# ON THE DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERION FOR TESTING LINEAR HYPOTHESES ON REGRESSION COEFFICIENTS

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## 1. Summary

This article gives the exact distribution of Wilks' likelihood ratio criterion for testing hypotheses about regression coefficients in the multivariate normal case. The exact density and the distribution function, in the most general case, are expressed in simple algebraic functions. The exact distribution is obtained with the help of inverse Mellin transform, Calculus of residues and the properties of Psi and Zeta functions. The exact expressions of the density and the distribution function and a detailed calculation of the various terms are given in Mathai and Rathie [8]. They obtained the results through the technique of partial fractions. This article gives a simpler alternate method and further, all the different cases are combined with the help of a unified notation. Tables of percentage points are also given at the end of this article.

## 2. Introduction

Let $x_1, x_2, \cdots, x_N$ be a set of $N$ observations, $x_\alpha$ being drawn from a multivariate normal population $N(\boldsymbol{\beta} Z_\alpha, \Sigma)$. The vectors $Z_\alpha$, with $t$ components, are known and the $p\times p$ matrix $\Sigma$ and the $p\times t$ matrix $\boldsymbol{\beta}$ are unknown. Let $N\geqq p+t$ and the rank of $\boldsymbol{Z}=(Z_1, \cdots, Z_N)$ be $t$. Let,

$$\boldsymbol{\beta}=(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \tag{2.1}$$

where $\boldsymbol{\beta}_1$ has $t_1$ columns and $\boldsymbol{\beta}_2$ has $t_2$ columns. Consider the hypothesis,

$$H: \quad \boldsymbol{\beta}_1=\boldsymbol{\beta}_1^* \tag{2.2}$$

where $\boldsymbol{\beta}_1^*$ is a given matrix. Let $U=\lambda^{2/N}$ where $\lambda$ is the likelihood ratio criterion for testing $H$. Under the hypothesis $H$, the moments of $U$ are available in Anderson ([1], p. 192–194). That is

$$(2.3) \qquad E(U^h)=\prod_{j=1}^{p}\{\Gamma[(n+1-j)/2+h]\Gamma[(n+t_1+1-j)/2]/ (\Gamma[(n+1-j)/2]\Gamma[(n+t_1+1-j)/2+h])\}$$

where $n=N-t$ and $E$ denotes 'mathematical expectation'. If $U$ is denoted as $U_{p,t_1,n}$ then the distribution of $U_{p,t_1,N-t}$ is the same as that of $U_{t_1,p,N-p-t_2}$, when the hypothesis is true, (Anderson [1], p. 193). Hence without loss of generality we need consider only the cases where $q\geqq p$ while considering the distribution of $U_{p,q,n}$.

When $p=1$, $n(1-U_{1,q,n})/\{qU_{1,q,n}\}$ has an $F$-distribution with $q$ and $n$ degrees of freedom. $(n+1-p)(1-U_{p,1,n})/(pU_{p,1,n})$ has an $F$-distribution with $p$ and $n+1-p$ degrees of freedom. Wilks [16] obtained the distributions for the cases $p=1, 2, 3$, $q=3$; $p=4$, $q=4$. Consul [5] gave the distributions for the cases $p=1, 2, 3, 4$ and $q=3, 4, 5, 6, 7, 8$ in infinite series and in algebraic expressions. Pillai and Gupta [11] gave the exact distributions in multiple sums for $p=3, 4, 5, 6$. Schatzoff [15] suggested a method of obtaining the exact distribution in the most general case. Mathai and Rathie [8] have given the exact distribution, in simple algebraic expressions for all the cases. Pillai and Gupta [11] also computed the percentage points for some particular cases which extended some results of Schatzoff [15] and supplemented some tables by Pillai [10]. Approximations are considered by Bartlett [2], Rao [12] and Box [3].

Wilks [16] used direct integration to obtain the particular distributions. Consul [5] used the technique of inverse Mellin transform and the properties of hypergeometric functions to obtain the particular distributions. Schatzoff [14], [15] used the representation of $(-\log U)$ as a sum of independently distributed random variables and then took successive convolutions. He has suggested a recursive technique to compute the percentage points. But practical difficulties of evaluating successive convolutions limit the uses of his method. Pillai and Gupta [11] considered Schatzoff's representation and obtained the distribution in multiple sums for some particular cases. The difficulty of computing multiple sums limit the uses of their method. Mathai and Rathie [8] used the technique of partial fractions and the inverse Mellin transform and obtained the distributions in the most general cases, expressed them in terms of simple algebraic functions and evaluated all the terms explicitly. In this article a simpler alternate method is suggested. With the help of the Calculus of residues, the exact distributions are obtained for all the different cases and are written in a very simple form with the help of some unified notations. The results agree with the results of Mathai and Rathie [8] and some particular cases are verified with the results of Consul [5]. The percentage points are also computed with the help of these representations.

## 3. The exact distribution

From (2.3) it follows that,

$$(3.1)\quad E(U^{s-1})=C\prod_{j=1}^{p}\{\Gamma[(n+1-j)/2+s-1]/\Gamma[(n+1+q-j)/2+s-1]\}$$

where

$$(3.2)\qquad C=\prod_{j=1}^{p}\{\Gamma[(n+1+q-j)/2]/\Gamma[(n+1-j)/2]\}\ .$$

Since $0<u<1$, the moment sequence in (3.1) uniquely determines the distribution, according to Rao ([13], p. 86 (a)). Hence the density of $U_{p,q,n}$, denoted by $f(u)$, is available as the inverse Mellin transform of (3.1). That is,

$$(3.3)\quad f(u)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\{E(U^{s-1})\}u^{-s}ds\ ,\qquad 0<u<1,\ i=(-1)^{1/2}\ .$$

But it is easy to see that $f(u)$ is a Meijer's $G$-function. For a definition of Meijer's $G$-function, see Erdélyi ([6], p. 207) and Braaksma [4]. According to Braaksma ([4], p. 278, (6.1)), $f(u)$ is available as the sum of the residues at the poles of the integrand in (3.3) and it does not depend upon the choice of the contour. Hence $f(u)$ will be evaluated with the help of the residue theorem.

The poles of (3.1) can be determined after cancelling all the common factors in (3.1) and then rearranging the factors. For convenience, the four different cases, namely, Case I: $p$-even, $q$-even; Case II: $p$-odd, $q$-even; Case III: $p$-even, $q$-odd; Case IV: $p$-odd, $q$-odd, will be considered separately, for $q\geqq p$. The simplification is achieved by the following procedure. For example, when $q$ is even,

$$(3.4)\qquad \begin{aligned}&\Gamma(s+n/2-1)/\Gamma(s+n/2-1+q/2)\\ &\quad=1/\{(s+n/2-1+q/2-1)(s+n/2-1+q/2-2)\\ &\qquad\cdots(s+n/2-1)\}\ .\end{aligned}$$

Hence, with the help of a unified notation, $E(U^{s-1})$ can be written in the following form, after cancelling all the common factors.

$$(3.5)\qquad E(U^{s-1})=C\left\{\prod_{j\in a}(\alpha-j)^{-a_j}\prod_{j\in b}(\alpha-1/2-j)^{-b_j}\right\}$$

for Cases I, II and III and for Case IV,

$$(3.6)\quad E(U^{s-1})=C\left\{[\Gamma(\alpha-1/2)/\Gamma(\alpha)]\prod_{j\in a}(\alpha-j)^{-a_j}\prod_{j\in b}(\alpha-1/2-j)^{-b_j}\right\}$$

where

$$a = s + n/2 + q/2 - 1 \tag{3.7}$$

and $a$, $b$, $a_j$, $b_j$ are all different for the different cases. These are given below. In Case IV, the Gammas are cancelled after leaving aside the first Gamma in the numerator and the last Gamma in the denominator of (3.1) and then these Gammas are simplified to obtain (3.6).

