CHARACTERISTIC FUNCTIONS SATISFYING A FUNCTIONAL EQUATION (II)

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Introduction and summary

In the previous sections, we derived a representation for the ch.f.'s $\varphi(t)$ which satisfy the relation

$$(0.1)\qquad \varphi(t)=\varphi(a_1t)\cdots\varphi(a_pt)\varphi(-a_{p+1}t)\cdots\varphi(-a_nt)\,,\quad 1>a_k>0,\quad k=1,2,\cdots,n\,.$$

We shall call such ch.f.'s *partially stable*. According to the results of the preceding sections, every partially stable ch.f. $\varphi(t)$ is, except for the factor of the form $e^{i\beta t}$, semi-stable in the sense of P. Lévy [2] (see Section 58). In fact, if $\varphi(t)$ is not a "stable" ch.f. then there exist a positive number ρ and mutually prime positive integers $l_1,\cdots,l_n$ (the g.c.d. of them is 1), such that

$$(0.2)\qquad \log a_k=-l_k\rho\,,\qquad k=1,\cdots,n\,.$$

We can easily verify that $\varphi(t)$ satisfies one of the relations

$$(0.3)\qquad \varphi(t)=\varphi^{\gamma}(ct)\,,\qquad (\text{when } (a_1,\cdots,a_n)\in B_n^p(\rho))\,,$$

and

$$(0.4)\qquad \varphi(t)=\varphi^{\gamma}(-ct)\,,\qquad (\text{when } (a_1,\cdots,a_n)\in C_n^p(\rho))\,,$$

where $c=e^{-\rho}$, $\gamma=c^{-\alpha}$ and α is the unique real zero of the entire function $a_1^z+\cdots+a_n^z-1$. In any case, $\varphi(t)$ is semi-stable. If conversely $\varphi(t)$ satisfies (0.3) or (0.4), then $\varphi(t)$ can be put in the form described in Theorem 4 and Theorem 3 respectively (see [2]). However, a semi-stable ch.f. is not necessarily partially stable. Whether it is partially stable depends on the value γ appearing in the defining equation

$$(0.5)\qquad \varphi(t)=\varphi^{\gamma}(\gamma^{-1/\alpha}t)\,,\qquad \gamma>1\,.$$

In Section 6, we shall show that the gap between semi-stability and partial stability will be filled with ch.f.'s satisfying the relation

(0.6) $$\varphi(t)=\prod_{1}^{\infty} \varphi(a_n t)\,, \qquad 1>|a_n|\,.$$

B. Ramachandran and C. R. Rao [9] introduced a class of generalized stable ch.f.'s which satisfy

(0.7) $$\prod_{j=1}^{n} \varphi^{\gamma_j}(a_j t)=\prod_{j=n+1}^{n+m} \varphi^{\gamma_j}(a_j t)\,, \qquad \gamma_j>0, \quad j=1, \cdots, n+m\,.$$

This is a very general relation containing (1.5), (0.1) as well as (0.5). They studied the case $a_{n+1}=1, a_{n+2}=\cdots=a_{n+m}=0$, i.e., the relation

(0.8) $$\varphi(t)=\varphi^{\gamma_1}(a_1 t)\cdots\varphi^{\gamma_n}(a_n t)\,, \qquad \gamma_j>0, \quad 1>|a_j|>0, \quad j=1, \cdots n\,.$$

In Section 7 we shall show that in spite of the apparent generality, every ch.f. satisfying (0.8) is, except for the factor $e^{i\beta t}$, semi-stable.

Every semi-stable ch.f. is infinitely divisible, and every infinitely divisible distribution has a non-empty domain of partial attraction. We derive in Section 8 the condition under which a distribution F belongs to the domain of partial attraction of a semi-stable distribution. Some properties of the semi-stable distributions, existence of moments, and differentiability of the distribution function, are derived in Section 9.

6. Semi-stability and partial stability

Let $\varphi(t)$ be a semi-stable ch.f. satisfying (0.5) and suppose that it is not stable. Then $\varphi(t)$ is infinitely divisible and is put in the form

(6.1) $$\log \varphi(t)=i\beta t+\int_0^\infty \left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x) + \int_{-\infty}^0 \left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x)\,,$$

where

$$M(x)=-\lambda(-\log x)/x^\alpha\,, \qquad x>0\,,$$
$$N(x)=\mu(-\log|x|)/|x|^\alpha\,, \qquad x<0\,,$$

and

(6.2) $$\lambda(t),\ \mu(t) \in \boldsymbol{P}^+((\log \gamma)/\alpha)\,.$$

($\boldsymbol{P}^+(\xi)$ is the set of all left continuous periodic functions with the period ξ. $P^+(0)$ is interpreted as the set of real numbers). For any $\gamma>1$ and $0<\alpha<2$, such ch.f.'s really exist (see [2], Section 58). If $\varphi(t)$ satisfies the farther condition (0.1) of partial stability, then a's are written as (0.2) and we have, in addition to (6.2), (6.3) $$\lambda(t),\ \mu(t) \in \boldsymbol{P}^{+}(\rho)\ ,$$

and

(6.4) $$a_1^{\alpha}+\cdots+a_n^{\alpha}-1=0$$

(see Theorems 1–4). It follows from (6.2) and (6.3) that $r=\rho\alpha/\log\gamma$ is a rational number since otherwise $\lambda(t)$ and $\mu(t)$ are constants and hence $\varphi(t)$ must be stable. Because of (0.2), the condition (6.4) is equivalent to the statement that γ^{-r} is a zero of the polynomial $x^{l_1}+\cdots+x^{l_n}-1$. But since γ^{-r} can not, for any choice of a rational number r, be a zero of a polynomial when γ is non-algebraic, semi-stability does not imply partial stability. However, the following lemma suggests that the semi-stability will be characterized by the relation (0.6).

