RESPONSE RELIABILITY AND ATTITUDE CHANGE —SUPPLEMENT TO RESPONSE ERRORS AND BIASED INFORMATION*—

CHIKIO HAYASHI

(Received Oct. 21, 1970)

In the previous paper*, response error is treated in the form of response reliability represented by a probabilistic model in the case where no attitude change exists at two time points. Here, the author considers the problem of estimation of probabilities of attitude change represented by a simple Markov process**, which is very popular, in the case where response error exists which is represented by a probabilistic model of reliability found in the previous paper. For simplicity, let the number of response categories be three, $+$, $\pm$, $-$ (see pp. 213–214 in the previous paper*). Let a reliability matrix be P. P is represented by the matrix the elements of which are p_{ij}, $i, j=+, \pm, -$. The number of response pairs $++$, $+\pm, \cdots$, etc. obtained by test-retest are m_{ij}, $i, j=+, \pm, -$. n_i is the number of true i responses, $i=+, \pm, -$ where $n=n_+ + n_\pm + n_-$, n being the size of sample. The equations which hold in the mean (expectation) are formally

$$(P\times P)'\mathfrak{N}=\mathfrak{M}$$

where $P\times P$ means Kronecker's product and $(\ )'$ means transposed matrix, $\mathfrak{N}$ being a column vector the elements of which are n_+, 0, 0, 0, $n_\pm$, 0, 0, 0, n_- and $\mathfrak{M}$ being a column vector the elements of which are m_{+j}, $j=+, \pm, -$; $m_{\pm j}$, $j=+, \pm, -$; m_{-j}, $j=+, \pm, -$, $\sum_{i,j} m_{ij}$ being equal to n. This expression was heuristically expressed in the previous paper. Here, we assume that P, representing the response reliability, is known and also constant at the two time points.

Let U be the transition matrix of attitude change the elements of which are u_{ij}, $i, j=+, \pm, -$, u_{ij} being probability under the condition of $\sum_j u_{ij}=1$ where i corresponds to the response category at the initial time point and j corresponds to that at the coming time point. The

* *Ann. Inst. Statist. Math.*, Vol. 20, 1968, 211-228.

** J. S. Coleman: Model of Change and Response Uncertainty, Prentice-Hall, 1964.

relation holds in the mean (expectation), neglecting the attitude change

$$(U)'\begin{pmatrix} n_+ \\ n_\pm \\ n_- \end{pmatrix}=\begin{pmatrix} l_+ \\ l_\pm \\ l_- \end{pmatrix},$$

where $\begin{pmatrix} n_+ \\ n_\pm \\ n_- \end{pmatrix}$ is a frequency vector to categories in the initial time point and $\begin{pmatrix} l_+ \\ l_\pm \\ l_- \end{pmatrix}$ is a frequency vector in the coming time point. Thus we have the following equation in the mean (expectation) in the case where the attitude change mentioned above exists, and the transition events and response events are assumed to be independent,

(1) $$(P\times P)'\tilde{\mathfrak{N}}=\mathfrak{M}$$

where $\tilde{\mathfrak{N}}$ is a column vector, the elements of which are n_+u_{++}, $n_+u_{+\pm}$, n_+u_{+-}, $n_\pm u_{\pm+}$, $n_\pm u_{\pm\pm}$, $n_\pm u_{\pm-}$, n_-u_{-+}, $n_-u_{-\pm}$, n_-u_{--}, and $(P\times P)$ is assumed to be non-singular (this is equivalent to assume that P is non-singular) and $\mathfrak{M}$ is a column vector the elements of which are m_{+j}, $j=+, \pm, -$; $m_{\pm j}$, $j=+, \pm, -$; m_{-j}, $j=+, \pm, -$, that represent the data observed at two time points. $\tilde{\mathfrak{N}}$ represents the result of attitude change. Practically, the estimator $\hat{\tilde{\mathfrak{N}}}$ of $\tilde{\mathfrak{N}}$ is given by the following formula,

(2) $$\hat{\tilde{\mathfrak{N}}}=(P\times P)'^{-1}\mathfrak{M}\,.$$

$\hat{\tilde{\mathfrak{N}}}$ is obviously unbiased. Thus, we can obtain the estimators $\hat{n}_+$, $\hat{n}_\pm$, $\hat{n}_-$, $\hat{u}_{ij}$, $i, j=+, \pm, -$ and $\sum_j \hat{u}_{ij}=1$ holds for $i=+, \pm, -$ respectively. For example, we have

$$\hat{n}_+\hat{u}_{++}=\hat{n}_{++}$$
$$\hat{n}_+\hat{u}_{+\pm}=\hat{n}_{+\pm}$$
$$\hat{n}_+\hat{u}_{+-}=\hat{n}_{+-}$$
$$\hat{u}_{++}+\hat{u}_{+\pm}+\hat{u}_{+-}=1\,,$$

where $\hat{n}_{+j}$ $(j=+, \pm, -)$ are obtained by the equations mentioned above which contain P_{ij} and m_{ij}, for $i, j=+, \pm, -$.