*Case* I: $p$-even, $q$-even, $(q \geqq p)$.

$$a_j = b_j = \begin{cases} j\,, & j = 1, 2, \cdots, p/2 - 1\,, \\ p/2\,, & j = p/2,\ p/2+1, \cdots, q/2\,, \\ p/2 - i\,, & j = q/2 + i, \quad i = 1, 2, \cdots, p/2 - 1\,. \end{cases} \tag{3.8}$$

$$a = \{1, 2, \cdots, (p+q)/2 - 1\} = b\,. \tag{3.9}$$

*Case* II: $p$-odd, $q$-even, $(q \geqq p)$.

$$\begin{aligned} a_j &= \begin{cases} j\,, & j = 1, 2, \cdots, (p-1)/2\,, \\ (p+1)/2\,, & j = (p+1)/2,\ (p+1)/2+1, \cdots, q/2\,, \\ (p+1)/2 - i\,, & j = q/2 + i, \quad i = 1, 2, \cdots, (p-1)/2\,, \end{cases} \\ b_j &= \begin{cases} j\,, & j = 1, 2, \cdots, (p-3)/2\,, \\ (p-1)/2\,, & j = (p-1)/2,\ (p-1)/2+1, \cdots, q/2\,, \\ (p-1)/2 - i\,, & j = q/2 + i, \quad i = 1, 2, \cdots, (p-3)/2\,. \end{cases} \end{aligned} \tag{3.10}$$

$$a = \{1, 2, \cdots, (p+q-1)/2\}\,, \qquad b = \{1, 2, \cdots, (p+q-3)/2\}\,. \tag{3.11}$$

*Case* III: $p$-even, $q$-odd, $(q \geqq p)$.

$$\begin{aligned} a_j &= \begin{cases} j\,, & j = 1, 2, \cdots, p/2 - 1\,, \\ p/2\,, & j = p/2,\ p/2+1, \cdots, (q+1)/2\,, \\ p/2 - i\,, & j = (q+1)/2 + i, \quad i = 1, 2, \cdots, p/2 - 1\,, \end{cases} \\ b_j &= \begin{cases} j\,, & j = 1, 2, \cdots, p/2 - 1\,, \\ p/2\,, & j = p/2,\ p/2+1, \cdots, (q-1)/2\,, \\ p/2 - i\,, & j = (q-1)/2 + i, \quad i = 1, 2, \cdots, p/2 - 1\,. \end{cases} \end{aligned} \tag{3.12}$$

$$a = \{1, 2, \cdots, (p+q-1)/2\}\,, \qquad b = \{1, 2, \cdots, (p+q-3)/2\}\,. \tag{3.13}$$

*Case* IV: $p$-odd, $q$-odd, $(q \geqq p)$.

$$(3.14)\quad a_j=\begin{cases} j-1\,, & j=2,3,\cdots,(p-1)/2\,,\\ (p-1)/2\,, & j=(p-1)/2+1,\cdots,(q+1)/2\,,\\ (p-1)/2-i\,, & j=(q+1)/2+i,\quad i=1,2,\cdots,(p-3)/2\,,\end{cases}$$

$$b_j=\begin{cases} j+1\,, & j=1,2,\cdots,(p-1)/2\,,\\ (p-1)/2+1\,, & j=(p-1)/2+1,\cdots,(q-1)/2\,,\\ (p-1)/2\,, & j=(q+1)/2\,,\\ (p-1)/2-i\,, & j=(q+1)/2+i,\quad i=1,2,\cdots,(p-3)/2\,.\end{cases}$$

$$(3.15)\quad a=\{2,3,\cdots,(p+q-2)/2\}\,,\qquad b=\{1,2,\cdots,(p+q-2)/2\}\,.$$

For the Cases I, II and III the poles are available by equating to zero the various factors of

$$(3.16)\quad \prod_{j\in a}(\alpha-j)^{a_j}\prod_{j\in b}(\alpha-1/2-j)^{b_j}$$

where the exponents denote the orders of the poles and for Case IV the poles are available from the factors of

$$(3.17)\quad \prod_{v=0}^{\infty}(\alpha-1/2+v)\prod_{j\in a}(\alpha-j)^{a_j}\prod_{j\in b}(\alpha-1/2-j)^{b_j}\,,$$

where the sets $a$, $b$ and the exponents $a_j$ and $b_j$ are available from (3.8) to (3.15).

Now the density $f(u)$ will be evaluated with the help of the following lemmas which are easy to prove.

LEMMA 3.1. *If $\phi(s)$ is a Gamma product with a pole of order $k$ at the point $s=d$ then the residue $R$ of $\phi(s)x^{-s}$ at $s=d$, is given by*

$$(3.18)\quad R=\frac{x^{-d}}{(k-1)!}\sum_{v=0}^{k-1}\binom{k-1}{v}(-\log x)^{k-1-v} \cdot\left\{\sum_{v_1=0}^{v-1}\binom{v-1}{v_1}G_0^{(v-1-v_1)}\sum_{v_2=0}^{v_1-1}\binom{v_1-1}{v_2}G_0^{(v_1-1-v_2)}\cdots\right\}H_0\,,$$

*where*

$$(3.19)\quad H_0=(s-d)^k\phi(s)\qquad at\ s=d\,,$$

*and*

$$(3.20)\quad G_0^{(r)}=\frac{\partial^{r+1}}{\partial s^{r+1}}\log\,[(s-d)^k\phi(s)]\,,\qquad at\ s=d,\ for\ r\geqq 0,\ G_0^{(0)}=G_0\,.$$

This is a known result and it can be easily seen from the following observations.

$$(3.21)\quad R=\lim_{s\to d}\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial s^{k-1}}[(s-d)^k\phi(s)x^{-s}]\,,\qquad \text{at } s=d\,,$$

$$(3.22)\quad =\frac{x^{-s}}{(k-1)!}\left\{\frac{\partial}{\partial s}+(-\log x)\right\}^{k-1}[(s-d)^k\phi(s)]\,,\qquad \text{at } s=d\,,$$

$$(3.23)\quad =\frac{x^{-s}}{(k-1)!}\sum_{v=0}^{k-1}\binom{k-1}{v}(-\log x)^{k-1-v}\frac{\partial^v}{\partial s^v}[(s-d)^k\phi(s)]\,,\qquad \text{at } s=d\,.$$

But,

$$(3.24)\quad \frac{\partial^v}{\partial s^v}[(s-d)^k\phi(s)]=\frac{\partial^{v-1}}{\partial s^{v-1}}\left[\{(s-d)^k\phi(s)\}\left\{\frac{\partial}{\partial s}\log(s-d)^k\phi(s)\right\}\right]\,.$$

