DISTRIBUTION OF CERTAIN FACTORS USEFUL IN DISCRIMINANT ANALYSIS*

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1. Introduction and summary

Let A be a $(p\times p)$ symmetric positive definite matrix having the noncentral Wishart density.

$$(1.1)\qquad f(A)=\exp\left[-\mathrm{tr}\,(\Sigma^{-1}\Omega)\right]{}_0F_1\left(\frac{1}{2}q,\ \frac{1}{2}\Sigma^{-1}\Omega\Sigma^{-1}A\right)W(A\,|\,\Sigma\,|\,q)$$

where

$$(1.2)\qquad W(A\,|\,\Sigma\,|\,q)=\frac{|\,A\,|^{(q-p-1)/2}\exp\left\{-\frac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}A\right\}}{2^{qp/2}\Gamma_p\left[\frac{1}{2}q\right]|\,\Sigma\,|^{q/2}}$$

and

$$(1.3)\qquad \Gamma_p\left[\frac{1}{2}q\right]=\pi^{(p)(p-1)/4}\prod_{i=1}^{p}\Gamma\left[\frac{1}{2}(q+1-i)\right],$$

and ${}_0F_1(q/2,\ \Sigma^{-1}\Omega\Sigma^{-1}A/2)$ is a hypergeometric function of matrix arguments, see ([7], p. 733). Let B be another $(p\times p)$ symmetric positive definite matrix, having central Wishart density

$$(1.4)\qquad f(B)=W(B\,|\,\Sigma\,|\,n-q)\ .$$

Assuming the matrix Ω to be of rank $s<p$ we make the transformations

$$(1.5)\qquad A=C(I-L)C'\ ,\qquad B=CC'\ ,$$

where C is a lower triangular matrix of order p. The noncentral multivariate beta density of the $(p\times p)$ matrix L is found by Radcliffe ([7], p, 734) to be

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(1.6) $$g(L)=|L|^{(n-p-q-1)/2}|I-L|^{(q-p-1)/2}\theta(L_{11}) ,$$

where $\theta(L_{11})$, see Radcliffe ([7], p. 734), involves only the elements of $s\times s$ matrix L_{11} and other parameters but not any other elements of L. The noncentral multivariate beta density of L is a direct generalization of the noncentral linear beta density of rank one of L as given by Kshirsagar [5], who used the density of L, to derive the distribution of the test criterion for testing the adequacy of a single hypothetical discriminant function. Radcliffe generalizes Kshirsagar's results and gives the test criterion for testing the adequacy of s ($<p$) hypothetical discriminant functions. If $\Gamma'x$, where Γ' is an $s\times p$ matrix of rank s, denote the s discriminant functions, then $\Lambda=|L|$ may be factorized as

(1.7) $$\Lambda=\Lambda_1\Lambda_2|L_{11}|$$

where the direction and collinearity factors Λ_1 and Λ_2 are

(1.8) $$\Lambda_1=\frac{|\Gamma'AB^{-1}(B-A)\Gamma||\Gamma'B\Gamma|}{|\Gamma'(B-A)\Gamma|}$$
$$\Lambda_2=\frac{\Lambda}{|L_{11}|}\;\frac{|\Gamma'(B-A)\Gamma||\Gamma'A\Gamma|}{|\Gamma'B\Gamma||\Gamma'AB^{-1}(B-A)\Gamma|} .$$

It may be noted that the factorization of Λ, given here, is a generalization of the factorization given by Bartlett [2].

By choosing $\Gamma'=(I, 0)$ where I is an $s\times s$ identity matrix and factorizing the density of L in terms of rectangular coordinates T, $L=TT'$, T a lower triangular, Radcliffe [7] expresses the densities of Λ_1 and Λ_2 in terms of the elements of T. He also gives another factorization of Λ as, Radcliffe ([7], p. 732),

(1.9) $$\Lambda=\Lambda_5\Lambda_6|L_{11}| ,$$

where

$$\Lambda_5=\frac{|B-A||\Gamma'A\Gamma+\Gamma'A(B-A)^{-1}A\Gamma|}{|B||\Gamma'A\Gamma|}$$
$$\Lambda_6=\frac{|\Gamma'B\Gamma||\Gamma'A\Gamma|}{|\Gamma'(B-A)\Gamma||\Gamma'A\Gamma+\Gamma'A(B-A)^{-1}A\Gamma|}$$

Λ_5 and Λ_6 are also useful for testing direction and collinearity of the hypothetical discriminant functions $\Gamma'x$. Following Kshirsagar's [6] method, Radcliffe expresses Λ_5 and Λ_6 as functions of the elements of T and obtains their distributions. We are giving here a shorter and neater proof, which might be of pedagogical interest. Also our main interest is to express Λ_1, Λ_2, Λ_5 and Λ_6 as functions of the elements of L, rather than functions of elements of T. All distributions are derived without the constant factor, K is used as a generic symbol for the constant factors of the density functions.

