THE LIMITING DISTRIBUTION OF THE SAMPLE OCCUPANCY NUMBERS FROM THE MULTINOMIAL DISTRIBUTION WITH EQUAL CELL PROBABILITIES

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1. Introduction

Assume that a random sample of n observations has been made from a multinomial population with uniform cell probabilities, that is, cell i has probability N^{-1}, $i=1, 2, \cdots, N$. Let s_i be the number of cells which occur exactly i times in the sample. Then, we clearly have

$$(1) \qquad \sum_{i=0}^{n} s_i = N \quad \text{and} \quad \sum_{i=0}^{n} i s_i = n .$$

The random variables s_i, $i=0, 1, \cdots, n$ will be called the (sample) occupancy numbers in agreement with usage in past publications of the authors. (Wilks [10] refers to these as the cell frequency counts).

Our interest in the behavior of the occupancy numbers is motivated by their significant role in non-parametric tests of the hypothesis $F(x) = F_0(x)$, where $F(x)$ is an absolutely continuous cumulative distribution function and $F_0(x)$ is a specified absolutely continuous cumulative distribution function. In particular, the χ^2 goodness of fit test, the empty cell test, and the likelihood ratio test (based on the multinomial distribution) all are expressible in terms of occupancy numbers. For each of these tests, the customary procedure (but not the only one possible) is to select an integer N in advance of the experiment; then divide the real line into N consecutive intervals each of which has probability N^{-1} under $F_0(x)$. Thus, when the hypothesis is true, the distribution of the observations, when classified only by the interval in which they fall and ignoring the natural ordering of the intervals, is the multinomial distribution with equal cell probabilities.

In this paper, we will study the limiting distribution of s_i, $i=1, 2, \cdots, k$; k fixed and independent of n and N, as $n, N \to \infty$ so that $n/N \to \alpha$, $0 < \alpha < \infty$.

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Under the hypotheses of this paper, I. Weiss [9] and M. Okamoto [6] established independently that $(s_0-E(s_0))/\sigma_{s_0}$ has a limiting standard normal distribution. Weiss and Okamoto both employed the method of moments in their investigation. Subsequently, Renyi [7] reexamined the limiting distribution of s_0 using generating functions. The limiting distribution of s_0 under alternative hypotheses was examined by S. Kitabatake [5] and V. P. Chistyakov [1].

Sevast'yanov and Chistyakov [8], using saddlepoint methods, established the joint asymptotic normality of any subset of $(s_0, s_1, \cdots, s_p)$ and this was extended to alternative hypotheses by Chistyakov and Viktorova [2].

In this paper, we study the asymptotic distribution of s_i by using the method of moments. Despite the fact that the asymptotic normality has been previously established, it was felt that information concerning the rate of convergence of the standardized central moments would prove useful and lead to improvements in probability estimates over those specifically given by the limiting normal distribution. In the Sevast'yanov and Chistyakov [8] and the Chistyakov and Viktorova [2] papers only the moments of order one and two are reported and for these only the leading terms of their asymptotic development are reported. The methods of this paper can be extended to exhibit the joint asymptotic normality of any subset of $(s_0, s_1, \cdots, s_p)$, but this extension would be very tedious. The complete asymptotic expansion of the standardized central moments of s_0 is implicit in Weiss's paper [9], but the specific details are not provided therein.

In another paper (Harris and Park [3]), we have studied the limiting distribution of linear combinations of the occupancy numbers, since this is precisely the form in which the occupancy numbers enter into various non-parametric tests. The results in this paper have been useful in pursuing that investigation.