LEMMA 3.2.

$$(3.25)\quad \frac{\partial^{r+1}}{\partial s^{r+1}}\left\{\log\prod_{j=1}^{k}\Gamma(h_j+s)\right\}=\sum_{j=1}^{k}\psi(h_j+s)\,,\qquad \textit{for } r=0\,,$$

$$(3.26)\quad =(-1)^{r+1}r!\left\{\sum_{j=1}^{k}\zeta(r+1,h_j+s)\right\}\,,\qquad \textit{for } r\geq 1\,,$$

*where $\psi(\cdot)$ and $\zeta(\cdot,\cdot)$ are the well known Psi and Riemann Zeta functions respectively. The definitions are given below for convenience.*

$$(3.27)\quad \psi(z)=\frac{d}{dz}\log\Gamma(z)=-\gamma+(z-1)\sum_{n=0}^{\infty}[(n+1)(z+n)]^{-1}\,,$$

*where $\gamma$ is the Euler's constant; $\gamma=0.577\cdots$*

$$(3.28)\quad \zeta(s,v)=\sum_{n=0}^{\infty}(n+v)^{-s}\,,\qquad v\neq 0,-1,-2,\cdots,R(s)>1\,,$$

*where $R(\cdot)$ denotes the real part of $(\cdot)$. Result* (3.26) *follows from the definition in* (3.27) *and* (3.28). *Also these results are mentioned in Mathai* [7].

By using Lemma 3.1, the density for the Cases I, II and III can be written as follows.

$$(3.29)\quad f(u)=C\left\{\sum_{j\in a}\frac{u^{n/2+q/2-1-j}}{(a_j-1)!}\sum_{v=0}^{a_j-1}\binom{a_j-1}{v}(-\log u)^{a_j-1-v}A_vY\right.$$
$$\left.+\sum_{j\in b}\frac{u^{n/2+q/2-3/2-j}}{(b_j-1)!}\sum_{v=0}^{b_j-1}\binom{b_j-1}{v}(-\log u)^{b_j-1-v}B_vW\right\}\,,$$

$0<u<1$, where

$$Y=\prod_{\substack{t\in a\\ t\neq j}}(j-t)^{-a_t}\prod_{t\in b}(j-1/2-t)^{-b_t}\,, \tag{3.30}$$

$$W=\prod_{t\in a}(j+1/2-t)^{-a_t}\prod_{\substack{t\in b\\ t\neq j}}(j-t)^{-b_t}\,, \tag{3.31}$$

$$A_v=\sum_{v_1=0}^{v-1}\binom{v-1}{v_1}A_0^{(v-1-v_1)}\sum_{v_2=0}^{v_1-1}\binom{v_1-1}{v_2}A_0^{(v_1-1-v_2)}\cdots\,, \tag{3.32}$$

$$B_v=\sum_{v_1=0}^{v-1}\binom{v-1}{v_1}B_0^{(v-1-v_1)}\sum_{v_2=0}^{v_1-1}\binom{v_1-1}{v_2}B_0^{(v_1-1-v_2)}\cdots\,, \tag{3.33}$$

$$A_0^{(r)}=(-1)^{r+1}r!\left\{\sum_{\substack{t\in a\\ t\neq j}}[a_t/(j-t)^{r+1}]+\sum_{t\in b}[b_t/(j-1/2-t)^{r+1}]\right\} \tag{3.34}$$

for $r\geqq 0$ and

$$B_0^{(r)}=(-1)^{r+1}r!\left\{\sum_{t\in a}[a_t/(1/2+j-t)^{r+1}]+\sum_{\substack{t\in b\\ t\neq j}}[b_t/(j-t)^{r+1}]\right\}\,,\qquad r\geqq 0\,. \tag{3.35}$$

$C$ is given in (3.2) and the quantities $a$, $b$, $a_j$, $b_j$, for the different cases, are available from (3.8) to (3.15). The density for Case IV is available from Lemmas 3.1 and 3.2, as

$$\begin{aligned} f(u)=C\Bigg\{&\sum_{v=0}^{\infty}\left[\frac{(-1)^v}{v!}\frac{u^{n/2+q/2-3/2+v}}{\Gamma(1/2-v)}\prod_{t\in a}(1/2-v-t)^{-a_t}\right.\\ &\cdot\left.\prod_{t\in b}(-v-t)^{-b_t}\right]+\sum_{j\in a}\frac{u^{n/2+q/2-1-j}}{(a_j-1)!}\sum_{v=0}^{a_j-1}\binom{a_j-1}{v}\\ &\cdot(-\log u)^{a_j-1-v}A_v'Y'+\sum_{j\in b}\frac{u^{n/2+q/2-3/2-j}}{(b_j-1)!}\\ &\cdot\sum_{v=0}^{b_j-1}\binom{b_j-1}{v}(-\log u)^{b_j-1-v}B_v'W'\Bigg\}\,,\qquad 0<u<1\,, \end{aligned} \tag{3.36}$$

where $A_v'$ and $B_v'$ have the same expressions in (3.32) and (3.33) respectively with $A_0$ and $B_0$ replaced by $A_0'$ and $B_0'$, and

$$Y'=\{\Gamma(j-1/2)/\Gamma(j)\}\,Y\,,\qquad W'=\{\Gamma(j)/\Gamma(1/2+j)\}\,W\,, \tag{3.37}$$

$$A_0'=\psi(j-1/2)-\psi(j)+A_0\,, \tag{3.38}$$

$$B_0'=\psi(j)-\psi(1/2+j)+B_0\,, \tag{3.39}$$

$$A_0'^{(r)}=(-1)^{r+1}r!\,\{\zeta(r+1,\,j-1/2)-\zeta(r+1,\,j)\}+A_0^{(r)}\,,\qquad r\geqq 1\,, \tag{3.40}$$

$$B_0'^{(r)}=(-1)^{r+1}r!\,\{\zeta(r+1,\,j)-\zeta(r+1,\,1/2+j)\}+B_0^{(r)}\,,\qquad r\geqq 1\,. \tag{3.41}$$

The expressions in (3.29) and (3.36) follow directly from Lemmas 3.1 and 3.2. The crucial steps in the derivations of (3.29) and (3.36) are the simplifications of $E(U^{s-1})$ into (3.5) and (3.6) with the various quantities in (3.8) to (3.15).

## 4. Verification

In order to illustrate the simplicity of the method, a few verifications will be given here. Since the Cases I, II and III are derived in a similar fashion, we will verify a particular case from one of the cases in I, II or III and a particular case from IV. The Particular cases $p=4$, $q=4$ and $p=3$, $q=3$ will be verified here.