LEMMA 8. *If $1>a>0$, then there exists a sequence $\{c_n\}$ of non-negative integers such that,*

(6.5) $$\sum_1^{\infty} c_n a^n=1$$

holds.

PROOF. The sequence $\{c_n\}$ is not unique, and we give a constructive proof to be used later. Let p and q be any positive integers such that

$$a>b\equiv 1-pa^q>0\ .$$

Let $\{c'_n\}$ be a sequence of non-negative integers defined by

$$c'_1=0\ ,\qquad c'_n=\left[\left(b-\sum_1^{n-1} c'_k a^k\right)\Big/ a^n\right].$$

Then we have

$$b-\sum_1^{n} c'_k a^k \geqq 0 \geqq b-\sum_1^{n} c'_k a^k-a^n\ ,$$

which implies that $b-\sum_1^{\infty} c'_k a^k$ exists and is equal to 0, i.e.,

$$1-pa^q-\sum_1^{\infty} c'_k a^k=0\ .$$

Then, put

$$c_n=\begin{cases} c'_n & \text{if } n\neq q\ ,\\ c'_q+p & \text{if } n=q\ .\end{cases}$$ (q.e.d)

Now we have

THEOREM 6. *If a ch.f. $\varphi(t)$ satisfies* (0.5), *then there exists a sequence*

$\{a_n\}$, $1>a_n\geqq 0$, $n=1,2,\cdots$, *such that the relation*

$$\varphi(t)=\prod_{n=1}^{\infty}\varphi(a_n t) \tag{6.6}$$

holds.

THEOREM 7. *Let* $\{a_n\}$ *be a given sequence of real numbers such that* $|a_n|<1$, $n=1,\cdots$; *if* $a_n=0$, *then* $a_{n+1}=a_{n+2}=\cdots=0$; *and such that*

$$\sum_1^{\infty}\left|a_n\right|^{\alpha}=1 \qquad \text{for some } \alpha>0\,. \tag{6.7}$$

If a non-degenerate ch.f. $\varphi(t)$ *satisfies equation* (0.6), *then there exists a real number* β *such that* $\psi(t)=e^{-i\beta t}\varphi(t)$ *is a semi-stable ch.f., and it satisfies the relation*

$$\psi(t)=\psi^{\gamma}(ct)\,, \qquad \text{when } a_k\geqq 0,\ k=1,\cdots \tag{6.8}$$

or

$$\psi(t)=\psi^{\gamma}(-ct)\,, \qquad \text{when } a_k<0 \text{ for some } k\,. \tag{6.9}$$

If especially for some i *and* j, $\log|a_i|/\log|a_j|$ *is irrational, then* $\psi(t)$ *is a stable ch.f. with the characteristic exponent* α. *If on the contrary there exist positive number* ρ *and a sequence* $l_1, l_2, \cdots$ *of mutually prime positive integers such that*

$$\log|a_k|=-l_k\rho\,, \qquad \text{if } a_k\neq 0\,, \tag{6.10}$$

then constants γ *and* c *of* (6.8) *and* (6.9) *are given by* $\gamma=c^{-\alpha}$, *and* $c=e^{-\rho}$.

PROOF OF THEOREM 6. If $\alpha\geqq 2$, $\varphi(t)$ is a degenerate or non-degenerate normal ch.f. (it is an infinitely divisible ch.f. and the Poisson spectra can not have any point of increase), and nothing is left to be proved. We assume therefore that $0<\alpha<2$ and consider only the case $0<\alpha<1$: the remaining cases will be treated similarly. Then (6.1) reduces to

$$\log\varphi(t)=\int_0^{\infty}(e^{itx}-1)dM(x)+\int_{-\infty}^{0}(e^{itx}-1)dN(x)\,. \tag{6.11}$$

Put $a=\gamma^{-1/\alpha}$, $c_0=0$ and let $\{c_n\}$ be a sequence of non-negative integers as defined by Lemma 8. Put

$$a_n=\begin{cases} a^{k/\alpha} & \text{if } n=c_0+\cdots+c_{k-1}+l,\ 1\leqq l\leqq c_k\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Then $\sum_1^{\infty}a_n^{\alpha}=1$, and from (6.11) and (6.2) we have

$$\log \varphi(a_n t) = a_n^\alpha \log \varphi(t)\,, \qquad n=1, 2, \cdots,$$

from which (6.6) follows.

