INVESTIGATION OF THE MEAN WAITING TIME FOR QUEUEING SYSTEM WITH MANY SERVERS

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1. Introduction

In this paper, we shall consider two models of queueing system such as shown in the figure below. Model 1 is a $GI/M/S(\infty)$ type queueing system with S equal exponential service channels in parallel, each having mean service rate μ/S. Model 2 is a $GI/M/S-1(\infty)$ type queueing system with $(S-1)$ equal exponential service channels in parallel, each having mean service rate $\mu/(S-1)$. So, the total value of the mean

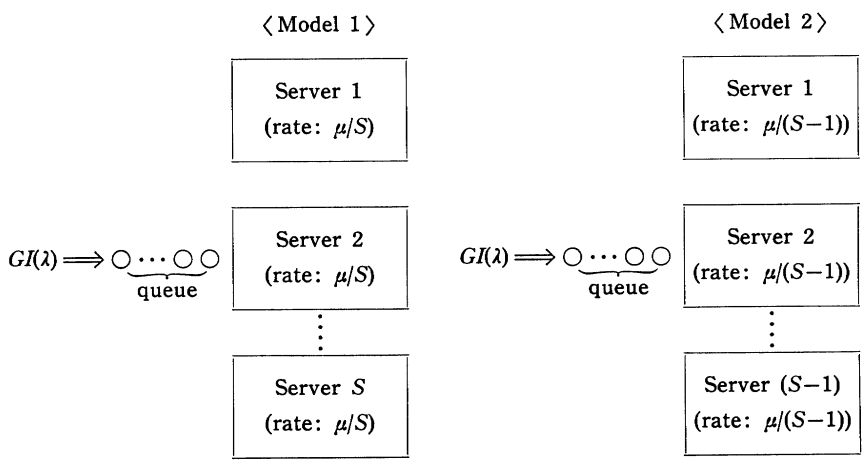

Fig.

service rate for Model 1 equals that for Model 2. The inter-arrival times of the customers are supposed to be independent and identically distributed with mean arrival rate λ. Customers are supposed to stay there in, and to make a single queue in the order of their arrival as the customer at the head of the queue enters the first queue that he finds unoccupied.

As to these systems it is known that the system has steady state probabilities if and only if $\lambda/\mu<1$ ([1], [3]).

In regard to the mean waiting time in the queue, $E[W_q^{(S)}]$ for Model 1 and $E[W_q^{(S-1)}]$ for Model 2, one may think it as likely that the inequality

$$E[W_q^{(S)}]<E[W_q^{(S-1)}]$$

holds. However, as for the mean waiting times in the systems, $E[W^{(S)}]$ for Model 1, and $E[W^{(S-1)}]$ for Model 2, we can show that the inequality

$$E[W^{(S)}]>E[W^{(S-1)}]$$

holds.

The inequalities mentioned above may easily be proved straightforwardly, in principle, by using the well-known result for the steady state probabilities. However, if one attempts to do this, one will face serious difficulties in calculation. Therefore, we shall compare the expected values of the waiting times in another way.

2. Comparison of the mean waiting times

Let us examine the queueing system of Model 1. First, notice that in a $GI/M/S$ type queueing system the epochs of arrival are points of regeneration. Let p_{ij} be the probability of transition from state E_i to state E_j. Here the system is said to be in state E_n if there are n customers at a given time. Then we have the transition matrix as follows;

$$P=\begin{bmatrix} p_{00} & p_{01} & 0 & \cdots\cdots & & & \\ p_{10} & p_{11} & p_{12} & \cdots\cdots & 0 & & \\ p_{20} & p_{21} & p_{22} & \ddots & \cdots & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ p_{S-1,0} & p_{S-1,1} & p_{S-1,2} & \cdots & k_1 & k_0 & 0 \cdots \\ p_{S,0} & p_{S,1} & p_{S,2} & \cdots & k_2 & k_1 & k_0 \cdots \\ p_{S+1,0} & p_{S+1,1} & p_{S+1,2} & \cdots & k_3 & k_2 & k_1 \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \end{bmatrix}$$

where $p_{ij}=k_{i-j+1}$ (for $i, j\geqq S-1$), i.e., k_r is the probability that precisely r customers depart during a single inter-arrival time. Let p_i $(i=0, 1, 2, \cdots)$ be the steady state probability, and consider the vector $\boldsymbol{p}'=(p_0, p_1, p_2, \cdots)$. Using the relation

$$\boldsymbol{p}'\boldsymbol{P}=\boldsymbol{p}' ,$$

we have the following equations:

$$p_0=\sum_{i=0}^{\infty} p_i p_{i0} , \tag{2.1}$$

(2.2) $$p_m=\sum_{i=m-1}^{\infty} p_i p_{i,m} \qquad (\text{for } S-2\geqq m\geqq 1),$$

