A CHARACTERIZATION OF THE NORMAL LAW

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Summary

If X, Y, Z are three random observations from a normal population with mean zero then the characteristic function of $(X/Z, Y/Z)$ is $\exp(-\sqrt{t^2+u^2})$. It is shown in this paper that this property characterizes the normal law.

1.

Let X_1, X_2 be two independent normal variates with zero mean and common variance. It is then well-known that the quotient X_1/X_2 follows the Cauchy law distributed symmetrically about the origin. It is also well-known that we cannot obtain a characterization of the normal distribution by this property of the quotient [1]. A characterization of the generalized normal law (g.n.l.), that is, a distribution with frequency function

$$(1) \qquad f(x)=\frac{\left(\frac{n}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)\sigma^n}|x|^{n-1}e^{-x^2n/2\sigma^2}, \qquad -\infty<x<\infty,\ n\geqq 1$$

(and hence the usual normal law) has been obtained in [2] where the following is proved: If $X, X_1, X_2, \cdots$ are independent observations from a population with a distribution function $F(x)$ assumed to be continuous at $x=0$ and if the frequency function of $t_k=X\Big/\sqrt{\frac{1}{k}\left(\sum_{i=1}^{k} X_i^2\right)}$ is

$$\frac{\Gamma\left(\frac{n(k+1)}{2}\right)}{\left(\frac{n}{k^2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{kn}{2}\right)}|x|^{n-1}\left(1+\frac{x^2}{k}\right)^{-n(k+1)/2} \qquad -\infty<x<\infty,\ n\geqq 1$$

then necessarily X will have the frequency function $f(x)$ defined in (1).

The object of this paper is to obtain a new characterization of the g.n.l. in terms of three independent observations from the population. Of course, when $n=1$ our characterization will be that of the ordinary normal law.

2.

LEMMA 1. *If X follows a g.n.l. then the characteristic function (c.f.) of $\log|X|$ can never vanish.*

PROOF. If possible, let the c.f. of $\log|X|$ vanish at a point w, that is,

$$0=\int_0^\infty \exp\{iw\log x\}\, x^{n-1}\exp\left\{-\frac{nx^2}{2\sigma^2}\right\}dx\ .$$

The transformation $x^2=2\sigma^2/n\cdot\alpha y$, $\alpha>0$ yields

$$0=\int_0^\infty \exp\left\{i\frac{w}{2}\log y\right\}y^{n/2-1}e^{-\alpha y}dy \qquad \text{for all } \alpha>0\ .$$

This identity in α contradicts the completeness of the family of frequency functions $e^{-\alpha x}$, $x\geqq 0$, $\alpha>0$. Hence the lemma.

COROLLARY. *If Z follows the generalized Cauchy law (g.c.l.) given by*

$$g(Z)=\frac{\Gamma(n)}{(\Gamma(n/2))^2}|Z|^{n-1}(1+Z^2)^{-n}\ , \qquad -\infty<Z<\infty,\ n\geqq 1$$

then the c.f. of $\log|Z|$ does not vanish.

PROOF. Observe first that if X_1, X_2 are independently distributed according to g.n.l. then their ratio is distributed according to g.c.l. Therefore distributionally Z can be conceived of as the ratio of two independent g.n.l.'s. Hence the c.f. of $\log|Z|$ has the form $|m(t)|^2$ where $m(t)$ is the c.f. of $\log|X|$, X being a random variable following g.n.l. That $m(t)$ does not vanish has been established in Lemma 1. The assertion of the corollary now follows.

LEMMA 2. *If X_1, X_2, X_3 are independent random variables and if the joint c.f. of (X_1-X_3, X_2-X_3) does not vanish, then it determines the distribution of the variables X_1, X_2 and X_3 except for additive constants.*

Note that the assumption that the joint c.f. of (X_1-X_3, X_2-X_3) does not vanish implies and is implied by the assumption that the c.f.'s of the variables X_k, $k=1,2,3$ do not vanish.

PROOF. Let $\phi_k(t)$, $k=1,2,3$ denote the c.f. of X_k, $k=1,2,3$ and let $\xi(t_1,t_2)$ denote the joint c.f. of (X_1-X_3, X_2-X_3):

$$\xi(t_1, t_2) = E\{e^{it_1X_1+it_2X_2+i(-t_1-t_2)X_3}\}$$
$$= \phi_1(t_1)\phi_2(t_2)\phi_3(-t_1-t_2) , \qquad -\infty < t_1, t_2 < \infty .$$

