty} n^{1/2}(\boldsymbol{J}^0-\boldsymbol{I})=\boldsymbol{0} .$$

Finally we note that for each y_i, $i=u+i,\cdots,v-1$, $y_u<y_i<y_v$, and y_u, y_v converge in probability to $\theta_2 F^{-1}(p)+\theta_1$, $\theta_2 F^{-1}(q)+\theta_1$, respectively. Then it follows from (5), (9) and (6) that for $r=1,2$

$$n^{1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta}^\sharp)=n^{-1/2}\{[\boldsymbol{I}+\boldsymbol{o}(n^{-1/2})]^{-1}[\|S_{ri}(\xi_i)\|+\boldsymbol{o}'(n^{-1/2})] \\ -[\boldsymbol{I}^{-1}\|S_{ri}(\xi_i)\|]\}\boldsymbol{y} \qquad i=u,\cdots,v ,$$

converge in probability to $\boldsymbol{0}$, where $\boldsymbol{o}(n^{-1/2})$ and $\boldsymbol{o}'(n^{-1/2})$ are, respectively, 2 by 2 and 2 by $v-u+1$ matrices whose elements are of order $o(n^{-1/2})$ *uniformly* as $n\to\infty$ and $\boldsymbol{y}^T=(y_u,\cdots,y_v)$. The theorem is thus proved.

3. Asymptotic equivalence of $\theta^\sharp$ and θ^*

The well known best linear unbiased estimate of $\boldsymbol{\theta}$ based on (1) is

(10) $$\boldsymbol{\theta}'=(\boldsymbol{A}^T\boldsymbol{V}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{V}^{-1}\boldsymbol{y}$$

where $\boldsymbol{A}=\|\boldsymbol{1}\boldsymbol{t}\|$ is a $(v-u+1)$ by 2 matrix with $\boldsymbol{1}^T=\|1,1,\cdots,1\|$, $\boldsymbol{t}^T=\|t_u,t_{u+1},\cdots,t_v\|$, and $\boldsymbol{V}=\|\mathrm{Cov}(x_i,x_j)\|$ is the covariance matrix of $x_u<x_{u+1}<\cdots<x_v$. Since $\partial[F(z_1)(1-F(z_2))/(f(z_1)f(z_2))]/\partial z_r$, $r=1,2$ are bounded on $[F^{-1}(p)-\varepsilon, F^{-1}(q)+\varepsilon]$ for some $\varepsilon>0$, by the Lemma we have

(11) $$\boldsymbol{A}=\boldsymbol{A}^*+\boldsymbol{o}^T(n^{-1/2}) \quad \text{and} \quad \boldsymbol{V}=\boldsymbol{V}^*+\boldsymbol{o}''(1/n)$$
uniformly for $np\leqq i, j\leqq nq$,

where

$$\boldsymbol{A}^*=\|\boldsymbol{1}\boldsymbol{\xi}\| , \qquad \boldsymbol{\xi}^T=\|\xi_u,\xi_{u+1},\cdots,\xi_v\| ,$$

$\boldsymbol{V}^*=(1/n)\|[K(i/(n+1), j/(n+1))]/[f(\xi_i)f(\xi_j)]\|$, and $\boldsymbol{o}(1/n)$ a $v-\mu+1$ by $v-\mu+1$ matrix whose elements are of order $o(1/n)$ uniformly as $n\to\infty$. Let

$$\boldsymbol{\theta}^*=(\boldsymbol{A}^{*T}\boldsymbol{V}^{*-1}\boldsymbol{A}^*)^{-1}\boldsymbol{A}^{*T}\boldsymbol{V}^{*-1}\boldsymbol{y}$$

and call it the approximately best linear estimate.

THEOREM 2. *If C_1 to C_6 are satisfied, $n^{1/2}(\boldsymbol{\theta}^\sharp-\boldsymbol{\theta}^*)$ converges in probability to* $\boldsymbol{0}$.