(2.3) $$p_{S-1}=p_{S-2}p_{S-2,S-1}+\sum_{i=S-1}^{\infty} p_i k_{i+2-S},$$

(2.4) $$p_n=\sum_{i=n-1}^{\infty} p_i k_{i+1-n} \qquad (\text{for } n\geqq S).$$

Moreover, we have the equations

$$\sum_{j=0}^{i+1} p_{ij}=1 \qquad (\text{for } S-2\geqq i\geqq 0),$$

$$\sum_{j=0}^{S-2} p_{S-1+i,j}=l_{i+1} \qquad (\text{for } i\geqq 0),$$

where

$$l_n\equiv 1-\sum_{i=0}^{n} k_i .$$

Concerning the steady state probability q_i ($i=0, 1, 2, \cdots$) for Model 2 with $(S-1)$ servers, we similarly obtain the following equations:

(2.5) $$q_0=\sum_{i=0}^{\infty} q_i q_{i,0},$$

(2.6) $$q_m=\sum_{i=m-1}^{\infty} q_i q_{i,m} \qquad (\text{for } S-3\geqq m\geqq 1),$$

(2.7) $$q_{S-2}=q_{S-3}q_{S-3,S-2}+\sum_{i=S-2}^{\infty} q_i k_{i+3-S},$$

(2.8) $$q_n=\sum_{i=n-1}^{\infty} q_i k_{i+1-n} \qquad (\text{for } n\geqq S-1).$$

Now we shall start with the following

LEMMA. *For $n\geqq S-1$, the ratios*

$$\alpha=\frac{q_n}{p_n}, \qquad \beta=\frac{q_{n-1}}{p_n}$$

remain constant irrespective of the value of n, and we have

$$\alpha<1, \qquad \beta>1.$$

PROOF. By using the equations (2.1), (2.2) and (2.3), we have

(2.9) $$k_0 p_{S-1}=\sum_{i=S}^{\infty} p_i l_{i+1-S}.$$

From the equations (2.5), (2.6), (2.7) and (2.8), we have

$$k_0 q_{S-1} = \sum_{i=S}^{\infty} q_i l_{i+1-S} \,. \tag{2.10}$$

On the other hand, p_n, q_n for $n \geqq S$ can be written as follows;

$$p_n = A w^n \,, \qquad q_n = B w^n \,,$$

where w is a constant on which is independent of the value S for the present models ([3]). Therefore, we have

$$\frac{q_n}{p_n} = \frac{B}{A} = \alpha \qquad (\text{for } n \geqq S) \,.$$

By using (2.9) and (2.10), we get

$$\frac{q_{S-1}}{p_{S-1}} = \alpha \,.$$

Next we shall prove that α is less than unity.

If the value of α is not less than unity, then we have

$$\begin{aligned} p_{S-2} q_{S-2,S-1} &< p_{S-2} \cdot p_{S-2,S-1} \\ &= \sum_{i=S-1}^{\infty} p_i l_{i+2-S} \\ &= \frac{1}{\alpha} \sum_{i=S-1}^{\infty} q_i l_{i+2-S} \\ &= \frac{1}{\alpha} q_{S-2} q_{S-2,S-1} \\ &< q_{S-2} q_{S-2,S-1} \,, \qquad (\text{where } q_{S-2,S-1} = k_0) \end{aligned}$$

by using the equations

$$p_{i,i+1} > k_0 \qquad (\text{for } S-2 \geqq i) \,,$$

and

$$q_{i,i+1} < p_{i,i+1} \qquad (\text{for } S-2 \geqq i) \,.$$

Thus we have

$$p_{S-2} < q_{S-2} \,.$$

By the same argument we can obtain

$$p_{S-3} q_{S-3,S-2} < p_{S-3} p_{S-3,S-2} = p_{S-2} \sum_{j=0}^{S-3} p_{S-2,j} + p_{S-1} \sum_{j=0}^{S-3} p_{S-1,j} + \cdots \,.$$