PROOF OF THEOREM 7. It was shown by R. G. Laha and E. Lukacs [4] that the ch.f. $\varphi(t)$ satisfying the relation (6.6) is infinitely divisible and that it is a degenerate or non-degenerate normal ch.f. whenever $\alpha \geqq 2$. We assume therefore that $0<\alpha<2$, and let the P. Lévy canonical representation of $\varphi(t)$ be

$$(6.12) \qquad \log \varphi(t) = i\beta t - \frac{1}{2}\sigma^2 t^2 + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x) + \int_{-\infty}^0 \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dN(x)\,.$$

We rewrite (0.6) and (6.7) as

$$(6.13) \qquad \varphi(t) = \prod_1^\infty \varphi(a_n t) \prod_1^\infty \varphi(-b_n t)\,, \qquad 1>a_n \geqq 0,\ 1>b_n \geqq 0\,,$$

and

$$(6.14) \qquad \sum_1^\infty a_n^\alpha + \sum_1^\infty b_n^\alpha = 1\,.$$

It follows from (6.12), (6.13) and from the assumption $0<\alpha<2$, that

$$(6.15) \qquad \sigma^2 = 0\,,$$

$$(6.16) \qquad M(x) = \sum_1^\infty M(x/a_n) - \sum_1^\infty N(-x/b_n)\,, \qquad x>0$$

and

$$(6.17) \qquad N(x) = \sum_1^\infty N(-x/a_n) - \sum_1^\infty M(-x/b_n)\,, \qquad x<0\,.$$

The functions $f(t)=M(e^{-t})$ and $g(t)=-N(-e^{-t})$ are non-positive, monotone non-increasing and satisfy $\lim_{t\to-\infty} f(t) = \lim_{t\to-\infty} g(t) = 0$. Equations (6.16) and (6.17) become

$$(6.18) \qquad f(t) = \sum_1^\infty f(t+A_n) + \sum_1^\infty g(t+B_n)\,,$$

and

$$(6.19) \qquad g(t) = \sum_1^\infty g(t+A_n) + \sum_1^\infty f(t+B_n)\,,$$

where $A_n = \log a_n$, and $B_n = \log b_n$.

We can solve these simultaneous equations by reducing to the finite sum case which was treated in Section 3. We shall therefore consider only the case $b_k=0$, $k=1, 2, \cdots$. Then our equations become mutually independent:

$$f(t)=\sum_1^\infty f(t+A_n) \tag{6.20}$$

and

$$g(t)=\sum_1^\infty g(t+A_n)\ . \tag{6.21}$$

For the proof of the theorem it suffices to prove the following lemma.

LEMMA 9. (6.20) *implies that* $f(t)$ *is of the form*

$$f(t)=-\lambda(t)e^{\alpha t}\ , \tag{6.23}$$

where

$$\lambda(t)\in \boldsymbol{P}^+(\rho)\ , \qquad \textit{for some } \rho\geqq 0\ .$$

PROOF. For any given $\varepsilon>0$ let n be a sufficiently large positive integer such that

$$0\geqq \Delta_n(t)\equiv\sum_{n+1}^\infty f(t+A_k)\geqq -\varepsilon\ , \qquad t<0\ , \tag{6.23}$$

and let it be fixed. The integral

$$\chi_f(z)=\int_{-\infty}^0 e^{-zt}f(t)dt \tag{6.24}$$

converges and is regular in the half plane $\operatorname{Re} z<0$. Put

$$\sigma_n(z)=1-a_1^z-\cdots-a_n^z\ , \tag{6.25}$$

$$E_n(z)=\sum_1^n a_j^z\int_0^{A_j} e^{-zt}f(t)dt\ , \tag{6.26}$$

and

$$\delta_n(z)=\sum_{n+1}^\infty\int_{-\infty}^0 e^{-zt}f(t+A_k)dt=\int_{-\infty}^0 e^{-zt}\Delta_n(t)dt\ , \tag{6.27}$$

the series being absolutely and uniformly convergent in the half plane $\operatorname{Re} z\leqq r<0$. It follows then that

$$\chi_f(z)\sigma_n(z)=E_n(z)+\delta_n(z)\ , \qquad \operatorname{Re} z<0\ . \tag{6.28}$$

By the inversion formula of the Laplace transform, we have for negative r, Λ and t,

$$\int_t^0 e^{-\Lambda\tau} f(\tau)d\tau = -\lim_{y\to 0}\frac{1}{2\pi i}\int_{r-iy}^{r+iy} e^{tz}\cdot\frac{\chi_f(z+\Lambda)}{z}dz \tag{6.29}$$

$$= \varepsilon_n(t) - \lim_{y\to\infty}\frac{1}{2\pi i}\int_{r-iy}^{r+iy} e^{tz}\frac{E_n(z+\Lambda)}{z\cdot\sigma_n(z+\Lambda)}dz \tag{6.30}$$

where

$$\varepsilon_n(t) = -\lim_{y\to\infty}\frac{1}{2\pi i}\int_{r-iy}^{r+iy} e^{tz}\frac{\delta_n(z+\Lambda)}{z\cdot\sigma_n(z+\Lambda)}dz\,. \tag{6.31}$$