*Particular case* $p=4$, $q=4$:

In this case the sets $a$, $b$ and the quantities $a_j$, $b_j$ are as follows.

$$a=\{1, 2, 3\}=b \tag{4.1}$$

$$a_j=b_j=\begin{cases} 1\,, & j=1 \\ 2\,, & j=2 \\ 1\,, & j=3\,. \end{cases} \tag{4.2}$$

Now (3.29) reduces to the form,

$$\begin{aligned} f(u)=C\{&u^{n/2}(-1)2^3/[(3^2)(5)]+u^{n/2-1}[-\log u-(1/1-1/1) \\ &-(1/(1/2)-2/(1/2)-1/(3/2))]2^4/3+u^{n/2-2}(-1)2^3/3 \\ &+u^{n/2-1/2}2^3/3+u^{n/2-3/2}[-\log u-(1/(3/2)+2/(1/2) \\ &-1/(1/2))-(1/1-1/1)]2^4/3+u^{n/2-5/2}2^3/[(3^2)(5)]\}\,, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} C=&\Gamma[(n+4)/2]\Gamma[(n+3)/2]\Gamma[(n+2)/2]\Gamma[(n+1)/2]/ \\ &\{\Gamma(n/2)\Gamma[(n-1)/2]\Gamma[(n-2)/2]\Gamma[(n-3)/2]\}\,. \end{aligned} \tag{4.4}$$

Now simplifying the Gammas in $C$ by using the duplication formula for Gamma functions, namely,

$$\Gamma(2z)=\pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z+1/2)\,, \tag{4.5}$$

$f(u)$ reduces to the form,

$$\begin{aligned} f(u)=&\{\Gamma(n+1)\Gamma(n+3)/[\Gamma(n-3)\Gamma(n-1)6!2]\} \\ &\cdot\{u^{(n-5)/2}[-u^{5/2}+30u^{3/2}(-\log u+8/3)-15u^{1/2}+15u^2 \\ &+30u(-\log u-8/3)+1]\}\,, \end{aligned} \tag{4.6}$$

which agrees with the result given by Consul [5].

*Particular case* $p=3$, $q=3$:

In this case, (3.36) reduces to the form,

$$(4.7)\quad \begin{aligned} f(u)=C\Big\{ &\sum_{v=0}^{\infty}(-1)^v u^{n/2+v}/[v!\Gamma(1/2-v)(1/2-v-2)(-v-1)^2 \\ &\cdot(-v-2)]+\Gamma(3/2)u^{n/2-3/2}/[\Gamma(2)(3/2-1)^2(3/2-2)] \\ &+\Gamma(1)u^{n/2-1}[-\log u+\psi(1)-\psi(3/2)-(1/(3/2-2) \\ &\quad +1/(1-2))]/[\Gamma(3/2)(3/2-2)(1-2)] \\ &+\Gamma(2)u^{n/2-2}/[\Gamma(5/2)(5/2-2)(2-1)^2]\Big\} . \end{aligned}$$

Now by using the results,

$$(4.8)\quad \Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z) ,$$

$$(4.9)\quad \psi(z+n)=1/z+1/(z+1)+\cdots+1/(z+n-1)+\psi(z) ,$$

$$(4.10)\quad \psi(1/2)=\psi(1)-2\log 2 ,$$

$$(4.11)\quad \Gamma(1/2)=\pi^{1/2} ,$$

(4.7) reduces to the form,

$$(4.12)\quad \begin{aligned} f(u)=C\Big\{ &\sum_{v=0}^{\infty}(\Gamma(1/2+v)u^{n/2+v}/[v!\pi(v+3/2)(v+1)^2(v+2)]) \\ &-\pi^{1/2}2^2u^{n/2-3/2}+(2^2/\pi^{1/2})u^{n/2-1} \\ &\cdot[-\log u+1+2\log 2]+2^3u^{n/2-2}/(3\pi^{1/2})\Big\} . \end{aligned}$$

It is easy to see that the coefficients of $u^{n/2-2}$, $u^{n/2-3/2}$, $u^{n/2-1}\log u$, agree with the corresponding results in Consul [5]. Due to the difference in the representations it is not easy to verify every term with out devoting too much space for the simplification. It may be noticed that Consul's representation is too complicated for the evaluation of the distribution function.

## 5. The distribution function

The cumulative distribution $F(x)$ is given by

$$(5.1)\quad F(x)=\int_0^x f(u)du , \qquad 0<x<1 .$$

Here (5.1) can be easily evaluated from (3.29) and (3.36) by using the following lemma. Since the result is obvious the expression for $F(x)$ is not explicitly given here. Lemma 5.1 is proved by successive integration by parts.

LEMMA 5.1. *For* $\beta>0,\ k=0,1,\cdots,0<x<1,$

$$(5.2)\quad \int_0^x u^{\beta}(-\log u)^k du=\frac{x^{\beta+1}}{k+1}\sum_{r=0}^{k}\left\{(k+1)k\cdots(k-r+1)\frac{(-\log x)^{k-r}}{(\beta+1)^{r+1}}\right\}.$$

It is easy to see that the method given in this article is simpler compared to any other method available so far, for solving this problem. Further, the representations in (3.29) and (3.36) are in better simplified forms compared to other representations. Also, it is interesting to notice that the representations in (3.29) and (3.36) are in convenient forms for programming. In Section 6 the percentage points are computed by using the cumulative distribution function corresponding to (3.29) and (3.36).

## 6. Tables of percentage points

In this section a short table of the percentage points is given. The entries are the values of $u$ corresponding to $F(u)=0.95$ and $F(u)=0.99$ and for the various values of $p$, $q$ and $n$. A detailed table of $F(u)$ is available from the author on request. These are computed by using the exact distributions given in this article and with the help of an IBM 360-75, O.S. System, digital computer.

