ON THE ε-ENTROPY OF DIFFUSION PROCESSES

KIMIO KAZI

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Summary

In the present paper we give an evaluation of ε-entropy of a one dimensional diffusion process $\xi(t)$, $0 \leqq t \leqq T$ whose generator is

$$\mathfrak{G} = \frac{1}{2} a(x)^2 \frac{d^2}{dx^2} \qquad (x \in R), \tag{1}$$

where the diffusion coefficient $a(x)^2$ satisfies

$$|a(x) - a(y)| \leqq L|x-y|, \tag{2}$$

$$0 < k \leqq a(x)^2 \leqq K \tag{3}$$

for every x and y (L, k and K are positive constants). We assume $\xi(0)=0$ for simplicity of calculation. Then we can prove the following:

THEOREM. *Under the conditions (2) and (3), ε-entropy $H_\varepsilon(\{\xi(t)\})$ of $\xi(t)$, $0 \leqq t \leqq T$ is asymptotically evaluated for small $\varepsilon > 0$,*

$$\frac{k}{eK} \cdot \frac{1}{\pi^2} \left\{ \int_0^T \sqrt{Ea(\xi(u))^2}\, du \right\}^2 \frac{1}{\varepsilon^2} \precsim H_\varepsilon(\{\xi(t)\}) \precsim \frac{2}{\pi^2} \left\{ \int_0^T \sqrt{Ea(\xi(u))^2}\, du \right\}^2 \frac{1}{\varepsilon^2}. \text{*}$$

Previously Kolmogorov stated in [3] without proof "For a diffusion process whose generator is given by (1) $H_\varepsilon(\{\xi(t)\})$ is calculated by the formula:

$$H_\varepsilon(\{\xi(t)\}) = \frac{4}{\pi} \left\{ \int_0^T Ea(\xi(u))^2 du \right\} \frac{1}{\varepsilon^2} + o\left(\frac{1}{\varepsilon^2}\right) \qquad (\varepsilon \to 0)$$

under certain natural conditions". However, in consideration of Pinsker's results for Gaussian processes [5] and our present theorem this formula appears inaccurate. For the proof of the theorem we use the well-known formula of ε-entropy for finite dimensional random variables (lemma 3).

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* $f(\varepsilon) \precsim g(\varepsilon)$ means $\overline{\lim_{\varepsilon \to 0}} \dfrac{f(\varepsilon)}{g(\varepsilon)} \leqq 1$.

1. In this section we prepare several lemmas for a general stochastic process $\xi(t)$ and indicate the general line of the proof of the theorem. The ε-entropy of a real-valued stochastic process $\boldsymbol{\xi}=\{\xi(t)\}_{0\leqq t\leqq T}$, $E\int_0^T \xi(u)^2 du < \infty$, is by definition

$$H_\varepsilon(\boldsymbol{\xi})=\inf_{\boldsymbol{\eta}=\{\eta(t)\}}\left\{I(\boldsymbol{\xi},\boldsymbol{\eta})\,;\,E\int_0^T|\xi(u)-\eta(u)|^2du\leqq\varepsilon^2\right\},$$

where $I(\boldsymbol{\xi},\boldsymbol{\eta})$ is Kolmogorov's amount of information [3] and the infimum is taken over all measurable processes $\boldsymbol{\eta}=\{\eta(t)\}_{0\leqq t\leqq T}$. If $\boldsymbol{\xi}=(\xi_1,\xi_2,\cdots)$ with $E\sum_{j=1}^\infty \xi_j^2<\infty$ is a l^2-valued random variable, $H_\varepsilon(\boldsymbol{\xi})$ is defined by

$$H_\varepsilon(\boldsymbol{\xi})=\inf_{\boldsymbol{\eta}=(\eta_1,\cdots)}\left\{I(\boldsymbol{\xi},\boldsymbol{\eta})\,;\,E\sum_{j=1}^\infty|\xi_j-\eta_j|^2\leqq\varepsilon^2\right\}.$$

In the case of a finite dimensional random variable $\boldsymbol{\xi}=(\xi_1,\cdots,\xi_n)$ with $E\sum_{j=1}^n\xi_j^2<\infty$, the infimum of $I(\boldsymbol{\xi},\boldsymbol{\eta})$ shall be taken over all $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_n)$ with $E\sum_{j=1}^n|\xi_j-\eta_j|^2\leqq\varepsilon^2$. We can reduce the calculation of the ε-entropy of a stochastic process to that of infinite dimensional random variable by the following lemma 1, which is due to Y. Baba.

