A REMARK ON THE INCOMPARABILITY OF TWO CRITERIA FOR A UNIFORM CONVERGENCE OF PROBABILITY MEASURES

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(Received Feb. 7, 1969)

Summary

This paper connects with Theorem 3 of the author's paper [1], in which two criteria for type $(\boldsymbol{B})_d$ convergence ([3]) are shown to be incomparable to each other by presenting two examples. However, the statement of the theorem is not complete. In the present paper, we shall modify the statement of the theorem and give a proof by presenting a new example.

THEOREM. Let $(R, \boldsymbol{B}, \mu)$ be any given abstract σ-finite measure space, and let $\{P_s\}$ $(s=1, 2, \cdots)$ and Q be probability measures over the measurable space $(R, \boldsymbol{B})$ which are absolutely continuous with respect to μ. Let f_s and g be their generalized probability density functions with respect to μ, (gpdf (μ), for short) respectively.

Uniform convergence $(\boldsymbol{B})$ of P_s to Q as $s\rightarrow\infty$ is usually defined by

$$(1) \qquad \int_R |f_s-g|\, d\mu\rightarrow 0\,, \qquad (s\rightarrow\infty)\,.$$

This is equivalent to type $(\boldsymbol{B})_d$ convergence of P_s to Q defined by

$$(2) \qquad \delta_d(P_s, Q: \boldsymbol{B})=\sup_{E\in\boldsymbol{B}}\left|P_s(E)-Q(E)\right|\rightarrow 0\,, \qquad (s\rightarrow\infty),\ ([3])$$

since it holds that

$$\int_R |f_s-g|\, d\mu=2\delta_d(P_s, Q: \boldsymbol{B})\,,$$

for each s.

For the uniform convergence given above, a useful criterion is given by H. Scheffé [4]:

$$(\mathrm{S}) \qquad f_s(z)\rightarrow g(z)\,, \quad (\text{a. e. } \mu)\,, \qquad (s\rightarrow\infty)\,,$$

which is usually called the Scheffé criterion.

In [1], the present author proposed another criterion for the convergence (1):

$$(3)\qquad I(P_s : Q)=\int_R f_s \log (f_s/g)d\mu \to 0\,, \qquad (s\to\infty)\,,$$

where $I(P_s : Q)$ is the so-called Kullback-Leibler mean information.

Theorem 3 of [1] states that the conditions (S) and (3) are not comparable to each other, i.e., (S) is not necessarily stronger than (3), and also (3) is not necessarily stronger than (S).

As was shown in [2], the condition

$$(4)\qquad I(Q : P_s)=\int_R g \log (g/f_s)d\mu \to 0\,, \qquad (s\to\infty)\,,$$

is also sufficient for the convergence (1). Hence, it is seen by (3) that the condition

$$(\mathrm{I})\qquad \min\{I(P_s : Q), I(Q : P_s)\}\to 0\,, \qquad (s\to\infty)\,,$$

is sufficient for (1).

We shall now modify Theorem 3 in [1] as follows:

THEOREM. (i) (S) *is not necessarily stronger than* (I), *and* (ii) (I) *is not stronger than* (S).

PROOF OF THE THEOREM. To prove the assertion (i) we introduce the following example.

Example 1. Let $(R, \boldsymbol{B}, \mu)$ be a σ-finite measure space such that the range of μ-measure, $M(\boldsymbol{B})=\{\mu(E) : E\in \boldsymbol{B}\}$, is identical with the interval $[0, \infty]$. Let $\{E_n\}$ $(n=2, 3, \cdots)$ be a sequence of disjoint subsets of R belonging to $\boldsymbol{B}$, such that

$$(5)\qquad \mu(E_n)=1/(n\log n)\,, \qquad n=2, 3, \cdots.$$

Let Q be a probability measure over $(R, \boldsymbol{B})$ whose gpdf (μ) is given by

$$(6)\qquad g(z)=\begin{cases} 1/(\alpha\log n)\,, & \text{on } E_n \text{ for which } n=2m\,;\ m=1, 2, \cdots, \\ 1/(\alpha n\log n)\,, & \text{on } E_n \text{ for which } n=2m+1\,;\ m=1, 2, \cdots, \\ 0\,, & \text{elsewhere}\,, \end{cases}$$

where

$$\alpha=\sum_{m=1}^{\infty}\left\{\frac{1}{2m(\log 2m)^2}+\frac{1}{(2m+1)^2(\log(2m+1))^2}\right\}.$$