Using the inequality

$$\sum_{k=0}^{\tau} p_{ik} < \sum_{k=0}^{\tau} q_{ik} \qquad (\text{for } \min(k, S-1) - 1 \geqq \tau \geqq 0) \,,$$

we have

$$p_{S-2}\sum_{j=0}^{S-3}p_{S-2,j}+p_{S-1}\sum_{j=0}^{S-3}p_{S-1,j}+p_S\sum_{j=0}^{S-3}p_{S,j}+\cdots$$
$$<p_{S-2}\sum_{j=0}^{S-3}q_{S-2,j}+p_{S-1}\sum_{j=0}^{S-3}q_{S-1,j}+p_S\sum_{j=0}^{S-3}q_{S,j}+\cdots$$
$$=p_{S-2}\sum_{j=0}^{S-3}q_{S-2,j}+\frac{1}{\alpha}\left\{q_{S-1}\sum_{j=0}^{S-3}q_{S-1,j}+q_S\sum_{j=0}^{S-3}q_{S,j}+\cdots\right\},$$

from which we get

$$p_{S-3}<q_{S-3}.$$

Repeating this process, we get

$$p_i<q_i \qquad (\text{for } S-2\geqq i\geqq 0).$$

Hence we have

$$1=\sum_{i=0}^{\infty}q_i>\sum_{i=0}^{S-2}p_i+\alpha\sum_{i=S-1}^{\infty}p_i=1+(\alpha-1)\sum_{i=S-1}^{\infty}p_i .$$

This is contradictory to the assumption $\alpha\geqq 1$. Accordingly, we have $\alpha<1$.

Now we shall prove $\beta>1$, since it is clear that β is a constant. First we can obtain the relation

$$q_{S-3}<\beta p_{S-2},$$

using the expression

$$q_{S-3,S-2}q_{S-3}=\sum_{i=S-2}^{\infty}q_il_{i+3-S}$$
$$=\beta\sum_{i=S-1}^{\infty}p_il_{i+2-S}=\beta p_{S-2}p_{S-2,S-1},$$

and the inequality

$$q_{S-3,S-2}>p_{S-2,S-1}.$$

Then we get the inequality

$$q_{S-4}<\beta p_{S-3}.$$

Because we have the relation

$$q_{S-4,S-3}q_{S-4}=q_{S-3}\sum_{j=0}^{S-4}q_{S-3,j}+q_{S-2}\sum_{j=0}^{S=4}q_{S-2,j}+q_{S-1}\sum_{j=0}^{S-4}q_{S-1,j}+\cdots$$
$$<\beta\left\{p_{S-2}\sum_{j=0}^{S-4}q_{S-3,j}+p_{S-1}\sum_{j=0}^{S-4}q_{S-2,j}+p_S\sum_{j=0}^{S-4}q_{S-1,j}+\cdots\right\}$$

$$<\beta\left\{p_{S-2}\sum_{j=0}^{S-3}p_{S-2,j}+p_{S-1}\sum_{j=0}^{S-3}p_{S-1,j}+p_S\sum_{j=0}^{S-3}p_{S,j}+\cdots\right\}$$
$$=\beta p_{S-3}p_{S-3,S-2}$$
$$<\beta p_{S-3}q_{S-4,S-3}$$

since the inequality

$$\sum_{j=0}^{S-4}q_{S-3+r,j}<\sum_{j=0}^{S-3}p_{S-2+r,j} \qquad (\text{for } r\geqq 0)$$

holds.

Continuing this iteration, we have for $S-3\geqq i\geqq 0$

$$q_i<\beta p_{i+1},$$

which yields

$$\beta>1.$$

COROLLARY. *For $n\geqq S-2$, the inequality*

$$q_n>p_{n+1}>q_{n+1}$$

holds.