A consequence of Theorem 2 is that $\boldsymbol{\theta}^*$ is asymptotically normal and efficient.

PROOF OF THEOREM 2. Similar to the proof of Theorem 1, it is sufficient to show that

$$n(\boldsymbol{A}^{*T}\boldsymbol{V}^{*-1}\boldsymbol{A}^*)^{-1}-\boldsymbol{I}^{-1}=\boldsymbol{o}(n^{-1/2}) \,, \tag{12}$$

and

$$\boldsymbol{A}^{*T}\boldsymbol{V}^{*-1}-\left\|\begin{matrix} S_{1u}(\xi_u),\cdots, S_{1v}(\xi_v) \\ S_{2u}(\xi_u),\cdots, S_{2v}(\xi_v)\end{matrix}\right\|=\boldsymbol{o}'(n^{-1/2}) \tag{13}$$

uniformly for $np\leqq i\leqq nq$.

Let $\boldsymbol{\alpha}^*=(\alpha_u^*, \alpha_{u+1}^*,\cdots, \alpha_v^*)$ be the second row of $\boldsymbol{A}^{*T}\boldsymbol{V}^{*-1}$. Now we proceed to show that

$$\boldsymbol{\alpha}^*-\| S_{2u}(\xi_u), S_{2,u+1}(\xi_{u+1}),\cdots, S_{2v}(\xi_v)\|=\boldsymbol{o}'''(n^{-1/2}) \tag{14}$$

uniformly for $np\leqq i\leqq nq$, where $\boldsymbol{o}'''(n^{-1/2})$ is a 1 by $v-u+1$ matrix whose elements are of order $\boldsymbol{o}(n^{-1/2})$ as $n\to\infty$. It may be derived that (c.f. Hammersly and Morton [7])

$$(n\boldsymbol{V}^*)^{-1}=\left\|\begin{matrix} c_u & d_u & 0 & 0 & \cdots & 0 \\ d_u & c_{u+1} & d_{u+1} & 0 & \cdots & 0 \\ 0 & d_{u+1} & c_{u+2} & d_{u+2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & \cdots & c_{v-1} & d_{v-1} \\ 0 & 0 & & \cdots & d_{v-1} & c_v \end{matrix}\right\| ,$$

where $c_i=(a_{i+1}b_{i-1}-a_{i-1}b_{i+1})/[(a_ib_{i+1}-a_{i+1}b_i)(a_{i-1}b_i-a_ib_{i-1})]$, $d_i=1/(a_ib_{i+1}-a_{i+1}b_i)$, $a_i=(i/(n+1))/f(\xi_i)$, $b_i=(1-i/(n+1))/f(\xi_i)$, $i=u, u+1,\cdots, v$, $a_{u-1}=0$, $a_{v+1}=1$, $b_{u-1}=1$, $b_{v+1}=0$. It may be checked from C_4 and C_6 that

$$d^2(xf)/dF^2\,, \qquad d^3(xf)/dF^3\,, \qquad dS_2(t)/dt$$

are bounded on the interval $[F^{-1}(p)-\varepsilon, F^{-1}(q)+\varepsilon]$ for some $\varepsilon>0$. Since $\xi_u<\xi_i<\xi_v$, $i=u+1,\cdots, v-1$, $\lim_{n\to\infty}\xi_u=F^{-1}(p)$, and $\lim_{n\to\infty}\xi_v=F^{-1}(q)$, ξ_i belong to this interval for sufficiently large n. After simple manipulation we have *uniformly* for $u\leqq i\leqq v$

$$\alpha_i^*=S_{2i}(\xi_i)+0(1/n)\,.$$

The proofs for the first row of (13) and for (12) follow from similar arguments.

Our proof of Theorem 2 is hinted by Plackett's expository paper [9]. However, our proof is more rigorous, and in addition, in Theorem 3 we shall show that the moments of $\boldsymbol{\theta}^*$ converge to those of its limiting normal distribution. Approximation to the best linear unbiased estimate by an estimate somehow similar to $\boldsymbol{\theta}^*$ was also considered by Blom [2] under different conditions.