On the other hand, let $\{P_s\}$ $(s=2, 3, \cdots)$ be a sequence of probability measures over $(R, \boldsymbol{B})$ such that P_s has the gpdf(μ) defined by

$$(7)\quad f_s(z)=\begin{cases} 1/(\alpha \log 2)-\alpha_s\,, & \text{on } E_2\,,\\ g(z)\,, & \text{on } E_n \text{ for which } 3\leqq n\leqq s-1\,,\\ 1/(\alpha n \log n)\,, & \text{on } E_n \text{ for which } n=2m\,;\\ & \quad m=[(s+1)/2],\ [(s+1)/2]+1, \cdots,\\ 1/(\alpha \log n)\,, & \text{on } E_n \text{ for which } n=2m+1\,;\\ & \quad m=[s/2],\ [s/2]+1, \cdots,\\ 0\,, & \text{elsewhere}\,, \end{cases}$$

where [ ] denotes the Gauss symbol and α_s is given by

$$\alpha_s=\frac{2\log 2}{\alpha}\left\{\sum_{m=[(s+1)/2]}^{\infty}\frac{2m-1}{(2m\log 2m)^2}-\sum_{m=[s/2]}^{\infty}\frac{2m}{((2m+1)\log(2m+1))^2}\right\}.$$

Note that $\alpha_s \to 0$ as $s\to\infty$.

Then, it is easy to see that

$$(8)\qquad \sup_{z\in R}\left|f_s(z)-g(z)\right|\leqq \max\{|\alpha_s|,\ 1/(\alpha \log s)\}\,,$$

from which it follows that $f_s(z)-g(z)$ tends to zero as $s\to\infty$ uniformly over R. Thus, the condition (S) is satisfied.

It can be seen, however, that

$$(9)\quad I(P_s: Q)=\frac{1}{2\log 2}\left(\frac{1}{\alpha\log 2}-\alpha_s\right)\log(1-\alpha\cdot\alpha_s\log 2) - \sum_{m=[(s+1)/2]}^{\infty}\frac{1}{\alpha(2m)^2\log(2m)}+\sum_{m=[s/2]}^{\infty}\frac{1}{\alpha(2m+1)\log(2m+1)}$$

and

$$(10)\quad I(Q: P_s)=-\frac{1}{2\alpha(\log 2)^2}\log(1-\alpha\cdot\alpha_s\log 2)+\sum_{m=[(s+1)/2]}^{\infty}\frac{1}{\alpha(2m)\log(2m)} - \sum_{m=[s/2]}^{\infty}\frac{1}{\alpha(2m+1)^2\log(2m+1)}\,,$$

from which it follows that both of these quantities tend to infinity as $s\to\infty$, or, more precisely, $I(P_s: Q)=\infty$ and $I(Q: P_s)=\infty$ for each $s\geqq 2$. Thus, the condition (I) is not satisfied.

To prove the assertion (ii) of the theorem, we introduce Example 2 of [1], in which some misprints will be corrected.

Example 2. Let $(R, \boldsymbol{B}, \mu)$ be a finite measure space such that $M(\boldsymbol{B})$

$=[0, 1]$, and suppose that a $(\boldsymbol{B})$-partition of R,

$$\Delta_n=\left\{A_{n,k}: R=\sum_{k=1}^{n^2} A_{n,k},\ A_{n,k} \in \boldsymbol{B},\ \mu(A_{n,k})=1/n^2,\ k=1, \cdots, n^2\right\},$$

exists for each positive integer n.

For each $A_{n,k}$, let $f_{n,k}(z)$ be a function defined by

$$f_{n,k}(z)=\begin{cases} n\,, & \text{if } z \in A_{n,k}\,, \\ n/(n+1)\,, & \text{otherwise}\,. \end{cases} \tag{11}$$

Renumbering the functions, $\{f_{n,k}(z): k=1, \cdots, n^2;\ n=1, 2, \cdots\}$, in such a way that $f_{n,k}(z)=f_s(z)$ when $s=n(n-1)(2n-1)/6+k$, we get a sequence of gpdf (μ), $\{f_s(z)\}$ $(s=1, 2, \cdots)$. Let $\{P_s\}$ $(s=1, 2, \cdots)$ be the sequence of corresponding probability measures.

On the other hand, let Q be a probability measure over $(R, \boldsymbol{B})$, whose gpdf (μ) is given by

$$g(z)=1\,, \qquad z \in R\,. \tag{12}$$

Then, the sequence $\{f_s\}$ $(s=1, 2, \cdots)$ does not converge a.e. μ to g, because it does not hold that $f_s(z)\rightarrow g(z)$ as $s\rightarrow\infty$, for any fixed $z \in R$. Thus, the condition (S) is not satisfied.

The condition (I), however, is satisfied, because for any given $k=1, 2, \cdots, n^2$,

$$I(P_s: Q)=\frac{\log n}{n^2}+\frac{n^2-1}{n^2}\log\frac{n}{n+1}\rightarrow 0\,, \qquad (s\rightarrow\infty)\,.$$

The above two examples prove the theorem.

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