Now we shall prove the following

THEOREM 1. *In regard to the mean waiting time for Model* 1 *and Model* 2, *we have $E[W^{(S)}]>E[W^{(S-1)}]$.*

PROOF. For the mean waiting time, in the systems we have

$$E[W^{(S)}]=\frac{1}{\mu}\left\{S\sum_{i=0}^{S-2}p_i+\sum_{i=S-1}^{\infty}(i+1)p_i\right\} \tag{2.11}$$

and

$$E[W^{(S-1)}]=\frac{1}{\mu}\left\{(S-1)\sum_{i=0}^{S-2}q_i+\sum_{i=S-1}^{\infty}(i+1)q_i\right\}. \tag{2.12}$$

When we consider the case

$$S\sum_{i=0}^{S-2}p_i\geqq(S-1)\sum_{i=0}^{S-2}q_i,$$

we have

$$E[W^{(S)}]>E[W^{(S-1)}],$$

since

$$q_n/p_n=\alpha<1 \qquad (\text{for } n\geqq S-1)$$

holds. So, it remains to investigate the case

$$S\sum_{i=0}^{S-2} p_i<(S-1)\sum_{i=0}^{S-2} q_i\,. \tag{2.13}$$

In this case, the inequality

$$S\sum_{i=S-1}^{\infty} p_i>S\sum_{i=S-1}^{\infty} q_i+\sum_{i=0}^{S-2} q_i \tag{2.14}$$

holds. Thus we have

$$\mu\{E[W^{(S)}]-E[W^{(S-1)}]\}>S\sum_{i=0}^{S-2} p_i-(S-2)\sum_{i=0}^{S-2} q_i$$

using the inequality (2.14) and $p_n>q_n$ (for $n\geqq S-1$).

Hence if the relation

$$S\sum_{i=0}^{S-2} p_i\geqq(S-2)\sum_{i=0}^{S-2} q_i$$

is satisfied, we obtain

$$E[W^{(S)}]>E[W^{(S-1)}]\,.$$

On the other hand, if

$$S\sum_{i=2}^{S-2} p_i<(S-2)\sum_{i=0}^{S-2} q_i\,,$$

we have

$$S\sum_{i=S-1}^{\infty} p_i>S\sum_{i=S-1}^{\infty} q_i+2\sum_{i=0}^{S-2} q_i\,.$$

Continuing this iteration, we can find the following fact;

In order to complete the proof, we have only to prove the relation

$$E[W^{(S)}]>E[W^{(S-1)}]$$

when

$$S\sum_{i=0}^{S-2} p_i<\sum_{i=0}^{S-2} q_i\,.$$

In this case, however, we have

$$S\sum_{i=S-1}^{\infty} p_i>S\sum_{i=S-1}^{\infty} q_i+(S-1)\sum_{i=0}^{S-2} q_i\,.$$

Hence we have

$$\begin{aligned}\mu\{E[W^{(S)}]-E[W^{(S-1)}]\}&>S\sum_{i=0}^{S-2} p_i-(S-1)\sum_{i=0}^{S-2} q_i+(S-1)\sum_{i=0}^{S-2} q_i\\&>0\,.\end{aligned}$$

THEOREM 2. *We have the inequality*

$$E[W_q^{(S)}] < E[W_q^{(S-1)}],$$

concerning the mean waiting time in the queue.

PROOF This follows from the relation

$$\begin{aligned}&\mu\{E[W_q^{(S-1)}]-E[W_q^{(S)}]\}\\&=\{(q_{S-1}-p_S)+2(q_S-p_{S+1})+3(q_{S+1}-p_{S+2})+\cdots\}\\&=(\beta-1)\{p_S+2p_{S+1}+3p_{S+2}+\cdots\}>0.\end{aligned}$$

3. The case of Poisson arrivals

In the case of Poisson arrivals, we can compare explicitly the variance of the waiting time as well as the mean waiting time in the systems.