But if $a_0=\min(a_1, \cdots, a_n)$, and if $x=\operatorname{Re} z<0$, then

$$|\sigma_n(z)| = a_0^x\left|\sum_1^n\left(\frac{a_k}{a_0}\right)^z - \left(\frac{1}{a_0}\right)^z\right| \sim p\cdot a_0^x \geqq a_n^x \qquad \text{as } x\to-\infty\,,$$

where $1\leqq p\leqq n$. Hence if $-r$ is a sufficiently large number possibly dependent on n, then,

$$|\sigma_n(z)| \geqq \frac{1}{2}\exp\{A_n r\}\,, \qquad \operatorname{Re} z\leqq r<0\,. \tag{6.32}$$

On the other hand, from (6.23) and (6.27) we have

$$|\delta_n(z)| = \left|-\frac{1}{z}\Delta_n(0) + \frac{1}{z}\int_{-\infty}^0 e^{-zt}d\Delta_n(t)\right| \leqq \frac{2\varepsilon}{|z|}\,, \qquad \operatorname{Re} z\leqq r<0\,. \tag{6.33}$$

It follows from (6.32) and (6.33) that

$$|\varepsilon_n(t)| \leqq C\cdot\varepsilon\,, \qquad A_n\leqq t<0\,, \tag{6.34}$$

where C is a positive constant independent of n.

Now the second term of (6.30) becomes

$$\sum_{-\infty}^{\infty}\frac{1}{2\pi i}\int_{L_k} e^{t(z-\Lambda)}\frac{E_n(z)}{(z-\Lambda)\sigma_n(z)}dz - \frac{E_n(\Lambda)}{\sigma_n(\Lambda)}\,, \tag{6.35}$$

where contours L_k, $k=0, \pm1, \pm2, \cdots$ are defined as in the proof of Lemma 1, but this time they may depend on n. If $(a_1, \cdots, a_n)\in A_n(\rho_n)$, $\rho_n\geqq 0$, then the first term of (6.35) is calculated to be

$$\lambda_{1,n}(t)e^{(\alpha_n-\Lambda)t}\,,$$

where α_n is the unique real zero of $\sigma_n(z)$, and where $\lambda_{1,n}(t)\in \boldsymbol{P}^+(\rho_n)$. Thus (6.30) reduces to

$$\int_{-\infty}^t e^{-\Lambda\tau}f(\tau)d\tau = \chi_f(\Lambda) - \frac{E_n(\Lambda)}{\sigma_n(\Lambda)} - \varepsilon_n(t) + \lambda_{1,n}(t)e^{(\alpha_n-\Lambda)t}\,. \tag{6.36}$$

It follows from (6.28), (6.33) and (6.34) that $E_n(\Lambda)/\sigma_n(\Lambda)\to\chi_f(\Lambda)$, and

$\varepsilon_n(t)\to 0$ as $\varepsilon\to 0$ and $n\to\infty$. We also have $\alpha_n\to\alpha$. Therefore for $t<0$ $\lambda_1(t)=\lim_{n\to\infty}\lambda_{1,n}(t)$ exists, the convergence being uniform in every compact subset of $(-\infty, 0)$, and (6.36) becomes

$$\int_{-\infty}^{t} e^{-\Lambda\tau}f(\tau)d\tau=\lambda_1(t)e^{(\alpha-\Lambda)t}\,. \tag{6.37}$$

Now $\{\rho_n\}$ is a monotone non-increasing sequence: more precisely for each n, there corresponds a positive integer p_n such that $\rho_{n+1}=\rho_n/p_n$. Hence $\rho=\lim\rho_n$ exists. If especially $\log a_i/\log a_j$ is irrational, then $\rho_n=0$ for all $n\geqq i, j$. Again if $\log a_k=-l_k\cdot\rho$, $k=1, \cdots, \rho>0$, l's being positive integers and if $l_1, \cdots, l_s$ are mutually prime, then $\rho_n=\rho$ for $n\geqq s$.

It follows from these considerations that there exists a function $\bar\lambda_1(t)\in \boldsymbol{P}^+(\rho)$ such that $\lambda_1(t)=\bar\lambda_1(t)$ for $t<0$. Differentiating (from the left) both sides of (6.37), we obtain

$$f(t)=-\lambda(t)e^{\alpha t}\,,\qquad t<0\,, \tag{6.38}$$

where

$$\lambda(t)=-(\alpha-\Lambda)\bar\lambda_1(t)-\bar\lambda_1'(t)\in \boldsymbol{P}^+(\rho)\,.$$

Considering the given equation (6.20), we conclude that (6.38) holds for all real t.