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| $n$ | $p=2,\ q=2$ | | $p=2,\ q=3$ | | $p=2,\ q=4$ | |
|---|---|---|---|---|---|---|
| | $F(x)=0.95$ | $F(x)=0.99$ | $F(x)=0.95$ | $F(x)=0.99$ | $F(x)=0.95$ | $F(x)=0.99$ |
| 2 | 0.60280 | 0.81000 | 0.39891 | 0.61551 | 0.27787 | 0.46747 |
| 3 | 0.74761 | 0.88566 | 0.56459 | 0.73811 | 0.43218 | 0.60517 |
| 4 | 0.81429 | 0.91776 | 0.65731 | 0.79988 | 0.53094 | 0.68380 |
| 5 | 0.85296 | 0.93570 | 0.71714 | 0.83769 | 0.60013 | 0.73570 |
| 6 | 0.87825 | 0.94719 | 0.75906 | 0.86340 | 0.65141 | 0.77270 |
| 7 | 0.89610 | 0.95518 | 0.79012 | 0.88202 | 0.69097 | 0.80054 |
| 8 | 0.90937 | 0.96107 | 0.81406 | 0.89615 | 0.72245 | 0.82226 |
| 9 | 0.91963 | 0.96559 | 0.83309 | 0.90721 | 0.74809 | 0.83968 |
| 10 | 0.92781 | 0.96916 | 0.84857 | 0.91617 | 0.76939 | 0.85397 |
| 11 | 0.93447 | 0.97206 | 0.86142 | 0.92355 | 0.78736 | 0.86591 |
| 12 | 0.94001 | 0.97447 | 0.87226 | 0.92973 | 0.80273 | 0.87602 |
| 13 | 0.94468 | 0.97649 | 0.88152 | 0.93496 | 0.81603 | 0.88473 |
| 14 | 0.94868 | 0.97821 | 0.88954 | 0.93950 | 0.82764 | 0.89230 |
| 15 | 0.95213 | 0.97970 | 0.89653 | 0.94343 | 0.83787 | 0.89893 |
| 16 | 0.95516 | 0.98100 | 0.90269 | 0.94688 | 0.84695 | 0.90479 |
| 17 | 0.95782 | 0.98215 | 0.90816 | 0.94993 | 0.85508 | 0.90999 |
| 18 | 0.96018 | 0.98316 | 0.91305 | 0.95265 | 0.86238 | 0.91469 |
| 19 | 0.96229 | 0.98406 | 0.91744 | 0.95509 | 0.86898 | 0.91888 |
| 20 | 0.96419 | 0.98487 | 0.92141 | 0.95729 | 0.87498 | 0.92269 |
| | $p=2,\ q=5$ | | $p=2,\ q=6$ | | $p=2,\ q=7$ | |
| 2 | 0.20315 | 0.36231 | 0.15448 | 0.28712 | 0.12122 | 0.23239 |
| 3 | 0.33849 | 0.49798 | 0.27112 | 0.41389 | 0.22153 | 0.34799 |
| 4 | 0.43392 | 0.58317 | 0.35963 | 0.49946 | 0.30211 | 0.43075 |
| 5 | 0.50518 | 0.64288 | 0.42910 | 0.56245 | 0.36799 | 0.49403 |
| 6 | 0.56045 | 0.68730 | 0.48506 | 0.61099 | 0.42274 | 0.54434 |
| 7 | 0.60460 | 0.72172 | 0.53108 | 0.64962 | 0.46890 | 0.58539 |
| 8 | 0.64067 | 0.74927 | 0.56956 | 0.68124 | 0.50830 | 0.61952 |
| 9 | 0.67072 | 0.77182 | 0.60224 | 0.70757 | 0.54230 | 0.64846 |
| 10 | 0.69612 | 0.79061 | 0.63032 | 0.72985 | 0.57194 | 0.67327 |
| 11 | 0.71789 | 0.80653 | 0.65470 | 0.74895 | 0.59800 | 0.69478 |
| 12 | 0.73674 | 0.82019 | 0.67607 | 0.76552 | 0.62109 | 0.71363 |
| 13 | 0.75323 | 0.83204 | 0.69495 | 0.78003 | 0.64167 | 0.73027 |
| 14 | 0.76778 | 0.84239 | 0.71176 | 0.79283 | 0.66015 | 0.74507 |
| 15 | 0.78070 | 0.85157 | 0.72681 | 0.80423 | 0.67682 | 0.75833 |
| 16 | 0.79227 | 0.85974 | 0.74037 | 0.81440 | 0.69193 | 0.77027 |
| 17 | 0.80267 | 0.86704 | 0.75265 | 0.82360 | 0·70570 | 0.78106 |
| 18 | 0.81208 | 0.87363 | 0.76382 | 0.83193 | 0.71829 | 0.79091 |
| 19 | 0.82063 | 0.87959 | 0.77402 | 0.83950 | 0.72985 | 0.79991 |
| 20 | 0.82844 | 0.88501 | 0.78338 | 0.84642 | 0.74050 | 0.80816 |
| | $p=2,\ q=8$ | | $p=2,\ q=9$ | | $p=2,\ q=10$ | |
| 2 | 0.09755 | 0.19159 | 0.08016 | 0.16042 | 0.06701 | 0.13619 |
| 3 | 0.18414 | 0.29529 | 0.15536 | 0.25431 | 0.13276 | 0.22073 |
| 4 | 0.25695 | 0.37423 | 0.22098 | 0.32755 | 0.19194 | 0.28874 |
| 5 | 0.31851 | 0.43613 | 0.27807 | 0.38710 | 0.24467 | 0.34543 |
| 6 | 0.37105 | 0.48663 | 0.32789 | 0.43678 | 0.29160 | 0.39367 |
| 7 | 0.41628 | 0.52869 | 0.37159 | 0.47890 | 0.33344 | 0.43521 |
| 8 | 0.45559 | 0.56425 | 0.41016 | 0.51511 | 0.37087 | 0.47141 |
| 9 | 0.49000 | 0.59484 | 0.44439 | 0.54654 | 0.40448 | 0.50318 |
| 10 | 0.52041 | 0.62138 | 0.47495 | 0.57417 | 0.43480 | 0.53140 |
| 11 | 0.54743 | 0.64463 | 0.50239 | 0.59861 | 0.46227 | 0.55657 |
| 12 | 0.57159 | 0.66518 | 0.52715 | 0.62038 | 0.48726 | 0.57917 |
| 13 | 0.59333 | 0.68347 | 0.54960 | 0.63991 | 0.51008 | 0.59958 |
| 14 | 0.61298 | 0.69986 | 0.57003 | 0.65752 | 0.53098 | 0.61810 |
| 15 | 0.63083 | 0.71464 | 0.58871 | 0.67348 | 0.55020 | 0.63498 |