From these, we have

$$\hat{n}_+=\hat{n}_{++}+\hat{n}_{+\pm}+\hat{n}_{+-}\,.$$

If we put

$$\hat{A}^{+\pm}_{++}=\hat{n}_{+\pm}/\hat{n}_{++} \qquad \text{and} \qquad \hat{A}^{+-}_{++}=\hat{n}_{+-}/\hat{n}_{++} ,$$

we have

$$\hat{u}_{++}=1/(1+\hat{A}^{+\pm}_{++}+\hat{A}^{+-}_{++})$$

from

$$\hat{u}_{++}+\hat{u}_{+\pm}+\hat{u}_{+-}=1$$

and so on. That is,

$$\hat{u}_{+\pm}=1/(1+\hat{A}^{++}_{+\pm}+\hat{A}^{+-}_{+\pm})$$

$$\hat{u}_{+-}=1/(1+\hat{A}^{++}_{+-}+\hat{A}^{+\pm}_{+-})$$

where

$$\hat{A}^{++}_{+\pm}=\hat{n}_{++}/\hat{n}_{+\pm} , \qquad \hat{A}^{+-}_{+\pm}=\hat{n}_{+-}/\hat{n}_{+\pm} ,$$

$$\hat{A}^{++}_{+-}=\hat{n}_{++}/\hat{n}_{+-} \qquad \text{and} \qquad \hat{A}^{+\pm}_{+-}=\hat{n}_{+\pm}/\hat{n}_{+-} .$$

Thus, similarly we have

$$\hat{u}_{\pm+}=1/(1+\hat{B}^{\pm\pm}_{\pm+}+\hat{B}^{\pm-}_{\pm+}) ,$$

$$\hat{u}_{\pm\pm}=1/(1+\hat{B}^{\pm+}_{\pm\pm}+\hat{B}^{\pm-}_{\pm\pm}) ,$$

$$\hat{u}_{\pm-}=1/(1+\hat{B}^{\pm+}_{\pm-}+\hat{B}^{\pm\pm}_{\pm-}) ,$$

$$\hat{u}_{-+}=1/(1+\hat{C}^{-\pm}_{-+}+\hat{C}^{--}_{-+}) ,$$

$$\hat{u}_{-\pm}=1/(1+\hat{C}^{-+}_{-\pm}+\hat{C}^{--}_{-\pm}) ,$$

$$\hat{u}_{--}=1/(1+\hat{C}^{-+}_{--}+\hat{C}^{-\pm}_{--}) ,$$

$$\hat{n}_{\pm}=\hat{n}_{\pm+}+\hat{n}_{\pm\pm}+\hat{n}_{\pm-} ,$$

$$\hat{n}_{-}=\hat{n}_{-+}+\hat{n}_{-\pm}+\hat{n}_{--} ,$$

where

$$\hat{B}^{kl}_{\pm j}=\hat{n}_{kl}/\hat{n}_{\pm j} ,$$

$$\hat{C}^{kl}_{-j}=\hat{n}_{kl}/\hat{n}_{-j} , \qquad j, k, l=+, \pm, - .$$

Thus we can estimate U and $\mathfrak{N}$ by the data and known parameters. It is practicable for large sample size to calculate their mean square errors and mean cross-product errors even approximately. This point of view is valid in data analysis and the rigorous calculation is too complicated without any assumption and it is too sophisticated. To obtain the matrix of mean square errors and mean cross-product errors we define the error column vector ∇ the elements of which are Δn_{+}, $\Delta n_{\pm}$, Δn_{-}, Δu_{++}, $\Delta u_{+\pm}$, Δu_{+-}, $\Delta u_{\pm+}$, $\Delta u_{\pm\pm}$, $\Delta u_{\pm-}$, Δu_{-+}, $\Delta u_{-\pm}$, Δu_{--}. We put