7. Generalization

Consider the ch.f.'s satisfying the equation

$$\varphi(t)=\prod_{1}^{p}\varphi^{\gamma_j}(a_jt)\prod_{p+1}^{n}\varphi^{\gamma_j}(-a_jt)\,, \tag{7.1}$$

$$\gamma_j>0,\ 1>a_j>0,\ j=1, 2, \cdots, n\,.$$

In order to determine $\varphi(t)$, we introduce the entire functions

$$\sigma_0^*(z)=1-\gamma_1a_1^z-\cdots-\gamma_na_n^z\,,$$

$$\sigma_1^*(z)=1-\gamma_1a_1^z-\cdots-\gamma_pa_p^z+\gamma_{p+1}a_{p+1}^z+\cdots+\gamma_na_n^z\,,$$

and

$$\sigma^*(z)=\begin{cases}\sigma_0^*(z)\sigma_1^*(z)\,, & \text{if } p<n\,,\\ \sigma_0^*(z) & \text{if } p=n\,.\end{cases}$$

Let α be the unique real zero of $\sigma_0^*(z)$. Now we have,

THEOREM 8. *Suppose a non-degenerate ch.f. $\varphi(t)$ satisfies the relation* (7.1), *then we have*

(i) *in any case*, $0<\alpha\leqq 2$. *If* $\alpha=2$, $\varphi(t)$ *is a normal ch.f.*: $\varphi(t)=\exp\{i\beta t-1/2\cdot\sigma^2t^2\}$,

(ii) *if* $0<\alpha<2$ *there exists a real number* β *such that* $\psi(t)=e^{-i\beta t}\varphi(t)$ *is semi-stable. It is stable if* $(a_1,\cdots,a_n)\in A_n(0)$. *If* $(a_1,\cdots,a_n)\in A_n(\rho)$, $\psi(t)$ *satisfies one of the following two equations*:

$$\psi(t)=\psi^{\gamma}(ct)\,,\qquad \textit{when } (a_1,\cdots,a_n)\in B_n^p(\rho)\,,$$

and,

$$\psi(t)=\psi^{\gamma}(-ct)\,,\qquad \textit{when } (a_1,\cdots,a_n)\in C_n^p(\rho)\,,$$

where $c=e^{-\rho}$, and $\gamma=c^{-\alpha}$.

PROOF. Statement (i) and the infinitely divisibility of $\varphi(t)$ were proved by B. Ramachandran and C. R. Rao [9]. We therefore assume that $0<\alpha<2$. Let (6.12) be the P. Lévy representation for $\varphi(t)$. From (7.1) we obtain

$$\sigma^2=0\,, \tag{7.2}$$

$$M(x)=\sum_1^p\gamma_jM(x/a_j)-\sum_{p+1}^n\gamma_jN(-x/a_j)\,,\qquad x>0\,, \tag{7.3}$$

and

$$N(x)=\sum_1^p\gamma_jN(x/a_j)-\sum_{p+1}^n\gamma_jM(-x/a_j)\,,\qquad x<0\,. \tag{7.4}$$

The functions $f(t)=M(e^{-t})$ and $g(t)=-N(-e^{-t})$ are monotone non-increasing and satisfy

$$f(t)=\sum_1^p\gamma_jf(t+A_j)+\sum_{p+1}^n\gamma_jg(t+A_j)\,, \tag{7.5}$$

$$g(t)=\sum_1^p\gamma_jg(t+A_j)+\sum_{p+1}^n\gamma_jf(t+A_j)\,. \tag{7.6}$$

We can show that Theorem 5 holds true even if we replace (3.1) and (3.2) by (7.5) and (7.6). Naturally some evident modifications are needed. Thus, for example, the functions $\sigma_0(z)$, $\sigma(z)$ must be replaced by $\sigma_0^*(z)$ and $\sigma^*(z)$, and so on. Lemma 7 is also true when (4.6) is replaced by

$$P(x)=\gamma_1P(x/a_1)+\cdots+\gamma_nP(x/a_n) \tag{7.7}$$

(note that $\alpha>0$ implies $\gamma_1+\cdots+\gamma_n>1$). Therefore Theorems 1–4 hold true even if we replace the statement "$\varphi(t)\in T_\alpha(a_1,\cdots,a_p,-a_{p+1},\cdots,-a_n)$" by "$\varphi(t)$ satisfies the relation (7.1)". The proof of our theorem is now immediately completed.

8. Domain of partial attraction

Let $X_1, X_2, \cdots$ be a system of indpendent random variables with common distribution F. If for some choice of sequences $\{A_n\}$ and $\{B_n\}$, the distribution of

$$\xi_n=\frac{X_1+\cdots+X_n}{B_n}-A_n$$

converges to a non-degenerate distribution G, then we say that F belongs to the domain of attraction of G. When a suitable subsequence of ξ_n converges, we say that F belongs to the domain of partial attraction of G. A distribution is infinitely divisible if and only if it has a non-empty domain of partial attraction (see [1]). In this section we shall derive a condition that a given distribution F belongs to the domain of partial attraction of a semi-stable distribution.

Let $\varphi(t)$ be an infinitely divisible ch.f. with the P. Lévy canonical form

$$\log \varphi(t)=i\beta_0 t+\int_0^\infty \left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x) \\ +\int_{-\infty}^0 \left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x)\,. \tag{8.1}$$

The following lemma is known (see [1], Section 25, Theorem 4).