Särndal ([10], p. 18), proposed the linear estimate

$$\boldsymbol{\theta}^{*\prime}=(A^T V^{*^{-1}}A)^{-1}A^T V^{*^{-1}}\boldsymbol{y}$$

which is strictly unbiased. Since $t_i=\xi_i+o(n^{-1/2})$ uniformly for $u\leqq i\leqq v$, it is clear that $n^{1/2}(\boldsymbol{\theta}^*-\boldsymbol{\theta}^{*\prime})$ converges in probability to $\mathbf{0}$ and hence $\boldsymbol{\theta}^{*\prime}$ is asymptotically normal and efficient.

4. The convergence of moments of $\boldsymbol{\theta}^0$, $\boldsymbol{\theta}^\dagger$, $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^{*\prime}$

THEOREM 3. *Let $\hat{\boldsymbol{\theta}}$ be any one of $\boldsymbol{\theta}^0$, $\boldsymbol{\theta}^\dagger$, $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^{*\prime}$. If C_1 to C_6 are satisfied, then for any fixed non-negative integers k and k', the left hand term of the following equation eventually exists when n becomes large and we have*

$$E\{[n^{1/2}(\hat{\theta}_1-\theta_1)]^k[n^{1/2}(\hat{\theta}_2-\theta_2)]^{k'}\}=\mu(k,k')+o(1) \qquad as\ n\to\infty ,$$

where $\mu(k,k')$ is the (k,k')th moment of the bivariate normal distribution $N(\mathbf{0},\theta_2^2\boldsymbol{I}^{-1})$.

PROOF. Let us demonstrate that

$$(15)\qquad E[n^{1/2}(\theta_1^0-\theta_1)]^2=\theta_2^2I_{22}(p,q)/[I_{11}(p,q)I_{22}(p,q)-I_{12}^2(p,q)]+o(1) \qquad \text{as } n\to\infty .$$

We know from [4] (p. 1881) that

$$n^{1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta})=\theta_2\boldsymbol{J}^{0^{-1}}\left\|\begin{matrix} n^{-1/2}L_{1n}^0 \\ n^{-1/2}L_{2n}^0\end{matrix}\right\|$$

converges in distribution to $N(\mathbf{0},\theta_2^2\boldsymbol{I}^{-1})$. Then by the Lemma

$$(16)\qquad E[n^{1/2}(\theta_1^0-\theta_1)]^2$$
$$=\theta_2^2[J_{11}^0J_{22}^0-J_{12}^0]^{-2}\Big\{n^{-2}\sum_{i,j=u}^{v}[K(i/(n+1),\ j/(n+1))/(f(\xi_i)f(\xi_j))]$$
$$\cdot\ [J_{22}^*S_{1i}(\xi_i)-J_{12}^*S_{2i}(\xi_i)]\,[J_{22}^*S_{1i}(\xi_j)-J_{12}^*S_{2i}(\xi_j)]\Big\}+o(1) \qquad \text{as } n\to\infty ,$$

where $J_{rs}^0=J_{rs}^0(n)$ and $J_{rs}^*=J_{rs}^*(n)$, $r,s=1,2$. Then following the manipulations similar to those given in p. 65 of [5], it can be shown that (15) holds. The remaining part of the theorem may be proved by analogous arguments. It should be noted that for values of (k,k') other than $(1,0)$, $(0,1)$, $(2,0)$, $(0,2)$, $(1,1)$, complicated manipulations may be needed in the proof of the theorem.

General convergence of moments of linear combinations of order statistics were also considered by Bickel [1] (Theorem 4.2).