Let Y_1, Y_2, Y_3 be any other set of independent random variables such that the c.f. of (Y_1-Y_3, Y_2-Y_3) is $\xi(t_1, t_2)$ defined earlier. Let $\psi_k(t)$, $k=1, 2, 3$ be the c.f. of Y_k, $k=1, 2, 3$. Hence $\psi_1(t_1)\psi_2(t_2)\psi_3(-t_1-t_2)=\phi_1(t_1)\phi_2(t_2)\phi_3(-t_1-t_2)$. From the assumption that $\xi(t_1, t_2)$ does not vanish it follows that none of the c.f.'s $\phi_k(t)$, $\psi_k(t)$, $k=1, 2, 3$ vanish. Writing $\psi_k(t)=Q_k(t)\phi_k(t)$, $k=1, 2, 3$ so that $Q_k(t)$, $k=1, 2, 3$ is a complex valued function defined for $-\infty<t<\infty$, non-vanishing and satisfying $Q_k(0)=1$, $k=1, 2, 3$, we get

$$(2) \qquad Q_1(t_1)Q_2(t_2)Q_3(-t_1-t_2)=1 .$$

Putting $t_1=t$, $t_2=0$, we get $Q_1(t)=1/Q_3(-t)$.
Putting $t_1=0$, $t_2=t$, we get $Q_2(t)=1/Q_3(-t)$.
Therefore

$$(3) \qquad Q_1(t)=Q_2(t)=\frac{1}{Q_3(-t)} .$$

Substituting in (2) we get

$$Q_3(t_1+t_2)=Q_3(t_1)Q_3(t_2) , \qquad -\infty<t_1, t_2<\infty .$$

The most general function $Q_3(t)$ continuous on the whole line $-\infty<t<\infty$, non-vanishing and satisfying the condition $Q_3(0)=1$ is the exponential function $Q_3(t)=e^{bt}$, $-\infty<t<\infty$ for some b, a complex number.

From (3) we obtain

$$Q_1(t)=Q_2(t)=Q_3(t)=e^{bt} .$$

Thus we have

$$\psi_k(t)=e^{bt}\phi_k(t) , \qquad k=1, 2, 3 .$$

Using the property of the c.f.'s $\phi(-t)=\bar{\phi}(t)$ (complex conjugate), we see that $\psi_k(t)=e^{ict}\phi_k(t)$, $k=1, 2, 3$ for some real c. Hence the assertion.

3.

Let $\eta(t)$ be the c.f. of $\log|X|$ where X follows g.n.l. with $\sigma=1$. Recall that $\eta(t)$ is non-vanishing. Let $G(x, y)$ be the joint distribution of $(X_1/X_3, X_2/X_3)$ where X_1, X_2, X_3 are independent observations on X. Notice that the joint c.f. of $(\log\{|X_1|/|X_3|\}, \log\{|X_2|/|X_3|\})$ is $\eta(t)\eta(-t-u)\eta(u)$.

Theorem. *Let Z_1, Z_2, Z_3 be three independent symmetric (about the* *origin) random variables with distribution functions continuous at zero. Then they all follow the g.n.l. if and only if the bivariate distribution function of* $(Z_1/Z_3, Z_2/Z_3)$ *is* G.

PROOF. Observe that the specification of the distribution of $\log|Z_k|$, $k=1, 2, 3$ determines the distribution of $|Z_k|$, $k=1, 2, 3$. Therefore, below we show that under the conditions mentioned in the theorem the distribution of $\log|Z_k|$, $k=1, 2, 3$ is uniquely determined except for an additive constant which is same for all the variables.

Suppose Z_1, Z_2, Z_3 satisfy the conditions of the theorem. Hence the joint c.f. of $(\log\{|Z_1|/|Z_3|\}, \log\{|Z_2|/|Z_3|\})$ is $\eta(t)\eta(-t-u)\eta(u)$. If $\theta_k(t)$ is the c.f. of $\log|Z_k|$, $k=1, 2, 3$, then we get

$$\theta_1(t)\theta_2(u)\theta_3(-t-u)=\eta(t)\eta(u)\eta(-t-u) .$$

Then $\theta_1, \theta_2, \theta_3$ are non-vanishing. That the claim now follows from Lemma 2 is seen by taking $\log|Z_k|=X_k$, $k=1, 2, 3$.

Note 1. If in the theorem the variables Z_1, Z_2, Z_3 are assumed to be identically distributed then the assumption of symmetry of their common distribution is not necessary. This property can be derived as a consequence of the fact that the distribution of Z_1/Z_2 is symmetric about the origin (c.f. [1]).

Note 2. Taking $n=1$ one observes that the c.f. of $G(x, y)$ is $\exp(-\sqrt{t^2+u^2})$. Thus the joint c.f. of $(Z_1/Z_3, Z_2/Z_3)$ is $\exp(-\sqrt{t^2+u^2})$ iff Z_1, Z_2, Z_3 follow the same normal law.

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