A QUEUEING SYSTEM WITH SEVERAL TYPES OF CUSTOMERS

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1. Introduction

Consider a single-server queueing system in which

(i) there are r types of customers

(ii) the arrivals are in batches such that each batch contains K_1 customers of the first type, K_2 customers of the second type, $\cdots$, K_r customers of the rth type with the total service times $X_1, X_2, \cdots, X_r$ respectively, where $\{K_1, K_2, \cdots, K_r, X_1, X_2, \cdots, X_r\}$ is a random vector

(iii) the structures $\{K_1, K_2, \cdots, K_r, X_1, X_2, \cdots, X_r\}$ for the different batches are mutually independent

(iv) the interarrival times of the batches are independent and identically distributed with distribution function $A(t)$

(v) the batch structures are independent of the interarrival times.

In the present paper the joint behaviour of the number of customers of each type served during a busy period and their total service time is studied. Furthermore, the explicit properties of the special cases with

$$dA(t)=\frac{\lambda^m}{(m-1)!}\exp(-\lambda t)t^{m-1}dt \qquad (t>0)$$

and

$$dA(t)=\exp(-\lambda t)\lambda dt \qquad (t>0)$$

are given.

Prior to the present work Welch [4] and the author [3] had studied the joint behaviour of the number of customers of each type served during a busy period and its length for the special case of the present queueing system with

$$dA(t)=\exp\left(-\sum_{i=1}^{r}\lambda_i t\right)\left(\sum_{i=1}^{r}\lambda_i\right)dt$$

$$P(K_j=1,\ K_i=0\ (i(\neq j)=1, 2, \cdots, r))=\frac{\lambda_j}{\sum_{i=1}^{r}\lambda_i} \qquad (j=1, 2, \cdots, r).$$

2. General queueing system

Let T_i $(i=1, 2, \cdots, r)$ be the total service time of the ith type in a busy period and N_i $(i=1, 2, \cdots, r)$ and N be the number of customers of the ith type and the number of batches served during it respectively. Then, defining $E(Q_1; Q_2)=E(Q_1 \mid Q_2)P(Q_2)$ if $P(Q_2)>0$ and $=0$ otherwise, we have for suitable θ_i and z_i $(i=1, 2, \cdots, r)$

$$(1) \qquad E\left(\prod_{i=1}^{r} e^{-\theta_i T_i} z_i^{N_i}; N=n\right)$$

$$=\begin{cases} \displaystyle\int\cdots\int dU(y_1;\cdot)dU(y_2;\cdot)\cdots dU(y_n;\cdot)dA(v_1)dA(v_2)\cdots dA(v_n) \\ \qquad y_1+y_2+\cdots+y_l-(v_1+v_2+\cdots+v_l)\geqq 0 \qquad (l=1, 2, \cdots, n-1) \\ \qquad y_1+y_2+\cdots+y_n-(v_1+v_2+\cdots+v_n)<0 \\ (G(\cdot))^n \displaystyle\int\cdots\int dB(y_1;\cdot)dB(y_2;\cdot)\cdots dB(y_n;\cdot)dA(v_1)dA(v_2)\cdots dA(v_n) \\ \qquad y_1+y_2+\cdots+y_l-(v_1+v_2+\cdots+v_l)\geqq 0 \qquad (l=1, 2, \cdots, n-1) \\ \qquad y_1+y_2+\cdots+y_n-(v_1+v_2+\cdots+v_n)<0 \\ (G(\cdot))^n P(N^*=n) \qquad (n=1, 2, \cdots), \end{cases}$$

where $U(y;\cdot)$ denotes

$$U(y; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)=E\left(\prod_{i=1}^{r} e^{-\theta_i X_i} z_i^{K_i}; X_1+X_2+\cdots+X_r \leqq y\right),$$

$G(\cdot)$ denotes

$$G(z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)=E\left(\prod_{i=1}^{r} e^{-\theta_i X_i} z_i^{K_i}\right),$$

$B(y;\cdot)$ denotes

$$B(y; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots \theta_r)=\frac{U(y; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)}{G(z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)},$$

and N^* denotes the number of customers served during a busy period for the queueing system $GI/G/1$ with the interarrival time distribution as $A(t)$ and the service time distribution as $B(t; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)$ (For $n=1$ $y_1+y_2+\cdots+y_l-(v_1+v_2+\cdots+v_l)\geqq 0$ $(l=1, 2, \cdots, n-1)$ is not to be read. Also note that for $t<0$ $A(t)$ and $U(t; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)$ equal 0).

Hence from Finch [1] it follows that the generating function of the Laplace-Stieltjes transforms $E\left(\prod_{i=1}^{r} e^{-\theta_i T_i}; N_i=n_i\ (i=1, 2, \cdots, r), N=n\right)$ is given by

$$(2)\qquad E\left\{\left(\prod_{i=1}^{r} e^{-\theta_i T_i} z_i^{N_i}\right) z^N\right\} = E\{(zG(z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r))^{N^*}\}$$

$$= 1 - \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{t=0}^{\infty} (1 - A_n(t)) dB_n(t; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)\right\}$$

where $A_n(t)$ is the n-fold convolution of $A(t)$ with itself, and

$$B_n(t; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)$$
$$= E\left(\prod_{i=1}^{r} e^{-\theta_i X_i^{(n)}} z_i^{K_i^{(n)}};\ X_1^{(n)} + X_2^{(n)} + \cdots + X_r^{(n)} \leqq t\right),$$

where $K_i^{(n)}$ $(i=1, 2, \cdots, r)$ denotes the number of customers of the ith type in a group of n batches and $X_i^{(n)}$ $(i=1, 2, \cdots, r)$ their total service time.