2. The moments of the occupancy numbers

In Wilks ([10], p. 433), the joint distribution of $s_0, s_1, s_2, \cdots, s_n$ is given by

$$p(s_0, s_1, \cdots, s_n)=\frac{n!N!}{N^n(0!)^{s_0}(1!)^{s_1}\cdots(n!)^{s_n}s_0!s_1!\cdots s_n!}, \tag{2}$$

where $s_i \geqq 0$, $\sum_{i=1}^{n} s_i=N$, $\sum_{i=1}^{n} is_i=n$. The νth factorial moment (Wilks [10], p. 153 or 433) is given by

$$E(s_i^{(\nu)})=\frac{\nu!(i\nu)!}{(i!)^\nu}\binom{N}{\nu}\binom{n}{i\nu}\left(\frac{1}{N}\right)^{i\nu}\left(1-\frac{\nu}{N}\right)^{n-i\nu}, \tag{3}$$

where $\nu \leqq N$, $i\nu \leqq n$. Thus, we can write

$$(4) \qquad E(s_i^{(\nu)}) = \frac{N^{(\nu)}}{(i!)^\nu}\left(\frac{n}{N}\right)^{i\nu}\left(1-\frac{\nu}{N}\right)^{n-i\nu} h(N, n, i, \nu) ,$$

where

$$h(N, n, i, \nu) = \begin{cases} 1 & i\nu = 0 \\ \prod_{t=0}^{i\nu-1}\left(1-\frac{t}{n}\right) & i\nu > 0 . \end{cases}$$

Let $\mu_k^{(i)}$ be the kth central moment of s_i and let $\alpha_{j,k}$ and $\beta_{j,k}$ be the Stirling numbers of the first and second kind respectively, defined by

$$x^{(k)} = \sum_{j=1}^{k} \alpha_{j,k} x^j$$

and

$$x^k = \sum_{j=1}^{k} \beta_{j,k} x^{(j)} ,$$

where $x^{(m)} = x(x-1)\cdots(x-m+1)$. We adopt the conventions that $\alpha_{j,k} = \beta_{j,k} = 0$ unless $j=k=0$, or $0<j\leqq k$. In particular $\alpha_{0,0}=\beta_{0,0}=1$. Then,

$$\begin{aligned}(5) \qquad \mu_k^{(i)} &= E(s_i - E(s_i))^k \\ &= \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{N^r}{(i!)^r}\left(\frac{n}{N}\right)^{ir}\left(1-\frac{1}{N}\right)^{r(n-i)} [h(N, n, i, 1)]^r \\ &\quad \cdot \sum_{p=0}^{k-r} \beta_{p,k-r} \frac{N^{(p)}}{(i!)^p}\left(\frac{n}{N}\right)^{ip}\left(1-\frac{p}{N}\right)^{n-ip} h(N, n, i, p) \\ &= \sum_{r=0}^{k}\sum_{p=0}^{k-r}\sum_{j=0}^{p} (-1)^r \binom{k}{r} \frac{N^{r+j}}{(i!)^{r+p}}\left(\frac{n}{N}\right)^{i(r+p)} \alpha_{j,p}\beta_{p,k-r} \\ &\quad \cdot \left(1-\frac{1}{N}\right)^{r(n-i)}\left(1-\frac{p}{N}\right)^{n-ip} [h(N, n, i, 1)]^r h(N, n, i, p) .\end{aligned}$$

We set $p+r=s$ and $j+r=l$ obtaining

$$\begin{aligned}(6) \qquad \mu_k^{(i)} &= \sum_{r=0}^{k}\sum_{s=r}^{k}\sum_{l=r}^{s} (-1)^r \binom{k}{r} \frac{N^l}{(i!)^s}\left(\frac{n}{N}\right)^{is} \alpha_{l-r,s-r}\beta_{s-r,k-r} \\ &\quad \cdot \left(1-\frac{1}{N}\right)^{r(n-i)}\left(1-\frac{s-r}{N}\right)^{n-i(s-r)} \\ &\quad \cdot [h(N, n, i, 1)]^r h(N, n, i, s-r) .\end{aligned}$$

In the asymptotic analysis of (6), we will frequently employ the following relationships. If N, $n\to\infty$ so that $n/N\to\alpha$, $0<\alpha<\infty$, then for each fixed u,

$$(7)\qquad \left(1-\frac{u}{N}\right)^n=\exp\left\{-n\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{u}{N}\right)^j\right\}$$
$$=\exp\left\{-n\sum_{j=1}^{\tau}\frac{1}{j}\left(\frac{u}{N}\right)^j\right\}(1+O(N^{-\tau}))\ .$$

We will also employ the convention that $\sum_{i=a}^{b}\alpha_i=0$, whenever $b<a$.