5. The asymptotic properties of the best linear unbiased estimate θ'

THEOREM 4. *If C_1 to C_6 are satisfied, then*

$$\lim_{n\to\infty} \| \mathrm{Cov}\,(n^{1/2}(\theta_r^0-\theta_r'), n^{1/2}(\theta_s^0-\theta_s')) \| = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \qquad r, s=1, 2 .$$

So $n^{1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta}')$ converges in probability to $\mathbf{0}$ and hence $\boldsymbol{\theta}'$ is asymptotically normal and efficient. Although many asymptotically normal and efficient linear estimates have been proposed, that the best linear unbiased estimate, which has the smallest variance in the class of all linear unbiased estimates for any sample size, has such optimal asymptotic properties seems has not been proved before.

PROOF OF THEOREM 4. Let us demonstrate that

$$\lim V(n^{1/2}(\theta_1^0-\theta_1'))=0 , \tag{17}$$

where $\lim = \lim_{n\to\infty}$. Let $I^{-1}=|I^{rs}(p, q)|$, $r, s=1, 2$. Since $I^{11}\theta_2^2/n$ is the Cramér-Rao lower bound and $V(\theta_1')\leqq V(\theta_1^0)$,

$$I^{11}\theta_2^2=\lim V(n^{1/2}\theta_1')=\lim V(n^{1/2}\theta_1^0) .$$

So it is sufficient to show that

$$\lim 2\,\mathrm{Cov}\,(n^{1/2}\theta_1^0, n^{1/2}\theta_1')=2I^{11}\theta_2^2 . \tag{18}$$

Let $g_n=n^{-1}\theta_2^2(I^{11}\partial\log L_n/\partial\theta_1+I^{12}\partial\log L_n/\partial\theta_2)$. By the regularity conditions C_1 to C_4 we have for $r, s=1, 2$

$$E\partial\log L_n/\partial\theta_r=0 , \tag{19}$$

$$\lim\ n^{-1}E((\partial\log L_n/\partial\theta_r)(\partial\log L_n/\partial\theta_s))=I_{rs}\theta_2^{-2} \tag{20}$$

and since $E\boldsymbol{\theta}^0=E\boldsymbol{\theta}'=\boldsymbol{\theta}$

$$\partial E\theta_r^0/\partial\theta_s=\partial E\theta_r'/\partial\theta_s=E(\theta_r'(\partial\log L_n/\partial\theta_s))=1 \text{ or } 0 \quad \text{according to } r=s \text{ or } r\neq s . \tag{21}$$

(19) (as well (21)) and (20), are, respectively, extensions to two-parameter case of (3.3.1) and (3.3) of [6]. It follows from (19) and (21) that

$$\begin{aligned} &2\,\mathrm{Cov}\,(n^{1/2}\theta_1^0, n^{1/2}\theta_1')-2I^{11}\theta_2^2 \\ &=nE((\theta_1^0-\theta_1)(\theta'-\theta_1-g_n))+nE((\theta_1'-\theta_1)(\theta_1^0-\theta_1-g_n)) . \end{aligned} \tag{22}$$

Using Schwartz inequality and (19) to (21) we have

$$\lim |nE((\theta_1^0-\theta)(\theta'-\theta_1-g_n)|^2$$
$$\leq \lim [(V(n^{1/2}\theta_1^0))(V(n^{1/2}\theta')+V(n^{1/2}g_n)-2nE(\theta_1' g_n))]=0 .$$

Similarly it can be shown that the second right-hand term tends to 0 and hence (18) holds. Following similar arguments we have

$$\lim V(n^{1/2}(\theta_2^0-\theta_2'))=0$$

and then by the Schwartz inequality

$$\lim \mathrm{Cov}\,(n^{1/2}(\theta_1^0-\theta_1'), n^{1/2}(\theta_2^0-\theta_2'))=0 .$$

6. Remarks

Theorems 1, 2, 3 and 4 may be extended to Type II multiply censored samples in which only the observations with ranks lying between and including $[np_i]$ and $[nq_i]$ are available, where $0<p_1<q_1<p_2<q_2<\cdots<p_k<q_k<1$, k a fixed integer.

Acknowledgement

The author wishes to thank Mr. G. Jarvis for helpful suggestions.

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