2. Distribution of Λ_1 and Λ_2

By partitioning L and $I-L$ as

$$L=\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \qquad I-L=\begin{pmatrix} I-L_{11} & -L_{12} \\ -L_{21} & I-L_{22} \end{pmatrix} \tag{2.1}$$

and using (1.6) we write the joint density of L_{11}, L_{21}, L_{22}, as,

$$\begin{aligned} f(L_{11}, L_{21}, L_{22}) = K\,|L_{11}|^{(n-q-p-1)/2}\,|I-L_{11}|^{(q-p-1)/2}\theta(L_{11}) \\ \cdot\,|L_{22}-L_{21}L_{11}^{-1}L_{12}|^{(n-q-p-1)/2} \\ \cdot\,|I-L_{22}-L_{21}(I-L_{11})^{-1}L_{12}|^{(q-p-1)/2}\,. \end{aligned} \tag{2.2}$$

On setting

$$\begin{cases} Z=L_{22}-L_{21}L_{11}^{-1}L_{12} \\ R=(I-L_{21}(L_{11}(I-L_{11}))^{-1}L_{12})^{-1/2}Z(I-L_{21}(L_{11}(I-L_{11}))^{-1}L_{12})^{-1/2}\,. \end{cases} \tag{2.3}$$

We find that the joint density of L_{11}, L_{21} and R is given by

$$\begin{aligned} f(L_{11}, L_{21}, R) = K\,|L_{11}|^{(n-q-p-1)/2}\,|I-L_{11}|^{(q-p-1)/2}\theta(L_{11}) \\ \cdot\,|I-L_{21}(L_{11}(I-L_{11}))^{-1}L_{12}|^{(n-p-s-1)/2} \\ \cdot\,|R|^{(n-q-p-1)/2}\,|I-R|^{(q-p-1)/2}\,. \end{aligned} \tag{2.4}$$

Again we set $\Delta=L_{21}(L_{11}(I-L_{11}))^{-1}L_{12}$ and assuming $(p-s)\leqq s$ we use Hsu's lemma, (Anderson [1], p. 319, Lemma 13.3.1) to integrate (2.4) with respect to the elements of L_{21} and find the joint density of L_{11}, Δ and R to be

$$\begin{aligned} f(L_{11}, \Delta, R) = K\,|L_{11}|^{(n-q-s-1)/2}\,|I-L_{11}|^{(q-s-1)/2}\theta(L_{11}) \\ \cdot\,|I-\Delta|^{(n-p-s-1)/2}\,|\Delta|^{(2s-p-1)/2} \\ \cdot\,|R|^{(n-q-p-1)/2}\,|I-R|^{(q-p-1)/2}\,. \end{aligned} \tag{2.5}$$

By setting $\Gamma'=(I, 0)$, it may be easily seen that

$$\Lambda_1=|I-\Delta| \qquad \text{and} \qquad \Lambda_2=|R|\,. \tag{2.6}$$

It follows from (2.5) that the densities of Λ_1 and Λ_2 are mutually independent. The densities of Λ_1 and Λ_2 are identical with those of a product of independent beta variates. This result agrees with the one given by Radcliffe ([7], p. 738), except the fact that we assume $p\leqq 2s$ and Radcliffe assumes $p\geqq 2s$.

3. Distribution of Λ_5 and Λ_6

Noting that,

$$(3.1) \qquad \Lambda_6 = \frac{|z|}{|z + L_{21}(L_{11}(I - L_{11}))^{-1}L_{12}|}$$

$\Lambda_5 = |z + L_{21}(L_{11}(I - L_{11}))^{-1}L_{12}|$, we set $z = PP'$, where P is a nonsingular matrix of order $(p-s)\times(p-s)$. The joint density of L_{11}, P and L_{12} may be obtained by using the result (2.3), and we find that

$$(3.2) \qquad f(L_{11}, P, L_{12}) = K|L_{11}|^{(n-q-p-1)/2}|I - L_{11}|^{(q-p-1)/2}\theta(L_{11}) \cdot |I - PP' - L_{21}(L_{11}(I - L_{11}))^{-1}L_{12}|^{(q-p-1)/2} \cdot |PP'|^{(n-q-p)/2} .$$