Now apply (7) to (6) and let n, $N\to\infty$, so that $n/N\to\alpha$, $0<\alpha<\infty$, obtaining for each fixed i, $1\leq i\leq q$,

$$(8)\qquad \left(1-\frac{1}{N}\right)^{r(n-i)}\left(1-\frac{s-r}{N}\right)^{n-i(s-r)}[h(N,n,i,1)]^r h(N,n,i,s-r)$$
$$=\exp\left\{-r(n-i)\sum_{t=1}^{\infty}\frac{1}{tN^t}-(n-i(s-r))\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{s-r}{N}\right)^t\right.$$
$$\left.-r\sum_{u=0}^{i-1}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{u}{n}\right)^t-\sum_{u=0}^{i(s-r)-1}\sum_{t=1}^{\infty}\frac{1}{t}\left(\frac{u}{n}\right)^t\right\}$$
$$=\exp\left\{-s\left(\frac{n}{N}\right)-\frac{n}{N}\sum_{t=1}^{\infty}\frac{r+(s-r)^{t+1}}{(t+1)N^t}+i\sum_{t=1}^{\infty}\frac{r+(s-r)^{t+1}}{tN^t}\right.$$
$$\left.-\sum_{t=1}^{\infty}\sum_{u=0}^{i-1}\frac{ru^t}{tN^t}\left(\frac{N}{n}\right)^t-\sum_{t=1}^{\infty}\sum_{u=0}^{i(s-r)-1}\frac{u^t}{tN^t}\left(\frac{N}{n}\right)^t\right\}\ .$$

Thus,

$$(9)\qquad \left(1-\frac{1}{N}\right)^{r(n-i)}\left(1-\frac{s-r}{N}\right)^{n-i(s-r)}[h(N,n,i,1)]^r h(N,n,i,s-r)$$
$$=\exp\left\{-s\left(\frac{n}{N}\right)-\sum_{t=1}^{\tau}\frac{1}{N^t}\left\{\left(\frac{n}{N}\right)\frac{r+(s-r)^{t+1}}{t+1}-i\frac{r+(s-r)^{t+1}}{t}\right.\right.$$
$$\left.\left.+\sum_{u=0}^{i-1}\left(\frac{N}{n}\right)^t\frac{ru^t}{t}+\sum_{u=0}^{i(s-r)-1}\left(\frac{N}{n}\right)^t\frac{u^t}{t}\right\}\right\}(1+O(N^{-\tau}))\ .$$

Observe that the exponent in (9) is of the form

$$-s\frac{n}{N}+\sum_{t=1}^{\tau}P_{t+1}(r)/N^t$$

where $P_{t+1}(r)$ is a polynomial in r of degree at most $t+1$ with coefficients depending on n/N, i, and s. Now we expand $\exp\left\{\sum_{t=1}^{\tau}P_{t+1}(r)/N^t\right\}$ obtaining

$$(10)\qquad \exp\left\{\sum_{t+1}^{\tau}P_{t+1}(r)/N^t\right\}=\sum_{j=0}^{\infty}\left(\sum_{t=1}^{\tau}P_{t+1}(r)/N^t\right)^j\Big/j!$$
$$=\sum_{j=0}^{\rho}\left(\sum_{t=1}^{\tau}P_{t+1}(r)/N^t\right)^j\Big/j!+R(\rho,N)\ .$$

We now estimate $R(\rho,N)$. Clearly, since $i\leq q$, $s-r\leq k$, $r\leq k$, we have

$$|P_{t+1}(r)| \leqq \frac{n}{N}\frac{(k+k^{t+1})}{t+1}+\frac{q(k+k^{t+1})}{t}+\left(\frac{N}{n}\right)^t\frac{k}{t}\frac{q^{t+1}}{t+1}$$
$$+\left(\frac{N}{n}\right)^t\frac{(qk)^{t+1}}{t(t+1)} .$$

