BOUNDEDNESS OF A MEASURABLE TRANSFORMATION AND A WEAKLY WANDERING SET

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1. Introduction

E. Hopf proved that the boundedness of a measurable transformation is a necessary and sufficient condition for the existence of a finite measure which is invariant under the transformation [3]. A. B. Hajian and S. Kakutani proved that the non-existence of a weakly wandering set of positive measure is also a necessary and sufficient condition [1]. It is obvious that the boundedness implies the non-existence of a weakly wandering set of positive measure. The purpose of this note is to prove directly that if there exists no weakly wandering set of positive measure, then the transformation is bounded. This problem was raised by Louis Sucheston as one of open problems [4]. As E. Hopf's condition is proved without Banach limit, by Theorem 2 we have another proof of A. B. Hajian and S. Kakutani's condition without Banach limit. So almost all propositions on the subject can be obtained without the axiom of choice.

2. Definition and lemmas

Let T be a non-singular bi-measurable transformation of a probability measure space $(\Omega, \mathfrak{B}, m)$ onto itself. From now on we fix $(\Omega, \mathfrak{B}, m)$ and T. A measurable set B is said to be equivalent to a measurable set B' (notation; $B \sim B'$), if there exist two decompositions $\{B_i\}$ and $\{B'_i\}$ $(i=1, 2, \cdots)$ of B and B' respectively and a sequence $\{n_i\}$ $(i=1, 2, \cdots)$ of integers such that $T^{n_i}B_i=B'_i$ $(i=1, 2, \cdots)$. (B_i or B'_i can be empty for some natural number i.) A measurable set B is said to be a weakly wandering set, if there exists a sequence $\{n_i\}$ $(i=1, 2, \cdots)$ of integers such that $T^{n_i}B \cap T^{n_j}B=\phi$ $(i \neq j)$. A measurable set B is said to be bounded, if $m(B-B')=0$ for any measurable set B' with $B' \sim B$ and $B' \subset B$. The transformation T is said to be bounded, if the whole space Ω is bounded.

The following lemmas are due to [1].

LEMMA 1. *Let* $(\Omega, \mathfrak{B}, m)$ *be a finite measure space and* λ *be a non-negative monotonic and subadditive set function defined on* $(\Omega, \mathfrak{B})$. *If a measure* m *is absolutely continuous with respect to* λ *then* m *is uniformly absolutely continuous with respect to* λ, *i.e., for any* $\varepsilon>0$, *there exists* $\delta>0$ *such that* $m(B)<\varepsilon$ *for any measurable set* B *with* $\lambda(B)<\delta$.

LEMMA 2. *If there exists no weakly wandering set of positive measure, then the following condition* (E.C.) *holds.*

(E.C.) *For any* $\varepsilon>0$, *there exists* $\delta>0$ *such that if* B *is a measurable set with* $m(B)<\delta$, *then* $m(T^n B)<\varepsilon$ $(n=0, \pm 1, \pm 2, \cdots)$.

If this condition (E.C.) holds, we say that $\{m_n\}$ is equi-uniformly absolutely continuous with respect to m, where $m_n(A)=m(T^n A)$ $(n=0, \pm 1, \pm 2, \cdots)$.

3. Results and their proofs

We put $m_n(A)=m(T^n A)$ $(n=0, \pm 1, \pm 2, \cdots)$, $\sigma_n(A)=\frac{1}{n}\sum_{k=0}^{n-1} m(T^k A)$ and $\sigma(A)=\limsup_{n\to\infty} \sigma_n(A)$.

LEMMA 3. *Let* $\{m_n\}$ $(n=0, \pm 1, \pm 2 \cdots)$ *be equi-uniformly absolutely continuous with respect to* m *and* A *be a measurable set. Assume that for any* $\varepsilon>0$, *there exists a measurable set* B *with* $B\sim A$ *and* $m(B)<\varepsilon$. *Then we have* $m(A)=0$.

