SOME KNOWN RESULTS CONCERNING ZERO-ONE SETS*

MORRIS SKIBINSKY

(Received Oct. 5, 1968)

1. Introduction and summary

Let $(\chi, \mathcal{F})$ be a measurable space,

$$\{Q_\theta : \theta \in \Lambda\},$$

a family of probability measures on $\mathcal{F}$ which are indexed by the points of a set Λ. Denote by $\mathcal{G}$, the smallest σ-field of Λ subsets relative to which all of the likelihood functions, $Q_{\cdot}(F)$ with F in $\mathcal{F}$ are measurable. Suppose g to be a real valued function on Λ measurable with respect to a σ-field $\tilde{\mathcal{G}}$ of Λ subsets that *contains* $\mathcal{G}$. In [2] necessary and sufficient conditions are given for the existence of an $\mathcal{F}$-measurable function f on χ with the property that

$$(1) \qquad Q_\theta(f = g(\theta)) = 1 ,$$

respectively

(i) for all θ in Λ,

(ii) for almost all θ in Λ relative to a probability measure m on $\tilde{\mathcal{G}}$.

These results are then applied to sequences of independent and identically distributed random variables whose common distribution belongs to a specified family.

In Section 2, we review the basic results of [2], with some changes in emphasis and notation. Our Lemma 1 is implicit in [2] (see the proof of Theorem 3), though not explicitly stated. In Section 3, the necessary and sufficient conditions above referred to are shown to be simple consequences of a lemma which appears in a 1954 paper by Bahadur.

Definitions and notation relating to σ-fields null sets and conditional expectation are standard.

---

* Research carried out at Brookhaven National Laboratory under contract with the U.S. Atomic Energy Commission.

2. The results of Breiman, Le Cam, and Schwarz

For $F \in \mathcal{F}$, $G \subset \Lambda$, write

$$F \sim G \text{ if } Q_{.}(F) = I_G\,; \qquad F \overset{m}{\sim} G \text{ if } Q_{.}(F) = I_G[m]\,.$$

I_G denotes the indicator function of G. Let $\mathcal{K}$ ($\mathcal{K}_m$) denote the collection of all pairs (F, G) with $F \in \mathcal{F}$, $G \subset \Lambda$ such that $F \sim G$ ($F \overset{m}{\sim} G$). Regard these collections as subsets of the product of $\mathcal{F}$ with the class of all Λ subsets, and define $\mathcal{F}^*$, $\mathcal{G}^*$ ($\mathcal{F}^*_m$, $\mathcal{G}^*_m$) to be the respective projections of $\mathcal{K}$ ($\mathcal{K}_m$) on these factor spaces. As noted in [2] each of these projections is a σ-field. We now restate, slightly modified and combined, the Theorems 1 and 2 of [2].

THEOREM 1. *Let $\sigma(g)$ denote the σ-field of Λ-subsets generated by the function g on Λ. There exists an $\mathcal{F}$-measurable function f on χ such that* (1) *holds for all (almost all* (m)*) θ in Λ, if and only if $\sigma(g) \subset \mathcal{G}^*$ ($\sigma(g) \subset \mathcal{G}^*_m$). In this case $\sigma(f) \subset \mathcal{F}^*$ ($\sigma(f) \subset \mathcal{F}^*_m$).*

The above result is applicable to sequences of independent, identically distributed random variables as follows. Let $\{\hat{Q}_\theta : \theta \in \Lambda\}$ be a family of probability measures on a measurable space $(\hat{\chi}, \hat{\mathcal{F}})$. Let $\hat{\mathcal{G}}$ be the σ-field of Λ subsets generated by the $\hat{Q}_{.}(F)$, $F \in \hat{\mathcal{F}}$. Take $\chi = \{x = (x_1, x_2, \cdots) : x_i \in \hat{\chi},\ i = 1, 2, \cdots\}$; $X_1, X_2, \cdots$, the coordinate functions on χ; and $\mathcal{F}$, the σ-field of χ-subsets which they generate. For each θ in Λ, let Q_θ denote the unique probability measure on $\mathcal{F}$ relative to which the coordinate functions are independent random variables that satisfy

$$Q_\theta(X_j \in \hat{F}) = \hat{Q}_\theta(\hat{F})\,, \qquad F \in \hat{\mathcal{F}},\ j = 1, 2, \cdots\,.$$