LEMMA 10. *In order that a distribution F belongs to the domain of partial attraction of the infinitely divisible distribution corresponding to* (8.1) *it is necessary and sufficient that there exist a sequence $\{B_n\}$ of positive numbers and a subsequence $\{N(n)\}$ of the sequence $\{1, 2, \cdots\}$ of all positive integers such that*

$$\lim_{n\to\infty} N(n)(1-F(B_n x))=-M(x)\,, \qquad x>0\,, \tag{8.2}$$

$$\lim_{n\to\infty} N(n)F(-B_n x)=N(-x)\,, \qquad x>0\,, \tag{8.3}$$

and

$$\lim_{\varepsilon\to 0}\overline{\lim_{n\to\infty}}\, N(n)\left\{\int_{|x|<\varepsilon} x^2 dF(B_n x)-\left(\int_{|x|<\varepsilon} x dF(B_n x)\right)^2\right\}=0\,. \tag{8.4}$$

Now we have

THEOREM 9. *In order that a distribution F belongs to the domain of partial attraction of a semi-stable distribution corresponding to* (0.5) *with $0<\alpha<2$, it is necessary and sufficient that there exists a sequence $\{a_n\}$ of positive numbers tending to infinity such that for $x>0$*

(8.5)
$$\lim_{n\to\infty} \frac{1-F(a_n x)}{1-F(a_n x)+F(-a_n x)} = \frac{f(-\log x)}{1+f(-\log x)}$$

where

$$f(t) \in \boldsymbol{P}^+((\log \gamma)/\alpha) ,$$

and

(8.6)
$$\lim_{n\to\infty} \frac{1-F(a_n r^{p/\alpha} x)+F(-a_n r^{p/\alpha} x)}{1-F(a_n x)+F(-a_n x)} = \gamma^{-p}$$

for each integer p.

PROOF. We first note that a complex valued function $\varphi(t)$ on the real line is a ch.f. which satisfies, except for the factor $e^{i\beta t}$, condition (0.5) if and only if it admits the representation (8.1) with $M(x)$ and $N(x)$ which are monotone non-decreasing and satisfy

(8.7)
$$M(x)=\gamma M(\gamma^{1/\alpha} x) , \qquad x>0 ,$$

(8.8)
$$N(x)=\gamma N(\gamma^{1/\alpha} x) , \qquad x<0 ,$$

(see [2], Section 58. See also Lemma 7 and the Proof of Theorem 8).

Necessity: A direct consequence of Lemma 10 and conditions (8.7) and (8.8).

Sufficiency: Write $\rho=(\log \gamma)/\alpha$ and $F_0(x)=1-F(x)+F(-x)$. Then the condition (8.6) becomes

(8.9)
$$F_0(a_n x e^{\rho p})=F_0(a_n x)e^{-\rho p\alpha}(1+o(1)) \qquad \text{as } n\to\infty .$$

Denote by M_n the integer part of $(F_0(a_n))^{-1}$. Then we have for $e^{\rho p}\leqq x<e^{\rho(p+1)}$,

$$M_n F_0(a_n e^{\rho(p+1)}) \leqq M_n F_0(a_n x) \leqq M_n F_0(a_n e^{\rho p}) .$$

Using (8.9) we obtain

(8.10)
$$x^{-\alpha}e^{-\rho\alpha} \leqq \underline{\lim}\, M_n F_0(a_n x) \leqq \overline{\lim}\, M_n F_0(a_n x) \leqq x^{-\alpha}e^{\rho\alpha} .$$

Making use of the diagonal method and the the fact the $M_n F_0(a_n x)$ $n=1, 2, \cdots$ are monotone, we can show that there exist a monotone non-increasing function $R(x)$ on $(0, \infty)$ and a subsequence $\{N(n)F_0(B_n x)\}$ of the sequence $\{M_n F_0(a_n x)\}$ such that at all points x of continuity of $R(x)$

(8.11)
$$R(x)=\lim_{n\to\infty} N(n)F_0(B_n x) .$$

It follows from condition (8.9) that $R(x)$ is of the form

(8.12)
$$R(x)=x^{-\alpha}\xi(-\log x) , \qquad \xi(t) \in \boldsymbol{P}^+(\rho) .$$

The functions

$$M(x)=-\frac{f(-\log x)}{1+f(-\log x)}R(x)\,, \qquad x>0$$

and

$$N(x)=\frac{1}{1+f(-\log|x|)}R(|x|)\,, \qquad x<0$$

satisfy equations (8.7) and (8.8), and we see from condition (8.5) that (8.2) and (8.3) are both fulfilled. The proof of (8.4) proceeds along the same line as that of Theorem 2 of Section 35, [1]. See also Theorem 9 of the next section.

9. Some properties of semi-stable distributions

THEOREM 10. *Suppose that F belongs to the domain of partial attraction of a semi-stable distribution corresponding to* (0.5).

Then the moment of positive order, $E(|X|^\beta)$ of F exists if and only if $\alpha>\beta$.