Now let $\max\left(\frac{n}{N}, \left(\frac{N}{n}\right)^t\right)=\beta$. Then

$$|P_{t+1}(r)| \leqq \frac{6\beta}{t}(qk)^{t+1} .$$

Thus, for N sufficiently large,

$$\left|\sum_{t=1}^{\tau} P_{t+1}(r)/N^t\right| \leqq 12q^2k^2\beta/N$$

and

$$|R(\rho, N)| = \left|\sum_{j=\rho+1}^{\infty}\left(\sum_{t=1}^{\tau} P_{t+1}(r)/N^t\right)^j\Big/j!\right| \leqq \sum_{j=\rho+1}^{\infty}\left(\frac{\gamma}{N}\right)^j\Big/j! ,$$

where $\gamma=12q^2k^2\beta$. Thus, we can easily establish that

$$|R(\rho, N)|=O(N^{-\rho-1}) . \tag{11}$$

Hence, combining (9), (10), and (11), we have in fact established the following lemma.

LEMMA 1. *If N, $n\to\infty$ so that $n/N\to\alpha>0$ and $i\leqq q$, then for any pair of positive integers ρ and τ,*

$$\left(1-\frac{1}{N}\right)^{r(n-i)}\left(1-\frac{s-r}{N}\right)^{n-i(s-r)}[h(N, n, i, 1)]^r h(N, n, i, s-r) \tag{12}$$
$$=\left\{\exp\left(-s\frac{n}{N}\right)\right\}\left[\sum_{j=0}^{\rho}\left(\sum_{t=1}^{\tau} P_{t+1}(r)/N^t\right)^j\Big/j!+O(N^{-\rho-1})\right][1+O(n^{-\tau})] ,$$

where $P_{t+1}(r)$ is a polynomial of degree at most $t+1$ in r.

We now establish the following.

LEMMA 2. *Under the hypotheses of Lemma 1,*

$$\sum_{j=0}^{\rho}\left(\sum_{t=1}^{\tau} P_{t+1}(r)/N^t\right)^j\Big/j! = \sum_{m=0}^{\tau\rho} K_m(r, s-r, i)/N^m , \tag{13}$$

where for each $m\leqq\rho$, $K_m(r, s-r, i)$ is a polynomial in r of exact degree $2m$ whenever $n\neq iN$. The coefficient of r^{2m} is

$$\frac{(-1)^m}{m!}\left(\frac{1}{2}\left(\frac{n}{N}\right)-i+\frac{i^2}{2}\left(\frac{N}{n}\right)\right)^m .$$

PROOF.

$$\sum_{j=0}^{\rho}\left(\sum_{t=1}^{\tau} P_{t+1}(r)/N^t\right)^j \Big/ j! = \sum_{j=0}^{\rho}\frac{1}{j!}\sum \frac{j!}{k_1!k_2!\cdots k_\tau!}\left(\frac{P_2(r)}{N}\right)^{k_1}\left(\frac{P_3(r)}{N^2}\right)^{k_2}\cdots$$
$$\left(\frac{P_{\tau+1}(r)}{N^\tau}\right)^{k_\tau}, \qquad k_\nu \geqq 0,\ \sum k_\nu = j\ .$$

Collecting terms by powers of N, we get

$$\sum_{m=0}^{\tau\rho}\frac{1}{N^m}\sum\frac{[P_2(r)]^{k_1}[P_3(r)]^{k_2}\cdots[P_{\tau+1}(r)]^{k_\tau}}{k_1!k_2!\cdots k_\tau!}\,,$$

the second sum running over $k_1, k_2, \cdots, k_\tau$ with $\sum_{j=1}^{\tau} jk_j = m$ and $\sum_{j=1}^{\tau} k_j \leqq \rho$. The degree κ of each K_m satisfies

$$\kappa \leqq \sum_{j=1}^{\tau}(j+1)k_j \leqq m+\rho \leqq 2m\ .$$

Further, for each $m\leqq\rho$, set $k_1=m$, $k_2=k_3=\cdots=k_\tau=0$ obtaining the term $[P_2(r)]^m$ which is of degree $2m$, since the coefficient of r^{2m} is $(-1)^m(n/2N-i+Ni^2/2n)^m$ and is non-zero by hypothesis. This is clearly the only term of degree $2m$.