With these definitions of χ, $\mathcal{F}$, Q_θ, and Λ, let $\mathcal{G}$, $\mathcal{G}^*$, $\mathcal{G}^*_m$, g be as already defined. Theorem 3 of [2] may now be put in the form of the following lemma and corollary.

LEMMA 1. $\mathcal{G} = \hat{\mathcal{G}} = \mathcal{G}^*$.

COROLLARY 1. *There exists an $\mathcal{F}$-measurable function f on χ such that* (1) *holds for all (almost all* (m)*) θ in Λ if and only if $\sigma(g) \subset \hat{\mathcal{G}}$ ($\sigma(g) \subset \hat{\mathcal{G}}[m]$).*

"$\sigma(g) \subset \hat{\mathcal{G}}[m]$" means that to each set G of $\sigma(g)$ there corresponds a set $\hat{G}$ in $\hat{\mathcal{G}}$ such that $m(G \Delta \hat{G}) = 0$. Δ denotes divided difference.

3. Theorem one as a consequence of Bahadur's lemma

In the following we take as given a measurable space $(\Omega, \mathcal{A})$ and a family $\mathcal{P}$ of probability measures P on $\mathcal{A}$. $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ with or without affixes denote sub-σ-fields of $\mathcal{A}$. Against this background, the statements of Theorem 1 are seen to be special cases of a more general proposition (Corollary 2) which in turn is itself an immediate consequence of a result due to Bahadur (Lemma 2).

Let

$$\mathcal{B}^*(\mathcal{C}) = \{B \in \mathcal{B} : \exists\, C \in \mathcal{C} \text{ such that } P(B \Delta C) = 0, \ \forall\, P \in \mathcal{P}\}$$

$$\mathcal{C}^*(\mathcal{B}) = \{C \in \mathcal{C} : \exists\, B \in \mathcal{B} \text{ such that } P(B \Delta C) = 0, \ \forall\, P \in \mathcal{P}\}.$$

These collections are easily seen to be sub-σ-fields of $\mathcal{B}$ and $\mathcal{C}$, respectively. In fact

$$\mathcal{B}^*(\mathcal{C}) = \mathcal{B} \cap (\mathcal{C} \vee \mathcal{N}), \qquad \mathcal{C}^*(\mathcal{B}) = (\mathcal{B} \vee \mathcal{N}) \cap \mathcal{C}$$

where $\mathcal{N}$ denotes the sub-σ-field of $\mathcal{A}$ generated by the class of $[\mathcal{A}, \mathcal{P}]$ null sets and $\vee$ denotes the smallest σ-field of subsets containing all of the sets in the collections which precede and follow it.

Let $\mathcal{C}_0 \subset \mathcal{C}$, then clearly

$$\mathcal{C}_0 \subset \mathcal{C}^*(\mathcal{B}) \Longleftrightarrow \mathcal{C}_0 \subset \mathcal{B}[\mathcal{P}]. \tag{2}$$

The right-hand side of (2) means that to each set of C of $\mathcal{C}_0$ there corresponds a set $B \in \mathcal{B}$ such that $P(B \Delta C) = 0$ for all $P \in \mathcal{P}$.