$$n_i = {}^0n_i(1+\Delta n_i)\ , \qquad i=+,\ \pm,\ -\ ,$$

$$u_{ij} = {}^0u_{ij}(1+\Delta u_{ij})\ , \qquad i,\ j=+,\ \pm,\ -$$

where ${}^0n_i$ and ${}^0u_{ij}$ mean true values, for $i, j=+, \pm, -$. Also we put $m_{ij}={}^0m_{ij}(1+\Delta m_{ij})$ where ${}^0m_{ij}$ means true value for all i, j: $i, j=+, \pm, -$ and we have $\sum_{i,j} m_{ij}=\sum_{i,j} {}^0m_{ij}=n$. From (1), we have approximately (neglecting the order of Δ^2 over)

$$(P\times P)'\Delta\tilde{\mathfrak{N}}=\Delta\mathfrak{M}$$

where $\Delta\tilde{\mathfrak{N}}$ is a column vector, the elements of which are ${}^0n_i{}^0u_{ij}(\Delta n_i+\Delta u_{ij})$; $i=+$, $j=+, \pm, -$; $i=\pm$, $j=+, \pm, -$; $i=-$, $j=+, \pm, -$, and $\Delta\mathfrak{M}$ is a column vector, the elements of which are Δm_{+j}, $j=+, \pm, -$, $\Delta m_{\pm j}$, $j=+, \pm, -$, Δm_{-j}, $j=+, \pm, -$. Since $(P\times P)'$ is not singular, we obtain

$$\Delta\tilde{\mathfrak{N}}=(P\times P)'^{-1}\Delta\mathfrak{M}\ .$$

Then we define the matrix A as below

$$\begin{pmatrix}
0 & 0 & 0 & {}^0u_{++} & {}^0u_{+\pm} & {}^0u_{+-} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & {}^0u_{\pm+} & {}^0u_{\pm\pm} & {}^0u_{\pm-} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {}^0u_{-+} & {}^0u_{-\pm} & {}^0u_{--} \\
{}^0n_+{}^0u_{++} & 0 & 0 & {}^0n_+{}^0u_{++} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^0n_+{}^0u_{+\pm} & 0 & 0 & 0 & {}^0n_+{}^0u_{+\pm} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{}^0n_+{}^0u_{+-} & 0 & 0 & 0 & 0 & {}^0n_+{}^0u_{+-} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & {}^0n_\pm{}^0u_{\pm+} & 0 & 0 & 0 & 0 & {}^0n_\pm{}^0u_{\pm+} & 0 & 0 & 0 & 0 & 0 \\
0 & {}^0n_\pm{}^0u_{\pm\pm} & 0 & 0 & 0 & 0 & 0 & {}^0n_\pm{}^0u_{\pm\pm} & 0 & 0 & 0 & 0 \\
0 & {}^0n_\pm{}^0u_{\pm-} & 0 & 0 & 0 & 0 & 0 & 0 & {}^0n_\pm{}^0u_{\pm-} & 0 & 0 & 0 \\
0 & 0 & {}^0n_-{}^0u_{-+} & 0 & 0 & 0 & 0 & 0 & 0 & {}^0n_-{}^0u_{-+} & 0 & 0 \\
0 & 0 & {}^0n_-{}^0u_{-\pm} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {}^0n_-{}^0u_{-\pm} & 0 \\
0 & 0 & {}^0n_-{}^0u_{--} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {}^0n_-{}^0u_{--}
\end{pmatrix}$$

Then we have also approximately (neglecting the order of Δ^2 over)

$$AV=\mathfrak{S}\ ,$$

where $\mathfrak{S}$ is a column vector the elements of which are 0, 0, 0 and the elements of $(P\times P)'\Delta\mathfrak{M}$. If A is assumed to be non-singular, we have $V=A^{-1}\mathfrak{S}$. Thus

$$\begin{aligned} E(VV') &= E(A^{-1}\mathfrak{S})(A^{-1}\mathfrak{S})' \\ &= A^{-1}E(\mathfrak{S}\mathfrak{S}')A^{-1\prime}\ , \end{aligned}$$

where $E(\mathfrak{S}\mathfrak{S}')$ is easily theoretically calculated and estimated by data. This is possible, complicated as it is, even when $\mathfrak{S}$ contains the sampling fluctuation of matrix $(P\times P)$.

THE INSTITUTE OF STATISTICAL MATHEMATICS