It now readily follows that the generating function of the Laplace-Stieltjes transforms $E(e^{-\theta T}; N_i = n_i\ (i=1, 2, \cdots, r), N=n)$, where $T = \sum T_i$ is the length of the busy period, is given by

$$(3)\qquad E\left\{\left(\prod_{i=1}^{r} z_i^{N_i}\right) \exp(-\theta T) z^N\right\}$$

$$= 1 - \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{t=0}^{\infty} (1 - A_n(t)) dB_n(t; z_1, z_2, \cdots, z_r; \theta, \theta, \cdots, \theta)\right\}.$$

3. Some special queueing systems

We shall now consider the following special cases:

(i) Special case with $dA(t) = \dfrac{\lambda^m}{(m-1)!} e^{-\lambda t} t^{m-1} dt$ $(t > 0)$

From Takács ([5] p. 108) it follows that the generating function

$$(4)\qquad E\left\{\left(\prod_{i=1}^{r} e^{-\theta_i T_i} z_i^{N_i}\right) z^N\right\}$$

$$= 1 - \prod_{k=1}^{m} (1 - \alpha_k(z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r; z))$$

$$(R(\theta_i) \geqq 0\ (i=1, 2, \cdots, r),\ |z_i| \leqq 1\ (i=1, 2, \cdots, r),\ |z| < 1)$$

where $\alpha_k(z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r; z)$ $(k=1, 2, \cdots, m)$ are the m distinct roots in the unit circle $|\alpha| < 1$ of the equation

$$\alpha^m = zG(z_1, z_2, \cdots, z_r; \theta_1 + \lambda - \lambda\alpha, \theta_2 + \lambda - \lambda\alpha, \cdots, \theta_r + \lambda - \lambda\alpha).$$

(It may be noted that the usage of the Lagrange expansion establishes that

$$\sum_{k=1}^{m} \log(1-\alpha_k(z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r; z))$$
$$=\sum_{n=1}^{\infty} \frac{mz^n}{(nm)!}\left(\frac{\partial}{\partial x}\right)^{nm-1}\left(G(z_1, z_2, \cdots, z_r; \theta_1+\lambda-\lambda x, \theta_2+\lambda-\lambda x, \cdots, \theta_r+\lambda-\lambda x)^n\left(\frac{-1}{1-x}\right)\right)\Big|_{x=0}$$

(For the details see Prabhu ([2] p. 154))). This result can also be obtained by using (2).

It follows immediately that the generating function

$$(5)\qquad E\left\{(e^{-\theta T}z^N)\prod_{i=1}^{r} z_i^{N_i}\right\}=1-\prod_{k=1}^{m}(1-\alpha_k(z_1, z_2, \cdots, z_r; \theta, \theta, \cdots, \theta; z)),$$
$$(R(\theta)\geqq 0,\ |z|<1,\ |z_i|\leqq 1\ (i=1, 2, \cdots, r)).$$

(ii) Special case with $dA(t)=\lambda\exp(-\lambda t)dt$ $(t>0)$

It is well known that for a queueing system $M/G/1$ with the inter-arrival time mean λ^{-1} and the service time distribution function $B(t)$ the probability that n customers are served during a busy period is given by

$$f_n=\int_0^\infty \exp(-\lambda y)\frac{(\lambda y)^{n-1}}{n!}dB_n(y)\,,\qquad (n\geqq 1)\,,$$

where $B_n(y)$ is the n-fold convolution of $B(y)$ with itself (see, for example, Finch [1]). Hence for $n=1, 2, \cdots$ the transform

$$(7)\qquad E\left(\prod_{i=1}^{r} e^{-\theta_i T_i} z_i^{N_i};\ N=n\right)$$
$$=\int_{y=0}^{\infty}\exp(-\lambda y)\frac{(\lambda y)^{n-1}}{n!}dB_n(y; z_1, z_2, \cdots, z_r; \theta_1, \theta_2, \cdots, \theta_r)$$
$$=\frac{\lambda^{n-1}}{n!}E\left(\left(\prod_{i=1}^{r} e^{-(\theta_i+\lambda)X_i^{(n)}} z_i^{K_i^{(n)}}\right)\left(\sum_{i=1}^{r} X_i^{(n)}\right)^{n-1}\right)$$

which establishes that for $n=1, 2, \cdots$

$$(8)\qquad P(N_i=n_i(i=1, 2, \cdots, r); T_i\leqq t_i\,(i=1, 2, \cdots, r); N=n)$$
$$=\frac{\lambda^{n-1}}{n!}E\left(e^{-\lambda\sum_{i=1}^{r} X_i^{(n)}}\left(\sum_{i=1}^{r} X_i^{(n)}\right)^{n-1};\ K_i^{(n)}=n_i\,(i=1, 2, \cdots, r); X_i^{(n)}\leqq t_i\,(i=1, 2, \cdots, r)\right).$$

It may further be seen that for $n=1, 2, \cdots$

$$(9)\qquad P(N_i=n_i\,(i=1, 2, \cdots, r); T\leqq t; N=n)$$
$$=\frac{\lambda^{n-1}}{n!}E(e^{-\lambda X(n)}(X(n))^{n-1}; X(n)\leqq t; K_i^{(n)}=n_i\,(i=1, 2, \cdots, r))$$

where $X(n)=\sum_{i=1}^{r} X_i^{(n)}$.

The reader may note that using the present method the joint behaviour of the number of customers of each type arriving during a busy period initiated by an occupation time of quantity x and their total service time can be studied.

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