The following lemma can now be established.

LEMMA 3. *For N, n sufficiently large, $n\neq iN$, and $m\leqq\rho$,*

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}\alpha_{u+m-r,s-r}\beta_{s-r,k-r}K_m(r, s-r, i)=\begin{cases}0 & u>k/2\\ ck! & u=k/2\ ,\end{cases} \tag{14}$$

where

$$c=\frac{C_{s-k/2-m}}{[2(s-k/2-m)]!}\,\frac{D_{k-s}}{[2(k-s)]!}\,\frac{(-1)^m}{m!}\left(\frac{1}{2}\left(\frac{n}{N}\right)-i+\frac{i^2}{2}\frac{N}{n}\right)^m, \tag{15}$$
$$C_\nu=(-1)^\nu \quad D_\nu=(-1)^\nu\prod_{j=1}^{\nu}(2j-1)\ .$$

PROOF. It is well known (see Jordan [4], p. 151 and p. 171) that $\alpha_{\eta-\nu,\eta}$ and $\beta_{\eta-\nu,\eta}$ are polynomials in η of degree 2ν with leading coefficients $C_\nu/(2\nu)!$ and $D_\nu/(2\nu)!$ respectively. Then,

$$f(r)=\alpha_{u+m-r,s-r}\,\beta_{s-r,k-r}\,K_m(r, s-r, i)$$

is a polynomial in r of degree $2(k-u)$. Thus, the left-hand side of (14) is

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}f(r)=(-1)^k\Delta^k f(0)=\begin{cases}0 & u>k/2\\ ck! & u=k/2\ .\end{cases}$$

We now return to our examination of the central moments and obtain the following theorem.

THEOREM 1. *If N, $n\to\infty$ so that $n/N\to\alpha>0$, then for every fixed $i\leqq q$ and each fixed k,*

$$\mu_k^{(i)}=N^{k/2}D_{k/2}\left(\frac{\alpha^i}{i!}e^{-\alpha}\right)^{k/2}\left[1-\left(\frac{\alpha+(i-\alpha)^2}{\alpha}\right)\frac{\alpha^i}{i!}e^{-\alpha}\right]^{k/2}+O(N^{k/2-1})\,, \tag{16}$$

k even;

$$\lim_{n,N\to\infty}\frac{\mu_k(i)}{[\mu_2(i)]^{k/2}}=\begin{cases}\prod_{\nu=1}^{k/2}(2\nu-1) & k \text{ even}\\ 0 & k \text{ odd}\,.\end{cases} \tag{17}$$

PROOF. From (6), (12), and (13), we have

$$\mu_k^{(i)}=\sum_{r=0}^{k}\sum_{s=r}^{k}\sum_{l=r}^{s}(-1)^r\binom{k}{r}\frac{N^l}{(i!)^s}\left(\frac{n}{N}\right)^{is}\alpha_{l-r,s-r}\beta_{s-r,k-r} \cdot\left\{e^{-sn/N}\left[\sum_{m=0}^{\tau\rho}\frac{K_m(r,s-r,i)}{N^m}+O(N^{-\rho-1})\right](1+O(N^{-\tau}))\right\}\,. \tag{18}$$

Letting $u=l-m$ and interchanging the order of summation, we have,

$$\mu_k^{(i)}=\sum_{u=-\tau\rho}^{k}\sum_{s=u}^{k}\sum_{m=\max(0,-u)}^{\min(s-u,\tau\rho)}\sum_{r=0}^{u+m}(-1)^r\binom{k}{r}\frac{N^u}{(i!)^s}\left(\frac{n}{N}\right)^{is} \cdot\alpha_{u+m-r,s-r}\beta_{s-r,k-r}e^{-sn/N}(K_m(r,s-r,i) +O(N^{m-\rho-1}))(1+O(N^{-\tau}))\,. \tag{19}$$