Further transforming L_{21} to η, where η is an $(p-s)\times s$, by the relation

$$(3.3) \qquad L_{21} = P\eta$$

the joint density of L_{11}, P and η is found to be

$$(3.4) \qquad f(L_{11}, P, \eta) = K|L_{11}|^{(n-q-p-1)/2}|I - L_{11}|^{(q-p-1)/2}\theta(L_{11}) \cdot |P(I + \eta(L_{11}(I - L_{11}))^{-1}\eta')P'|^{(n-q-p+s)/2} \cdot |I + \eta(L_{11}(I - L_{11}))^{-1}\eta'|^{-(n-q-p+s)/2} \cdot |I - P(I + \eta(L_{11}(I - L_{11}))^{-1}\eta')P'|^{(q-p-1)/2} .$$

Now we set

$$(3.5) \qquad P(I + \eta(L_{11}(I - L_{11}))^{-1}\eta')P' = W$$

and using Hsu's lemma (Anderson [1], Lemma 13.3.1) we find the joint density of W, η and L_{11} to be

$$(3.6) \qquad f(L_{11}, W, \eta) = K|L_{11}|^{(n-q-p-1)/2}|I - L_{11}|^{(q-p-1)/2}\theta(L_{11}) \cdot |I + \eta(L_{11}(I - L_{11}))^{-1}\eta'|^{-(n-q)/2} \cdot |W|^{(n-q-p+s-1)/2}|I - W|^{(q-p-1)/2} .$$

Further setting

$$(3.7) \qquad \eta(L_{11}(I - L_{11}))^{-1}\eta' = G$$

and using (3.6) and Hsu's lemma we get

$$(3.8) \qquad f(L_{11}, G, W) = K|L_{11}|^{(n-q-s-1)/2}|I - L_{11}|^{(q-s-1)/2}\theta(L_{11}) \cdot |I + G|^{-(n-q)/2}|G|^{(2s-p-1)/2} \cdot |W|^{(n-q-p+s-1)/2}|I - W|^{(q-p-1)/2}$$

Again transforming G to H by the transformation

$$H=(I+G)^{-1} \tag{3.9}$$

and noting that the Jacobian of the transformation from H to G is $|I+G|^{-(p-s+1)}$ we obtain the joint density of L_{11}, H and W to be

$$\begin{aligned} f(L_{11}, H, W)=K|L_{11}|^{(n-q-p-1)/2}|I-L_{11}|^{(q-s-1)/2}\theta(L_{11}) \\ \cdot |H|^{(n-p-s-1)/2}|I-H|^{(2s-p-1)/2} \\ \cdot |W|^{(n-q-p+s-1)/2}|I-W|^{(q-p-1)/2} . \end{aligned} \tag{3.10}$$

Here we note that $\Lambda_6=|H|$ and $\Lambda_5=|W|$. It, thus, follows that the densities of Λ_6 and Λ_5 are independent. This result agrees with the one given by Radcliffe ([7], p. 739), except that we assume $p\leqq 2s$ while Radcliffe assumes $p\geqq 2s$.

4. Distribution of Λ_3 and Λ_4

We have noted above that the Λ_1 is distributed as a product of $(p-s)$ independent beta variables and as such we must be able to factorize Λ_1 into $(p-s)$ mutually independent beta variables. Consider the factorization of Λ_1 into two parts

$$\Lambda_1=\Lambda_3\Lambda_4 , \tag{4.1}$$

where $\Lambda_3=\Delta_{11}$, Δ_{11} being the first element of the matrix $\Delta=L_{21}(L_{11}(I-L_{11}))^{-1}L_{12}$. From (2.5) we find the density of the $(p-s)\times(p-s)$ matrix Δ to be

$$f(\Delta)=K|I-\Delta|^{(n-q-s)/2}|\Delta|^{(2s-p-1)/2} . \tag{4.2}$$

Partitioning Δ and $I-\Delta$ as

$$\Delta=\begin{pmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{pmatrix} \qquad I-\Delta=\begin{pmatrix}1-\Delta_{11} & -\Delta_{12}\\ -\Delta_{21} & I-\Delta_{22}\end{pmatrix} \tag{4.3}$$

where Δ_{11} is 1×1, Δ_{12} is $1\times(p-s)$, Δ_{22} is $(p-s-1)\times(p-s-1)$, the joint density of Δ_{11}, Δ_{12} and Δ_{22} can be written as

$$\begin{aligned} f(\Delta_{11}, \Delta_{12}, \Delta_{22})=K\Delta_{11}^{(2s-p-1)/2}\left|\Delta_{22}-\frac{\Delta_{21}\Delta_{12}}{\Delta_{11}}\right|^{(2s-p-1)/2} \\ \cdot (1-\Delta_{11})^{(n-q-s)/2}\left|I-\Delta_{22}-\frac{\Delta_{21}\Delta_{12}}{1-\Delta_{11}}\right|^{(n-q-s)/2} , \end{aligned} \tag{4.4}$$