PROOF. Let ε be an arbitrary positive number. From the assumption of the equi-uniform absolute continuity, there exists $\delta>0$ such that if $m(E)<\delta$, then $\sigma(E)<\varepsilon$. For this δ, there exists a measurable set B with $B\sim A$ and $m(B)<\delta$. Therefore we have a countable decomposition $\{A_i\}$ of A such that $\sum_{i=1}^{\infty} m(T^{n_i} A_i)<\delta$ and $T^{n_i}A_i \cap T^{n_j}A_j=\phi$ $(i\neq j)$. We choose a natural number k such that $m\left(A-\bigcup_{i=1}^{k} A_i\right)<\delta$. Then we have

$$\sigma\left(A-\bigcup_{i=1}^{k} A_i\right)<\varepsilon \quad \text{and} \quad \sigma\left(\bigcup_{i=1}^{k} T^{n_i}A_i\right)<\varepsilon .$$

Noting that σ is a subadditive and monotonic set function and the equality $\sigma\left(\bigcup_{i=1}^{k} T^{n_i}A_i\right)=\sigma\left(\bigcup_{i=1}^{k} A_i\right)$, which will be shown immediately later, we have $\sigma(A)<2\varepsilon$. Hence, as ε is an arbitrary positive number, $\sigma(A)=0$. If $m(A)>0$, then from the equi-uniform continuity there exists $\delta>0$ such that $m(T^n A)>\delta$ $(n=0, \pm 1, \pm 2, \cdots)$. Then we have $\sigma(A)\geqq\delta$, which is a contradiction.

PROOF OF THE EQUALITY. $\sigma\left(\bigcup_{i=1}^{k} T^{n_i}A_i\right)=\sigma\left(\bigcup_{i=1}^{k} A_i\right)$ $(A_i \cap A_j=\phi$, $T^{n_i}A_i \cap T^{n_j}A_j=\phi$ $(i \neq j))$. From the definition of σ_n we have

$$\left|\sigma_n\left(\bigcup_{i=1}^{k} T^{n_i}A_i\right)-\sigma_n\left(\bigcup_{i=1}^{k} A_i\right)\right| \leqq \sum_{i=1}^{k} 2|n_i|/n .$$

We obtain easily the equality from this.

THEOREM 1. *If $\{m_n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ is equi-uniformly absolutely continuous with respect to m, then the transformation T is bounded.*

PROOF. It is sufficient to show the following: $m(A-B)=0$ for any two measurable sets A, B with $A \sim B$ and $A \supset B$. Now we fix a decomposition $\{A_i\}$ of A such that $B=\bigcup_{i=1}^{\infty} T^{n_i}A_i$ and $A=\bigcup_{i=1}^{\infty} A_i$. We define a transformation S of A into itself by

$$S\omega=T^{n_i}\omega \qquad (\text{if } \omega \in A_i) .$$

It is easy to verify that S is a one-to-one non-singular bi-measurable transformation of A onto $B=SA$. We have

$$\lim_{n\to\infty} m(S^nE)=0 , \qquad \text{where } E=A-SA .$$

From the definition of S, the set E is equivalent to S^nE $(n=0, 1, 2, \cdots)$. By Lemma 3, we conclude $m(E)=0$, which completes the proof.

Since it is obvious that the boundedness of T implies the nonexistence of a weakly wandering set of positive measure, Lemma 2 and Theorem 1 yield the following conclusion.

THEOREM 2. *The following three conditions are equivalent.*
(1) *There exists no weakly wandering set of positive measure.*
(2) *$\{m_n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ is equi-uniformly absolutely continuous with respect to m.*
(3) *T is bouded.*

THEOREM 3. *The following conditions are equivalent.*
(1) *$\{m_n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ is equi-uniformly absolutely continuous with respect to m.*
(2) *For any $\varepsilon>0$, there exists $\delta>0$ such that if $m(B)<\delta$, then $m(B')<\varepsilon$ for any measurable set B' with $B' \sim B$.*

PROOF. Since the condition (1) is obviously implied by (2), we prove that (2) is satisfied if (1) holds. Assume that the condition (1) holds but the condition (2) does not hold. Then there exists $\delta>0$ such that for any $\varepsilon>0$, there exist two measurable sets B, B' with $B \sim B'$, $m(B)>\delta$

and $m(B')<\varepsilon$. From the proof of Lemma 3, we conclude $\sigma(B)<\varepsilon'$, where ε' tends to 0 when ε tends to 0. (Note that ε and ε' play the same role with δ and 2ε in the proof of Lemma 3 respectively.) This means that the measure m is not uniformly absolutely continuous with respect to the non-negative monotonic and subadditive set function σ. Consequently there exists a measurable set A such that $\sigma(A)=0$ and $m(A)>0$ (Lemma 1). But this is a contradiction to the condition (1).

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