| n | p=2, q=8 | | p=2, q=9 | | p=2, q=10 | |
|---|---|---|---|---|---|---|
| | F(x)=0.95 | F(x)=0.99 | F(x)=0.95 | F(x)=0.99 | F(x)=0.95 | F(x)=0.99 |
| 16 | 0.64712 | 0.72802 | 0.60585 | 0.68803 | 0.56793 | 0.65043 |
| 17 | 0.66204 | 0.74020 | 0.62163 | 0.70132 | 0.58433 | 0.66462 |
| 18 | 0.67575 | 0.75130 | 0.63620 | 0.71353 | 0.59954 | 0.67771 |
| 19 | 0.68840 | 0.76152 | 0.64970 | 0.72478 | 0.61369 | 0.68981 |
| 20 | 0.70010 | 0.77094 | 0.66224 | 0.73515 | 0.62688 | 0.70104 |
| | p=2, q=11 | | p=2, q=12 | | p=2, q=13 | |
| 2 | 0.05683 | 0.11705 | 0.04880 | 0.10157 | 0.04236 | 0.08899 |
| 3 | 0.11470 | 0.19320 | 0.10007 | 0.17042 | 0.08805 | 0.15139 |
| 4 | 0.16818 | 0.25621 | 0.14852 | 0.22873 | 0.13209 | 0.20536 |
| 5 | 0.21683 | 0.30985 | 0.19340 | 0.27929 | 0.17352 | 0.25291 |
| 6 | 0.26086 | 0.35627 | 0.23464 | 0.32372 | 0.21210 | 0.29526 |
| 7 | 0.30068 | 0.39683 | 0.27240 | 0.36302 | 0.24783 | 0.33315 |
| 8 | 0.33674 | 0.43258 | 0.30697 | 0.39804 | 0.28086 | 0.36725 |
| 9 | 0.36948 | 0.46434 | 0.33865 | 0.42944 | 0.31140 | 0.39808 |
| 10 | 0.39928 | 0.49271 | 0.36774 | 0.45772 | 0.33966 | 0.42606 |
| 11 | 0.42649 | 0.51827 | 0.39450 | 0.48339 | 0.36583 | 0.45162 |
| 12 | 0.45143 | 0.54137 | 0.41919 | 0.50675 | 0.39013 | 0.47503 |
| 13 | 0.47435 | 0.56237 | 0.44202 | 0.52809 | 0.41272 | 0.49654 |
| 14 | 0.49548 | 0.58153 | 0.46319 | 0.54768 | 0.43377 | 0.51637 |
| 15 | 0.51501 | 0.59908 | 0.48284 | 0.56571 | 0.45341 | 0.53470 |
| 16 | 0.53311 | 0.61523 | 0.50115 | 0.58236 | 0.47178 | 0.55171 |
| 17 | 0.54993 | 0.63012 | 0.51823 | 0.59779 | 0.48898 | 0.56753 |
| 18 | 0.56560 | 0.64391 | 0.53420 | 0.61212 | 0.50513 | 0.58227 |
| 19 | 0.58023 | 0.65671 | 0.54916 | 0.62547 | 0.52032 | 0.59604 |
| 20 | 0.59392 | 0.66861 | 0.56321 | 0.63793 | 0.53461 | 0.60894 |
| | p=2, q=14 | | p=2, q=15 | | p=2, q=16 | |
| 2 | 0.03710 | 0.07857 | 0.03277 | 0.06988 | 0.02915 | 0.06255 |
| 3 | 0.07806 | 0.13534 | 0.06967 | 0.12169 | 0.06256 | 0.11002 |
| 4 | 0.11822 | 0.18532 | 0.10640 | 0.16803 | 0.09627 | 0.15325 |
| 5 | 0.15653 | 0.23000 | 0.14189 | 0.21000 | 0.12925 | 0.19348 |
| 6 | 0.19262 | 0.27027 | 0.17566 | 0.24824 | 0,16105 | 0.23233 |
| 7 | 0.22638 | 0.30668 | 0.20755 | 0.28313 | 0.19163 | 0.27587 |
| 8 | 0.25788 | 0.33973 | 0.23755 | 0.31507 | | |
| 9 | 0.28723 | 0.36985 | 0.26570 | 0.34438 | | |
| 10 | 0.31457 | 0.39736 | 0.29209 | 0.37137 | | |
| 11 | 0.34007 | 0.42267 | 0.31685 | 0.39625 | | |
| 12 | 0.36386 | 0.44597 | 0.34007 | 0.41932 | | |
| 13 | 0.38611 | 0.46749 | 0.36189 | 0.44072 | | |
| 14 | 0.40693 | 0.48742 | 0.38240 | 0.46064 | | |
| 15 | 0.42644 | 0.50592 | 0.40170 | 0.47920 | | |
| 16 | 0.44477 | 0.52315 | 0.41990 | 0.49655 | | |
| 17 | 0.46200 | 0.53923 | 0.43706 | 0.51279 | | |
| 18 | 0.47822 | 0.55428 | 0.45328 | 0.52803 | | |
| 19 | 0.49352 | 0.56837 | 0.46862 | 0.54235 | | |
| 20 | 0.50798 | 0.58161 | 0.48315 | 0.55585 | | |
| | p=3, q=4 | | p=3, q=6 | | p=3, q=8 | |
| 3 | 0.12755 | 0.25059 | 0.05440 | 0.11887 | 0.02787 | 0.06463 |
| 4 | 0.24215 | 0.38221 | 0.12077 | 0.20962 | 0.06807 | 0.12502 |
| 5 | 0.33336 | 0.47369 | 0.18449 | 0.28528 | 0.11139 | 0.18159 |
| 6 | 0.40624 | 0.54137 | 0.24252 | 0.34870 | 0.15441 | 0.23327 |
| 7 | 0.46535 | 0.59361 | 0.29442 | 0.40239 | 0.19561 | 0.27992 |
| 8 | 0.51406 | 0.63515 | 0.34058 | 0.44820 | 0.23435 | 0.32196 |

| n | p=3, q=4 | | p=3, q=6 | | p=3, q=8 | |
|---|---|---|---|---|---|---|
| | F(x)=0.95 | F(x)=0.99 | F(x)=0.95 | F(x)=0.99 | F(x)=0.95 | F(x)=0.99 |
| 9 | 0.55481 | 0.66894 | 0.38163 | 0.48768 | 0.27043 | 0.35981 |
| 10 | 0.58934 | 0.69705 | 0.41823 | 0.52195 | 0.30387 | 0.39396 |
| 11 | 0.61898 | 0.72075 | 0.45097 | 0.55206 | 0.33482 | 0.42486 |
| 12 | 0.64466 | 0.74100 | 0.48039 | 0.57866 | 0.36344 | 0.45288 |
| 13 | 0.66713 | 0.75852 | 0.50694 | 0.60231 | 0.38992 | 0.47844 |
| 14 | 0.68695 | 0.77382 | 0.53099 | 0.62347 | 0.41447 | 0.50180 |
| 15 | 0.70455 | 0.78729 | 0.55287 | 0.64251 | 0.43725 | 0.52321 |
| 16 | 0.72029 | 0.79925 | 0.57284 | 0.65974 | 0.45842 | 0.54291 |
| 17 | 0.73444 | 0.80991 | 0.59114 | 0.67539 | 0.47814 | 0.56107 |
| 18 | 0.74724 | 0.81952 | 0.60797 | 0.68967 | 0.49653 | 0.57788 |
| 19 | 0.75886 | 0.82821 | 0.62348 | 0.70276 | 0.51372 | 0.59347 |
| 20 | 0.76947 | 0.83610 | 0.63783 | 0.71479 | 0.52981 | 0.60796 |

| n | p=3, q=10 | | p=3, q=12 | | p=3, q=14 | |
|---|---|---|---|---|---|---|
| 3 | 0.01610 | 0.03879 | 0.01012 | 0.02502 | 0.00677 | 0.01706 |
| 4 | 0.04193 | 0.07996 | 0.02759 | 0.05404 | 0.01910 | 0.03815 |
| 5 | 0.07202 | 0.12175 | 0.04913 | 0.08526 | 0.03496 | 0.06189 |
| 6 | 0.10382 | 0.16234 | 0.07294 | 0.11703 | 0.05313 | 0.08693 |
| 7 | 0.13582 | 0.20084 | 0.09786 | 0.14833 | 0.07272 | 0.11237 |
| 8 | 0.16720 | 0.23700 | 0.12311 | 0.17870 | 0.09310 | 0.13771 |
| 9 | 0.19750 | 0.27071 | 0.14818 | 0.20782 | 0.11382 | 0.16256 |
| 10 | 0.22646 | 0.30204 | 0.17276 | 0.23555 | 0.13454 | 0.18671 |
| 11 | 0.25399 | 0.33113 | 0.19663 | 0.26186 | 0.15505 | 0.21005 |
| 12 | 0.28006 | 0.35815 | 0.21970 | 0.28676 | 0.17519 | 0.23248 |
| 13 | 0.30471 | 0.38322 | 0.24189 | 0.31026 | 0.19486 | 0.25401 |
| 14 | 0.32797 | 0.40660 | 0.26318 | 0.33252 | 0.21398 | 0.27454 |
| 15 | 0.34993 | 0.42837 | 0.28355 | 0.35355 | 0.23253 | 0.29415 |
| 16 | 0.37066 | 0.44869 | 0.30306 | 0.37345 | 0.25046 | 0.31291 |
| 17 | 0.39024 | 0.46768 | 0.32169 | 0.39225 | 0.26781 | |
| 18 | 0.40873 | 0.48547 | 0.33952 | 0.41007 | 0.28452 | |
| 19 | 0.42622 | 0.50215 | 0.35654 | 0.42693 | 0.30060 | |
| 20 | 0.44278 | 0.51783 | 0.37291 | 0.44375 | | |