Since $\alpha_{p,q}=0$ for $p<0$, we can extend the upper limit of the sum on r to k, obtaining

$$\mu_k^{(i)}=\sum_{u=-\tau\rho}^{k}N^u\sum_{s=u}^{k}\sum_{m=\max(0,-u)}^{\min(s-u,\tau\rho)}\left(\frac{n}{N}\right)^{is}\frac{e^{-sn/N}}{(i!)^s} \cdot\sum_{r=0}^{k}(-1)^r\binom{k}{r}\alpha_{u+m-r,s-r}\beta_{s-r,k-r}K_m(r,s-r,i) +\sum_{u=-\tau\rho}^{k}O(N^{u-\tau})+\sum_{u=-\tau\rho}^{k}\sum_{m=\max(0,-u)}^{\min(\tau\rho,k-u)}O(N^{u+m-\rho-1})\,. \tag{20}$$

Let

$$a_{s,u}(i,k)=\sum_{m=\max(0,-u)}^{\min(s-u,\tau\rho)}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\alpha_{u+m-r,s-r}\beta_{s-r,k-r}K_m(r,s-r,i)\,,$$

then, since $a_{s,u}(i,k)=0$ for $s<0$

$$\mu_k^{(i)}=\sum_{u=-\tau\rho}^{k}N^u\sum_{s=\max(0,u)}^{k}\left(\frac{n}{N}\right)^{is}\frac{e^{-sn/N}}{(i!)^s}a_{s,u}(i,k)+O(N^{k-\tau})+O(N^{k-\rho-1})\,. \tag{21}$$

If $n \neq iN$, we can apply Lemma 3. Here we choose τ and ρ larger than k so that the upper limit of summation on m is $s-u$ for $u>0$. Then, $a_{s,u}(i, k)=0$, $u>k/2$. Thus, the upper summation limit of u becomes $[k/2]$. Hence, for k even

$$\mu_k^{(i)}=N^{k/2}\sum_{s=k/2}^{k}\frac{e^{-sn/N}}{(i!)^s}\left(\frac{n}{N}\right)^{is}a_{s,k/2}(i, k)+R(k, N, n, i)+O(N^{k-\lambda}) \tag{22}$$

where $\lambda=\min(\tau, \rho+1)$ and

$$R(k, N, n, i)=\sum_{u=-\tau\rho}^{[k/2]-1}N^u\sum_{s=u}^{k}\frac{e^{-sn/N}}{(i!)^s}\left(\frac{n}{N}\right)^{is}a_{s,u}(i, k)\,.$$

For k odd, we have,

$$\mu_k^{(i)}=N^{(k-1)/2}\sum_{s=(k-1)/2}^{k}\frac{e^{-sn/N}}{(i!)^s}\left(\frac{n}{N}\right)^{is}a_{s,(k-1)/2}(i, k) +R(k, N, n, i)+O(N^{k-\lambda})\,. \tag{23}$$

From the proof of Lemma 3, since $a_{s,u}(i, k)$ is a sum of polynomials of degree $\leqq 2(k-u)$, $a_{s,u}(i, k)$ is itself a polynomial of degree $\leqq 2(k-u)$. Further for n and N both sufficiently large, $a_{s,u}(i, k)$ is uniformly bounded in u, $-\tau\rho \leqq u \leqq k$. Hence $R(k, N, n, i)=O(N^{[k/2]-1})$. Choose $\lambda>2k+1$. Then, for k even

$$\begin{aligned}\lim_{N,n\to\infty}a_{s,k/2}(i, k)&=k!\sum_{m=0}^{s-k/2}\frac{(-1)^{s-k/2-m}}{2^{s-k/2-m}(s-k/2-m)!\,2^{k-s}(k-s)!}\cdot\frac{(-1)^m}{m!}\left(\frac{\alpha}{2}-i+\frac{i^2}{2\alpha}\right)^m\\&=k!\sum_{m=0}^{s-k/2}\frac{(-1)^{s-k/2}(\alpha-2i+i^2/\alpha)^m}{2^{k/2}(s-k/2-m)!(k-s)!m!}\\&=\frac{k!(-1)^{s-k/2}}{2^{k/2}(k-s)!(s-k/2)!}\sum_{m=0}^{s-k/2}\binom{s-k/2}{m}(\alpha-2i+i^2/\alpha)^m\\&=D_{k/2}\binom{k/2}{s-k/2}(-1)^{s-k/2}\left(\frac{\alpha+(\alpha-i)^2}{\alpha}\right)^{s-k/2}.\end{aligned} \tag{24}$$