PROOF. We assume without loss of generality that the distribution F is symmetric about zero. Then the condition (8.6) of Theorem 9 takes on the form

$$1-F(a_n xe^{\rho p})=e^{-\rho p\alpha}(1-F(a_n x))(1+o(1)) \qquad \text{as } n\to\infty\,, \tag{9.1}$$

where

$$\rho=(\log\gamma)/\alpha\,.$$

Let ε be a sufficiently small positive number. Since $e^{-\rho p\alpha}\to 0$ as $p\to\infty$, we have, for sufficiently large n,

$$1-F(a_n e^{\rho p})=e^{-\rho p\alpha}(1-F(a_n))(1+\varepsilon_p)\,, \tag{9.2}$$

where

$$0\leqq|\varepsilon_p|\leqq\varepsilon\,, \qquad p=1,2,\cdots.$$

We assume (9.2) and let ε and n be fixed.

On the one hand,

$$\begin{aligned}\int_{a_n e^{\rho p}}^{a_n e^{\rho(p+1)}} x^\beta dF(x) &\leqq a_n^\beta e^{\rho(p+1)\beta}(1-F(a_n e^{\rho p}))\\ &=a_n^\beta e^{\rho\beta}(1-F(a_n))(1+\varepsilon_p)e^{\rho p(\beta-\alpha)}\\ &\leqq C(e^{\rho(\beta-\alpha)})^p\,,\end{aligned} \tag{9.3}$$

where

$$C=a_n^\beta e^{\rho\beta}(1-F(a_n))(1+\varepsilon)\,,$$

and on the other,

$$(9.4)\qquad \int_{a_n e^{\rho p}}^{a_n e^{\rho(p+1)}} x^\beta dF(x) \geqq a_n^\beta e^{\rho\beta}[(1-F(a_n e^{\rho p}))-(1-F(a_n e^{\rho(p+1)}))]$$
$$=a_n^\beta e^{\rho\beta p}\{(1-F(a_n))e^{-\rho p\alpha}(1+\varepsilon_p) - (1-F(a_n))e^{-\rho\alpha(p+1)}(1+\varepsilon_p)\}$$
$$\geqq D(e^{\rho(\beta-\alpha)})^p\,,$$

where

$$D=(1-F(a_n))a_n^\beta(1-e^{-\rho\alpha}-2\varepsilon)>0\,.$$

The theorem is a consequence of (9.3) and (9.4).

COROLLARY. *The semi-stable distribution F corresponding to* (0.5) *has a moment $E(|X|^\beta)$ of positive order if and only if $\alpha>\beta$.*

PROOF. Let $N(n)$ be the integer part of γ^n-1: $N(n)=[\gamma^n-1]=[\gamma^n]-1$. Then we have from (0.5)

$$\varphi(t)=\varphi^{\gamma^n}(\gamma^{-n/\alpha}t)$$
$$=\varphi^{N(n)}(\gamma^{-n/\alpha}t)\cdot\psi_n(t)$$

where

$$\psi_n(t)=\varphi^{\gamma^n-N(n)}(\gamma^{-n/\alpha}t)\,.$$

Clearly as $n\to\infty$, $|\psi_n(t)|\leqq|\varphi(\gamma^{-n/\alpha}t)|\to1$. It follows that

$$\lim_{n\to\infty}\varphi^{N(n)}(\gamma^{-n/\alpha}t)=\varphi(t)\,.$$

This means that F belongs to the domain of partial attraction of itself. (See also [7] and [9].) (q.e.d.)

The following theorem is essentially contained in [7].

THEOREM 11. *Distribution function F of a non-degenerate semi-stable distribution corresponding to* (0.5) *is differentiable infinitely many times. If $\alpha=1$ it is analytic in a neighbourhood of $x=0$, while if $\alpha>1$ it is an entire function.*

PROOF. It follows from (0.5) that

$$(9.5)\qquad |\varphi(t)|=\exp\{-\xi(t)|t|^\alpha\}\,,$$

where

$$\xi(t)=\xi(-t)=\xi(\gamma^{-1/\alpha}t)>0 ,$$

and that

(9.6) $$N\equiv\inf_{|t|\geq 1}\xi(t)>0 .$$

By the inversion formula,

(9.7) $$F(x)-F(x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx_0}-e^{-itx}}{it}\varphi(t)dt .$$

Differentiating both sides of (9.7) formally $n+1$ times,

(9.8) $$p^{(n)}(x)\equiv F^{(n+1)}(x)=\frac{(-1)^n}{2\pi}\int_{-\infty}^{\infty}t^n e^{-itx}\varphi(t)dt .$$

Using (9.5) and (9.6) we obtain,

$$|p^{(n)}(x)|\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}t^n e^{-\xi(t)|t|^\alpha}dt\leqq\frac{1}{\pi}\left[\int_0^1 t^n dt+\int_1^\infty t^n e^{-Nt^\alpha}dt\right]$$
$$\leqq\frac{1}{\pi\alpha}\Gamma\left(\frac{n+1}{\alpha}\right)N_0^{-(n+1)/\alpha}<\infty ,$$

where N_0 is taken to be small. By the Stirling formula,

$$\lim_{n\to\infty}\frac{1}{n!}\left[\frac{1}{\pi\alpha}\Gamma\left(\frac{n+1}{\alpha}\right)N_0^{-(n+1)/\alpha}\right]^{1/n}=\begin{cases}0 & \text{if } \alpha>1\\ N_0^{-1} & \text{if } \alpha=1 .\end{cases}$$

(q.e.d.)