| n | p=3, q=16 | | p=4, q=4 | | p=4, q=5 | |
|---|---|---|---|---|---|---|
| 3 | 0.00474 | 0.01214 | | | | |
| 4 | 0.01375 | 0.02791 | 0.06539 | 0.14144 | 0.03668 | 0.08466 |
| 5 | 0.02574 | 0.04628 | 0.14302 | 0.24472 | 0.08904 | 0.16172 |
| 6 | 0.03986 | 0.06624 | 0.21515 | 0.32765 | 0.14364 | 0.23029 |
| 7 | 0.05546 | 0.08702 | 0.27899 | 0.39507 | 0.19604 | 0.29074 |
| 8 | 0.07204 | 0.10818 | 0.33475 | 0.45074 | 0.24462 | 0.34347 |
| 9 | 0.08922 | 0.12933 | 0.38333 | 0.49728 | 0.28902 | 0.38957 |
| 10 | 0.10671 | 0.15024 | 0.42580 | 0.53670 | 0.32933 | 0.43000 |
| 11 | 0.12428 | 0.17074 | 0.46311 | 0.57044 | 0.36586 | 0.46567 |
| 12 | 0.14178 | 0.19076 | 0.49605 | 0.59972 | 0.39899 | 0.49725 |
| 13 | 0.15909 | 0.21024 | 0.52533 | 0.62530 | 0.42908 | 0.52548 |
| 14 | 0.17611 | 0.22915 | 0.55150 | 0.64783 | 0.45650 | 0.55078 |
| 15 | 0.19288 | 0.24581 | 0.57499 | 0.66782 | 0.48154 | 0.57357 |
| 16 | 0.20917 | | 0.59620 | 0.68568 | 0.50448 | 0.59421 |
| 17 | | | 0.61543 | 0.70173 | 0.52554 | 0.61297 |
| 18 | | | 0.63293 | 0.71623 | 0.54495 | 0.63010 |
| 19 | | | 0.64894 | 0.72938 | 0.56288 | 0.64579 |
| 20 | | | 0.66362 | 0.74138 | 0.57948 | 0.66022 |

| n | p=4, q=6 | | p=4, q=7 | | p=4, q=8 | |
|---|---|---|---|---|---|---|
| | $F(x)$=0.95 | $F(x)$=0.99 | $F(x)$=0.95 | $F(x)$=0.99 | $F(x)$=0.95 | $F(x)$=0.99 |
| 4 | 0.02208 | 0.05345 | 0.01406 | 0.03526 | 0.00936 | 0.02415 |
| 5 | 0.05806 | 0.10991 | 0.03936 | 0.07708 | 0.02759 | 0.05550 |
| 6 | 0.09904 | 0.16521 | 0.07029 | 0.12100 | 0.05117 | 0.09037 |
| 7 | 0.14102 | 0.21688 | 0.10371 | 0.16425 | 0.07781 | 0.12626 |
| 8 | 0.18194 | 0.26415 | 0.13768 | 0.20543 | 0.10589 | 0.16166 |
| 9 | 0.22087 | 0.30707 | 0.17113 | 0.24409 | 0.13435 | 0.19581 |
| 10 | 0.25739 | 0.34590 | 0.20342 | 0.28002 | 0.16254 | 0.22836 |
| 11 | 0.29141 | 0.38104 | 0.23424 | 0.31329 | 0.19002 | 0.25911 |
| 12 | 0.32300 | 0.41288 | 0.26345 | 0.34404 | 0.21656 | 0.28804 |
| 13 | 0.35227 | 0.44177 | 0.29102 | 0.37246 | 0.24201 | 0.31520 |
| 14 | 0.37939 | 0.46815 | 0.31698 | 0.39870 | 0.26633 | 0.34061 |
| 15 | 0.40456 | 0.49226 | 0.34139 | 0.42305 | 0.28950 | 0.36449 |
| 16 | 0.42793 | 0.51435 | 0.36434 | 0.44563 | 0.31152 | 0.38683 |
| 17 | 0.44965 | 0.53466 | 0.38593 | 0.46659 | | |
| 18 | 0.46989 | 0.55339 | 0.40623 | 0.48611 | | |
| 19 | 0.48876 | 0.57071 | 0.42534 | 0.50432 | | |
| 20 | 0.50640 | 0.58676 | 0.44335 | 0.52129 | | |
| | p=4, q=9 | | p=4, q=10 | | p=4, q=11 | |
| 4 | 0.00647 | 0.01708 | 0.00461 | 0.01240 | 0.00338 | 0.00922 |
| 5 | 0.01989 | 0.04090 | 0.01469 | 0.03077 | 0.01108 | 0.02357 |
| 6 | 0.03809 | 0.06872 | 0.02892 | 0.05311 | 0.02233 | 0.04165 |
| 7 | 0.05944 | 0.09843 | 0.04614 | 0.07775 | 0.03634 | 0.06216 |
| 8 | 0.08265 | 0.12865 | 0.06538 | 0.10349 | 0.05237 | 0.08408 |
| 9 | 0.10679 | 0.15854 | 0.08586 | 0.12951 | 0.06977 | 0.10671 |
| 10 | 0.13123 | 0.18764 | 0.10701 | 0.15535 | 0.08805 | 0.12955 |
| 11 | 0.15553 | 0.21566 | 0.12838 | 0.18063 | 0.10682 | 0.15223 |
| 12 | 0.17938 | 0.24244 | 0.14969 | 0.20512 | 0.12579 | 0.17441 |
| 13 | 0.20260 | 0.26793 | 0.17071 | 0.22868 | 0.14472 | 0.19585 |
| 14 | 0.22507 | 0.29215 | 0.19130 | 0.25113 | 0.16347 | 0.21537 |
| 15 | 0.24673 | 0.31501 | 0.21136 | 0.27156 | 0.18187 | 0.23592 |
| 16 | 0.26753 | 0.33673 | | | | |
| 17 | 0.28745 | 0.35675 | | | | |
| 18 | 0.30644 | 0.37598 | | | | |
| 19 | 0.32461 | 0.39337 | | | | |
| 20 | 0.34123 | | | | | |
| | p=4, q=12 | | p=4, q=13 | | p=4, q=14 | |
| 4 | 0.00253 | 0.00699 | 0.00193 | 0.00540 | 0.00150 | 0.00424 |
| 5 | 0.00851 | 0.01835 | 0.00664 | 0.01449 | 0.00526 | 0.01176 |
| 6 | 0.01751 | 0.03310 | 0.01392 | 0.02662 | 0.01131 | |
| 7 | 0.02900 | 0.05025 | 0.02341 | 0.04103 | 0.02054 | |
| 8 | 0.04241 | 0.06897 | 0.03470 | 0.05706 | | |
| 9 | 0.05724 | 0.08863 | 0.04738 | 0.07417 | | |
| 10 | 0.07307 | 0.10879 | 0.06111 | 0.09195 | | |
| 11 | 0.08954 | 0.12907 | 0.07559 | 0.10993 | | |
| 12 | 0.10640 | 0.14907 | 0.09055 | | | |
| 13 | 0.12340 | 0.16830 | 0.10573 | | | |
| 14 | 0.14032 | | | | | |
| | p=5, q=6 | | p=5, q=8 | | p=5, q=10 | |
| 5 | 0.01005 | 0.02610 | 0.00358 | 0.00996 | 0.00152 | 0.00442 |
| 6 | 0.02999 | 0.06044 | 0.01218 | 0.02618 | 0.00566 | 0.01270 |
| 7 | 0.05585 | 0.09858 | 0.02497 | 0.04683 | 0.01243 | 0.02431 |