Now we set

$$M=\Delta_{22}-\frac{\Delta_{21}\Delta_{12}}{\Delta_{11}} \tag{4.5}$$

and find the joint density of Δ_{11}, M and Δ_{21} to be

$$(4.6)\qquad f(\Delta_{11}, \Delta_{21}, M)=K\Delta_{11}^{(2s-p-1)/2}|M|^{(2s-p-1)/2} \cdot (1-\Delta_{11})^{(n-q-s)/2}\left|I-M-\frac{\Delta_{21}\Delta_{12}}{\Delta_{11}(1-\Delta_{11})}\right|^{(n-q-s)/2}$$

substitute

$$\Delta_{21}=\Delta_{11}(1-\Delta_{11})^{1/2}(I-M)^{1/2}\delta ,$$

we obtain the joint density of Δ_{11}, M and δ as

$$(4.7)\qquad f(\Delta_{11}, M, \delta)=K\Delta_{11}^{(2s-p-1)/2}\Delta_{11}^{(p-s-1)/2} \cdot (1-\Delta_{11})^{(n-q-s)/2}(1-\Delta_{11})^{(p-s-1)/2} \cdot |M|^{(2s-p-1)/2}|I-M|^{(n-q-s+1)/2} \cdot |I-\delta\delta'|^{(n-q-s)/2}$$

from (4.7) we see that the densities of $\Lambda_3=\Delta_{11}$ and $\Lambda_4=|M|$ are independent. We also note that $\Delta_{11}|M|=\Lambda_3\Lambda_4=\Lambda_1$. Radcliffe derives the distribution of Λ_3 and Λ_4 for the particular case $s=2$. We also proceed to obtain the results for $s=2$. In this case we proceed as follows. From equation (2.4) the density of L_{11} and L_{12}, for $s=2$, is

$$(4.8)\qquad f(L_{11}, L_{12})=\theta(L_{11})|L_{11}|^{(n-q-p-1)/2}|I-L_{11}|^{(q-p-1)/2} \cdot \left\{\frac{|L_{11}(I-L_{11})-L_{12}L_{21}|}{|L_{11}||I-L_{11}|}\right\}^{(n-p-3)/2} .$$

Let $L_{12}L_{21}=V$, using Hsu's lemma, the joint density of L_{11} and V is

$$(4.9)\qquad f(L_{11}, V)=K|L_{11}|^{(n-p-q-1)/2}|I-L_{11}|^{(q-p-1)/2}\theta(L_{11}) \cdot \left\{\frac{|L_{11}(I-L_{11})-V|}{|L_{11}||I-L_{11}|}\right\}^{(n-p-3)/2}|V|^{(p-5)/2} .$$

Further setting

$$(4.10)\qquad \begin{cases} L_{11}(I-L_{11})-V=R \\ L_{11}(I-L_{11})=UU' \end{cases}$$

where U is a lower triangular matrix and $R=UFU'$ we find that the density of the matrix F is independent of L and is given by

$$(4.11)\qquad f(F)=K|F|^{(n-p-3)/2}|I-F|^{(p-5)/2} .$$

We further note that $\Lambda_1=|F|$, $\Lambda_3=f_{11}$ where f_{11} is the first element of F. Proceeding on similar lines as in (4.3) and (4.4) and setting $x_{22}=f_{22}-f_{12}^2/f_{12}$ and $f_{12}=(1-x_{22})^{1/2}(1-f_{11})^{1/2}f_{11}^{1/2}x_{12}$ the joint density of f_{12}, x_{22} and x_{12} can be expressed as

$$f(f_{11}, x_{22}, x_{12}) = Kf_{11}^{(n-p-1)/2}(1-f_{11})^{(p-4)/2}x_{22}^{(n-p-3)/2} \cdot (1-x_{22})^{(p-4)/2}(1-x_{12})^{(p-5)/2} .$$

It follows from (4.12) that beta densities of $f_{11}=\Lambda_3$, $x_{22}=\Lambda_4$ are independent. This result agrees with the one given by Radcliffe ([7], p. 740).

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