First, let us consider the queueing system of Model 1. Then we have the moment generating function for the waiting time W,

$$M_W(\theta)=\left(\frac{\mu}{\mu-S\theta}\right)\left\{(1-\pi^{(S)})+\frac{\mu(1-\rho)\pi^{(S)}}{\mu(1-\rho)-\theta}\right\}, \tag{3.1}$$

where

$$\pi^{(S)}=\frac{p_0^{(S)}(S\rho)^S}{S!(1-\rho)},$$

$$p_0^{(S)}=\frac{1}{\sum_{n=0}^{S-1}\frac{(S\rho)^n}{n!}+\frac{(S\rho)^S}{S!(1-\rho)}},$$

$$\rho=\lambda/\mu.$$

Using (3.1) we get the expected value $E[W^{(S)}]$ and the variance $V[W^{(S)}]$ of the waiting time,

$$E[W^{(S)}]=\frac{(S\rho)^S p_0^{(S)}}{S!(1-\rho)^2\mu}+\frac{S}{\mu}, \tag{3.2}$$

$$V[W^{(S)}]=\frac{S^2(1-\rho)^2+\pi^{(S)}(2-\pi^{(S)})}{\mu^2(1-\rho)^2}. \tag{3.3}$$

Putting

$$S!\sum_{n=0}^{S-1}\frac{(S\rho)^n}{n!}=B^{(S)},$$

$$(S-1)!\sum_{n=0}^{S-2}\frac{\{(S-1)\rho\}^n}{n!}=B^{(S-1)},$$

$$B^{(S)}(1-\rho)=A^{(S)},$$

$$B^{(S-1)}(1-\rho)=A^{(S-1)},$$

we have

$$E[W^{(S)}]=\frac{1}{\mu(1-\rho)}\left\{\frac{(S\rho)^S}{A^{(S)}+(S\rho)^S}+(1-\rho)S\right\}$$

and

$$E[W^{(S-1)}]=\frac{1}{\mu(1-\rho)}\left\{\frac{((S-1)\rho)^{S-1}}{A^{(S-1)}+((S-1)\rho)^{S-1}}+(1-\rho)(S-1)\right\}.$$

By simple calculation, we obtain

$$\begin{aligned}&E[W^{(S)}]-E[W^{(S-1)}]\\&=\left[S!\left\{\sum_{n=0}^{S-2}\frac{(S(S-1)\rho)^n(S^{S-n-1}-(S-1)^{S-n-1})}{n!}\right\}\rho^S\right.\\&\quad\left.+\{B^{(S-1)}(S\rho)^S(1-\rho)+A^{(S)}A^{(S-1)}\}\right]\\&\quad\times[\mu\{A^{(S)}+(S\rho)^S\}\{A^{(S-1)}+((S-1)\rho)^{S-1}\}]^{-1}\\&>0.\end{aligned}$$

Next we shall show

$$V[W^{(S)}]>V[W^{(S-1)}].$$

For that, it is sufficient to prove

$$\frac{\pi^{(S)}(2-\pi^{(S)})}{(1-\rho)^2}>-2S+1+\frac{\pi^{(S-1)}(2-\pi^{(S-1)})}{(1-\rho)^2},\tag{3.4}$$

since

$$V[W^{(S)}]=\frac{S^2(1-\rho)^2+\pi^{(S)}(2-\pi^{(S)})}{\mu^2(1-\rho)^2}.$$

Using

$$E[W^{(S)}]>E[W^{(S-1)}],$$

we have

$$\pi^{(S-1)}-\pi^{(S)}<1-\rho.\tag{3.5}$$

Therefore, if we show

$$2S-1 \geqq \frac{2-(\pi^{(S)}+\pi^{(S-1)})}{1-\rho},$$

then the proof is completed. For that, it is sufficient to prove

$$(S-1)(1-\rho) \geqq 1-\pi^{(S-1)}.$$

By simple calculation, we obtain

$$\begin{aligned} &1-\pi^{(S-1)}-(S-1)(1-\rho) \\ &=(1-\rho)\{(S-1)!\}\left\{\sum_{n=0}^{S-2} \frac{((S-1)\rho)^n}{n!}(n+2-S)\right\} \\ &\quad \times\left\{\sum_{n=0}^{S-2} \frac{((S-1)\rho)^n}{n!}(S-1)!(1-\rho)+((S-1)\rho)^{S-1}\right\}^{-1}. \end{aligned}$$

This means that

$$(S-1)(1-\rho) \geqq 1-\pi^{(S-1)},$$

from which the inequality

$$V[W^{(S)}] > V[W^{(S-1)}]$$

follows.

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