Thus for k even,

$$\begin{aligned}\lim_{n,N\to\infty}\frac{\mu_k^{(i)}}{N^{k/2}}&=D_{k/2}\sum_{s=k/2}^{k}\frac{e^{-\alpha s}}{(i!)^s}\alpha^{is}\binom{k/2}{s-k/2}(-1)^{s-k/2}\left(\frac{\alpha+(\alpha-i)^2}{\alpha}\right)^{s-k/2}\\&=D_{k/2}\left(\frac{e^{-\alpha}\alpha^i}{i!}\right)^{k/2}\left(1-\left(\frac{\alpha+(\alpha-i)^2}{\alpha}\right)\frac{e^{-\alpha}\alpha^i}{i!}\right)^{k/2}.\end{aligned} \tag{25}$$

For k odd, the conclusion for $N \neq in$ follows from (23) and $R(k, N, n, i) =O(N^{[k/2]-1})$.

For $N=in$, the conclusion follows from the continuity of $\mu_k^{(i)}/N^{k/2}$ in α.

To see this, observe that (21) is a finite sum and that for N sufficiently large, $n=\alpha N+o(N)$. Substitution of this into (21) and application of some elementary analysis permits one to verify the continuity of the limit (25) in α.

COROLLARY. *Under the hypotheses of Theorem* 1, $(s_i-E(s_i))/\sigma_{s_i}$ *has a limiting standard normal distribution.*

PROOF. This is immediate upon noting that $\lim_{n,N\to\infty} \mu_k^{(i)}/[\mu_2^{(i)}]^{k/2}$ are the moments of the standard normal distribution.

Remark. The methods of this section are a direct extension of those used by I. Weiss [9]. We have however extended the analysis to s_i, $i\neq 0$, whereas Weiss restricted his attention to s_0. The procedure used herein also gives a complete asymptotic expansion for $\mu_k^{(i)}$ and thus contains additional information on the limiting behavior beyond the statement of the corollary.

REFERENCES

[1] Chistyakov, V. P. (1964). On the calculation of the power of the test of empty boxes, *Theory of Probability and its Applications*, **9**, 648-653.

[2] Chistyakov, V. P. and Viktorova, I. I. (1965). Asymptotic normality in a problem of balls when the probabilities of falling into different boxes are different, *Theory of Probability and its Applications*, **10**, 149-154.

[3] Harris, B. and Park, C. J. The limiting distribution of linear combinations of the sample occupancy numbers from the multinomial distribution with equal cell probabilities, *MRC Technical Summary Report* (to appear).

[4] Jordan, C. (1950). *Calculus of Finite Differences*, Chelsea Publishing Company, New York.

[5] Kitabatake, S. (1958). A remark on a non-parametric test, *Math. Japan*, **4**, 45-49.

[6] Okamoto, M. (1952). On a non-parametric test, *Osaka Math. Jnl.*, **4**, 77-85.

[7] Rényi, A. (1962). Three new proofs and a generalization of a theorem of Irving Weiss, *Magyar Tud. Akad. Mat, Kutató Int. Közl. A*, **7**, 203-214.

[8] Sevast'yanov, B. A. and Chistyakov, V. P. (1964). Asymptotic normality in the classical ball problem, *Theory of Probability and its Applications*, **9**, 198-211.

[9] Weiss, I. (1958). Limiting distributions in some occupancy problems, *Ann. Math. Statist.*, **29**, 878-884.

[10] Wilks, S. S. (1962). *Mathematical Statistics*, John Wiley, New York.