LEMMA 2 (Bahadur [1], Lemma 7.1, p. 442). *Let c and d be extended real valued constants such that $-\infty \leqq c < d \leqq \infty$; $\mathcal{D}_1$, $\mathcal{D}_2$, arbitrary sub-σ-fields of $\mathcal{A}$. Then*

$$\mathcal{D}_1 \subset \mathcal{D}_2[\mathcal{P}]$$

if and only if to each $\mathcal{D}_1$-measurable function u on Ω such that $c \leqq u \leqq d$, there corresponds a $\mathcal{D}_2$-measurable function v such that $c \leqq v \leqq d$ and such that

$$u = v[\mathcal{P}].$$

COROLLARY 2. *Let ζ be a $\mathcal{C}$-measurable function on Ω. There exists a $\mathcal{B}$-measurable function ξ on Ω such that*

$$P(\xi = \zeta) = 1, \qquad \forall\, P \in \mathcal{P}$$

if and only if

$$\sigma(\zeta) \subset \mathcal{C}^*(\mathcal{B}).$$

ξ then satisfies

$$\sigma(\xi)\subset\mathcal{B}^*(\sigma(\zeta))\,.$$

We now show that the statements of Theorem 1 may be viewed as simple consequences of Corollary 2. For the first statement let $\Omega=\chi\times\Lambda$, $\mathcal{A}=\mathcal{F}\times\bar{\mathcal{G}}$, where $\bar{\mathcal{G}}$ is the smallest σ-field of Λ-subsets containing $\tilde{\mathcal{G}}$ and the singleton subsets of Λ. Let $\mathcal{B}=\mathcal{F}\times\{\phi,\Lambda\}$, $C=\{\phi,\chi\}\times\tilde{\mathcal{G}}$ and define X and Θ on Ω by $X(x,\theta)=x$, $\Theta(x,\theta)=\theta$. Define the probability measure P_θ on $\mathcal{A}$ for each θ in Λ by

$$P_\theta(\Theta=\theta)=1\,,\quad P_\theta(X\in F)=Q_\theta(F)\,,\qquad \forall\, F\in\mathcal{F}$$

and take

$$\mathcal{P}=\{P_\theta\colon\theta\in\Lambda\}\,.$$

We need only note that there can exist an $\mathcal{F}$-measurable function f on χ such that (1) holds for all θ in Λ if and only if

$$P(fX=g\Theta)=1\,,\qquad \forall\, P\in\mathcal{P}\,.$$

By Corollary 2, this is true if and only if $\sigma(g\Theta)\subset C^*(\mathcal{B})$. But in the present case

$$\sigma(g\Theta)=\chi\times\sigma(g)\quad\text{and}\quad C^*(\mathcal{B})=\chi\times\mathcal{G}^*$$

so that the statement follows. For the second statement of Theorem 1 take $\mathcal{A}=\mathcal{F}\times\tilde{\mathcal{G}}$; Ω, $\mathcal{B}$, C, X, and Θ, as above. Let $\mathcal{P}$ consist of a single probability measure P on $\mathcal{A}$ where P is uniquely defined on $\mathcal{A}$ (e.g. see [3], p. 137) by the two properties: (i) $P(\Theta\in G)=m(G)$, $\forall\, G\in\tilde{\mathcal{G}}$, (ii) $Q_\theta(F)$ (for each F in $\mathcal{F}$) is a version of $E_P(I_{X\in F}\mid C)$. Thus, there can exist an $\mathcal{F}$-measurable function f on χ such that (1) holds for almost all θ in Λ relative to m if and only if

$$E_P(I_{fX=g(\theta)}\mid C)=1[C,P]\,,$$

i.e. if and only if

$$P(fX=g\Theta)=1\,.$$

Here also we have that

$$\sigma(g\Theta)=\chi\times\sigma(g)\quad\text{and}\quad C^*(\mathcal{B})=\chi\times\mathcal{G}^*_m$$

so that the statement is a consequence of Corollary 2.

UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS

References

[1] R. R. Bahadur, "Sufficiency and statistical decision functions," *Ann. Math. Statist.*, 25 (1954), 423-462.

[2] L. Breiman, L. Le Cam and L. Schwartz, "Consistent estimates and zero-one sets," *Ann. Math. Statist.*, 35 (1964), 157-161.

[3] M. Loeve, *Probability Theory* (2nd edition), Van Nostrand, New York, 1960.