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REFERENCES

[1] B. V. Gnedenko and A. N. Kolomogorov, *Limit distributions for sum of independent random variables*, Moscow 1949; English translation, Addison-Wesley, Cambridge, Mass., 1954.

[2] P. Lévy, *Théorie de l'Addition des variables aléatoires*, Gauthier-Villars, Paris, 1937.

[3] Yu V. Linnik, "Linear forms and statistical criteria," *Ukrain. Math. Zurnal*, 5 (1953), English translation; *Selected translations in Math. Statist. Prob.*, 3 (1962), 1-90.

[4] R. G. Laha and E. Lukacs, "On a linear form whose distribution is identical with that of a monomial," *Pacific Journal of Math.*, 15 (1965), 207-214.

[5] E. Lukacs, *Characteristic Functions*, Griffin's Statist. Monos., London, 1960.

[6] E. Lukacs and R. G. Laha, *Application of Characteristic Functions*, Griffin's Statist. Monos., London, 1964.

[7] R. Shimizu, "Certain class of infinitely divisible characteristic functions," *Ann. Inst. Statist. Math.*, 17 (1965), 115-132.

[8] R. Shimizu, "Characteristic functions satisfying a functional equation (I)," *Ann. Inst. Statist. Math,*, 20 (1968), 187-209.

[9] B. Ramachandran and C. R. Rao, "Some results on characteristic functions and characterizations of the normal and generalized stable law," *Sankhyā*, 30 (1968), 125-140.

CORRECTIONS TO

"CHARACTERISTIC FUNCTIONS SATISFYING A FUNCTIONAL EQUATION (II)"

Ryoichi Shimizu

The author have found some errors in his paper with the above mentioned title, published in the *Ann. Inst. Statist. Math.*, 21 (1969), 391–405.

1. In order to obtain (6.35), it was implicitly supposed that $\varepsilon_n(t)\leqq\delta_n(A)/\sigma_n(A)$, which is not true (or at least needs proof). Lemma 9 and Theorem 7 should be omitted. The author is much indebted to Prof. B. Ramachandran for his attention to this point.

2. Equation (9.2), on which the proofs of Theorems 9–10 depend, is not true. So we replace them by

THEOREM 9′. *Suppose that a distribution F satisfies conditions* (8.5) *and* (8.6) *with* $a_n=\gamma^{n/\alpha}$, $0<\alpha<2$. *Then F belongs to the domain of partial attraction of a semi-stable distribution corresponding to* (0.5). *F has a moment* $E(|X|^\beta)$ *of positive order if* $\beta<\alpha$, *while* $E(|X|^\beta)=\infty$ *if* $\beta>\alpha$.

Outline of the proof. Derivation of (8.2)–(8.3) needs no change. (8.6) with $a_n=\gamma^{n/\alpha}$ implies that for sufficiently large n,

$$(*)\qquad F_0(a^{n+p+1})=a^{-\alpha}F_0(a^{n+p})(1+\varepsilon_p)$$

or

$$(**)\qquad F_0(a^p x_0)=a^{-p\alpha}F_0(x_0)(1+\varepsilon_j)\,,$$

where $a=\gamma^{1/\alpha}=e^p$ (>1), $x_0\equiv a^n$, and $|\varepsilon_p|\leqq\varepsilon$, $p=1,2,\cdots$. Let ε be so small that $c\equiv(1+\varepsilon)/a^{\alpha-\beta}<1$ (when $\beta<\alpha$) and $d=(1-\varepsilon)/a^{\alpha-\beta}>1$ (when $\beta>\alpha$). Then we have $(a^px_0)^\beta F_0(a^px_0)\geqq x_0^\beta F_0(x_0)c^p$ (when $\beta<\alpha$) and, $(a^px_0)^\beta F_0(a^px_0)\leqq x_0^\beta F_0(x_0)d^p$ (when $\beta>\alpha$). The last part of the Theorem follows from these inequalities. This together with (**), in turn, proves that under the assumption of the Theorem, (8.4) holds.

3. Corollary to Theorem 10 needs some more explanation.

$\varphi(t)=\lim\limits_{n\to\infty}\varphi^{[\gamma^n]-1}(\gamma^{-n/\alpha}t)=\lim\limits_{n\to\infty}\varphi^{[\gamma^n]}(a^{-n}t)$ implies that $\lim\,[\gamma^n](1-F(a^{-n}x))=$

$-M(x)$ and $\lim\,[\gamma^n]F(-a^{-n}x)=N(-x)$, from which (8.5) and (8.6) with $a_n=\gamma^{n/\alpha}=a^n$ follow. Hence we have only to show that $E(|X|^\alpha)=\infty$. This follows from

$$a^{n\alpha}F_0(a^n)\sim[\gamma^n]F_0(a^n)\to -M(1)+N(-1)>0\,.$$

For further detailed discussion on this subject, see the subsequent paper "On the domain of partial attraction of semi-stable distributions" submitted to the present journal.