| n | p=5, q=6 | | p=5, q=8 | | p=5, q=10 | |
|---|---|---|---|---|---|---|
| | F(x)=0.95 | F(x)=0.99 | F(x)=0.95 | F(x)=0.99 | F(x)=0.95 | F(x)=0.99 |
| 8 | 0.08493 | 0.13748 | 0.04098 | 0.07024 | 0.02159 | 0.03852 |
| 9 | 0.11539 | 0.17557 | 0.05928 | 0.095172 | 0.03275 | 0.05466 |
| 10 | 0.14604 | 0.21198 | 0.07906 | 0.12073 | 0.04550 | 0.07214 |
| 11 | 0.17618 | 0.24642 | 0.09971 | 0.14632 | 0.05947 | 0.09036 |
| 12 | 0.20536 | 0.27872 | 0.12077 | 0.17166 | 0.07442 | |
| 13 | 0.23336 | 0.30890 | 0.14191 | 0.19674 | | |
| 14 | 0.26006 | 0.33704 | 0.16292 | 0.22314 | | |
| 15 | 0.28541 | 0.36327 | | | | |
| 16 | 0.30944 | 0.38777 | | | | |
| 17 | 0.33218 | 0.41062 | | | | |
| 18 | 0.35369 | 0.43214 | | | | |
| 19 | 0.37404 | 0.45240 | | | | |
| 20 | 0.39331 | 0.47131 | | | | |

| n | p=5, q=12 | | p=5, q=14 | | p=5, q=16 | |
|---|---|---|---|---|---|---|
| 5 | 0.00073 | 0.00220 | 0.00038 | 0.00119 | 0.00022 | 0.00068 |
| 6 | 0.00291 | 0.00673 | 0.00161 | 0.00383 | 0.00095 | 0.00230 |
| 7 | 0.00673 | 0.01357 | 0.00389 | 0.00803 | 0.00237 | 0.00499 |
| 8 | 0.01220 | 0.02245 | 0.00730 | 0.01375 | 0.00458 | 0.00879 |
| 9 | 0.01922 | 0.03304 | 0.01185 | 0.02078 | 0.00762 | |
| 10 | 0.02758 | | 0.01745 | | | |
| 11 | 0.03706 | | | | | |

| n | p=6, q=6 | | p=6, q=7 | | p=6, q=8 | |
|---|---|---|---|---|---|---|
| 6 | 0.00501 | 0.01371 | 0.00268 | 0.00763 | 0.00152 | 0.00448 |
| 7 | 0.01648 | 0.03483 | 0.00955 | 0.02095 | 0.00578 | 0.01306 |
| 8 | 0.03295 | 0.06072 | 0.02024 | 0.03864 | 0.01286 | 0.02525 |
| 9 | 0.05293 | 0.08901 | 0.03404 | 0.05929 | 0.02252 | 0.04031 |
| 10 | 0.07512 | 0.11863 | 0.05020 | 0.08178 | 0.03430 | 0.05731 |
| 11 | 0.09862 | 0.14823 | 0.06800 | 0.10526 | 0.04779 | 0.07576 |
| 12 | 0.12266 | 0.17726 | 0.08687 | 0.12907 | 0.06264 | |
| 13 | 0.14674 | 0.20533 | 0.10638 | | 0.07802 | |
| 14 | 0.17047 | 0.23229 | | | | |
| 15 | 0.19382 | 0.25807 | | | | |
| 16 | 0.21634 | 0.28220 | | | | |
| 17 | 0.23834 | 0.30348 | | | | |
| 18 | 0.25888 | 0.32156 | | | | |
| 19 | 0.27781 | 0.33962 | | | | |

| n | p=6, q=9 | | p=6, q=10 | | p=6, q=11 | |
|---|---|---|---|---|---|---|
| 6 | 0.00091 | 0.00273 | 0.00056 | 0.00173 | 0.00036 | 0.00113 |
| 7 | 0.00364 | 0.00843 | 0.00236 | 0.00558 | 0.00158 | 0.00381 |
| 8 | 0.00844 | 0.01699 | 0.00568 | 0.01165 | 0.00392 | 0.00820 |
| 9 | 0.01528 | 0.02801 | 0.01061 | 0.01981 | 0.00752 | 0.01423 |
| 10 | 0.02397 | 0.04096 | 0.01706 | 0.02994 | 0.01236 | |
| 11 | 0.03423 | | 0.02492 | | | |

| n | p=6, q=12 | | p=6, q=13 | | p=6, q=14 | |
|---|---|---|---|---|---|---|
| 6 | 0.00024 | 0.00076 | 0.00016 | 0.00052 | 0.00011 | |
| 7 | 0.00109 | 0.00281 | 0.00076 | 0.00188 | | |
| 8 | 0.00291 | | 0.00198 | 0.00427 | | |
| 9 | | | 0.00398 | | | |
| 10 | | | 0.00678 | | | |

| $n$ | $p=7,\ q=8$ | | $p=7,\ q=10$ | | $p=7,\ q=12$ | |
|---|---|---|---|---|---|---|
| | $F(x)=0.95$ | $F(x)=0.99$ | $F(x)=0.95$ | $F(x)=0.99$ | $F(x)=0.95$ | $F(x)=0.99$ |
| 7 | 0.00070 | 0.00215 | 0.00023 | 0.00073 | 0.00008 | 0.00029 |
| 8 | 0.00292 | 0.00688 | 0.00106 | 0.00262 | 0.00043 | 0.00111 |
| 9 | 0.00696 | 0.01421 | 0.00275 | 0.00587 | 0.00121 | 0.00266 |
| 10 | 0.01287 | 0.02391 | 0.00545 | 0.01046 | 0.00253 | |
| 11 | 0.02052 | | | | | |
| | $p=7,\ q=14$ | | $p=8,\ q=8$ | | $p=,8\ q=9$ | |
| 7 | 0.00003 | 0.00012 | | | | |
| 8 | 0.00019 | 0.00052 | 0.00035 | 0.00110 | 0.00018 | 0.00059 |
| 9 | 0.00057 | | 0.00155 | 0.00378 | 0.00087 | 0.00217 |
| 10 | | | 0.00392 | 0.00825 | 0.00230 | 0.00497 |
| 11 | | | 0.00761 | 0.01448 | 0.00464 | |
| 12 | | | 0.01257 | 0.01971 | | |
| | $p=8,\ q=10$ | | | | | |
| 8 | 0.00010 | 0.00033 | | | | |
| 9 | 0.00050 | 0.00144 | | | | |
| 10 | 0.00156 | | | | | |

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