LEMMA 1. *If $\xi(t)$ $(0\leqq t\leqq T)$, $E\xi(t)=0$, $E\int_0^T\xi(u)^2du<\infty$, is a mean continuous process with covariance function $r(t,s)=E\{\xi(t)\xi(s)\}$, then*

$$H_\varepsilon(\{\xi(t)\})=H_\varepsilon((\xi_1,\xi_2,\cdots)),$$

where $\xi_j=\int_0^T\xi(u)\varphi_j(u)du$ and $\varphi_j(t)$ is the jth eigenfunction of the integral equation:

$$\int_0^T r(t,s)\varphi(s)ds=\sigma\varphi(t)\,. \tag{4}$$

PROOF. As is well known, the integral operator R with kernel $r(t,s)$ defines a symmetric, positive definite, completely continuous operator in $L^2[0,T]$. If we arrange the eigen values of R as $\sigma_1\geqq\sigma_2\geqq\cdots\to0$, we know immediately from general theory

$$E\xi_j=0\,,\qquad E\xi_i\xi_j=\sigma_j\delta_{ij}\,, \tag{5}$$

$$E\sum_{j=1}^\infty\xi_j^2=\sum_{j=1}^\infty\sigma_j=E\int_0^T\xi(u)^2du<\infty\,.$$

Now take an arbitrary measurable process $\eta(t)$ $(0\leqq t\leqq T)$ such that

$$E\int_0^T|\xi(u)-\eta(u)|^2du\leqq\varepsilon^2\,,$$

and define

$$\eta_j=\int_0^T \eta(u)\varphi_j(u)du\,.$$

Obviously, it holds that

$$(6)\qquad E\sum_{j=1}^{\infty}|\xi_j-\eta_j|^2=E\int_0^T |\xi(u)-\eta(u)|^2du\leqq\varepsilon^2\,.$$

Since the linear operation from $L^2[0,T]$ to l^2:

$$\zeta(\cdot)\to(\zeta_1,\zeta_2,\cdots)\,,\qquad \zeta_j=\int_0^T \zeta(u)\varphi_j(u)du$$

is one to one, onto and bimeasurable mapping*, we have

$$(7)\qquad I(\{\xi(t)\},\{\eta(t)\})=I((\xi_1,\xi_2,\cdots),(\eta_1,\eta_2,\cdots))$$

in view of a fundamental property of amount of information: a one to one, onto and bimeasurable mapping preserves amount of information [3]. From (6) and (7) the result of the lemma follows at once.

As to a l^2-valued random variable we have the following:

LEMMA 2. *Let a random variable* $(\xi_1,\xi_2,\cdots)$ *satisfy* (5) *and* $\sum_{j=1}^{\infty}\sigma_j<\infty$. *If* $\varepsilon^2\leqq\sum_{j=1}^{\infty}\sigma_j$, *following inequalities hold;*

$$H_\varepsilon((\xi_1,\xi_2,\cdots))\geqq H_\varepsilon((\xi_1,\cdots,\xi_n))\qquad \textit{for every } n,$$

$$H_\varepsilon((\xi_1,\xi_2,\cdots))\leqq H_\theta((\xi_1,\cdots,\xi_m))\,,$$

where m *is an arbitrary integer such that* $\varepsilon^2-\sum_{j=m+1}^{\infty}\sigma_j=\theta^2>0$.

PROOF. The first inequality is trivial according to the definition of ε-entropy and a property of amount of information [3]. As to the second, we see easily from the definition

$$\begin{aligned}&H_\varepsilon((\xi_1,\xi_2,\cdots))\\ &\leqq \inf_{(\eta_1,\cdots,\eta_m)}\Big\{I((\xi_1,\xi_2,\cdots),(\eta_1,\cdots,\eta_m,0,0,\cdots));\\ &\qquad\qquad\qquad E\sum_{j=1}^{m}|\xi_j-\eta_j|^2\leqq\varepsilon^2-\sum_{j=m+1}^{\infty}\sigma_j\Big\}\\ &\leqq \inf_{(\eta_1,\cdots,\eta_m)}\Big\{I((\xi_1,\xi_2,\cdots),(\eta_1,\cdots,\eta_m));\ E\sum_{j=1}^{m}|\xi_j-\eta_j|^2\leqq\theta^2\\ &\qquad \text{and random variables } (\xi_1,\xi_2,\cdots),\ (\xi_1,\cdots,\xi_m),\ (\eta_1,\cdots,\eta_m)\end{aligned}$$

* It is easily proved that we may consider $\zeta(t)$ $(0\leqq t\leqq T)$ with $E\int_0^T \zeta(t)^2dt<\infty$ to be a $L^2[0,T]$ valued random variable.

$$\left. \text{form a Markov chain} \right\}^{(\sharp)}$$

$$= \inf_{(\eta_1,\cdots,\eta_m)} \left\{ I((\xi_1,\cdots,\xi_m),(\eta_1,\cdots,\eta_m)) ; E\sum_{j=1}^{m}|\xi_j-\eta_j|^2 \leqq \theta^2 \right\}$$

$$= H_\theta((\xi_1,\cdots,\xi_m)) .$$

Here we note that the range of the infimum of ($\sharp$) is not empty. In fact we may give an example of the conditional density function of $(\eta_1,\cdots,\eta_m)$ with respect to $(\xi_1,\xi_2,\cdots)$:

$$p(y_1,\cdots,y_m|x_1,x_2,\cdots)=\frac{1}{(2\pi\theta^2\alpha m^{-1})^{m/2}}\exp\left\{-\frac{m}{2\alpha\theta^2}\sum_{j=1}^{m}(y_j-\alpha x_j)^2\right\},$$

$$\alpha=1-\frac{\theta^2}{\sum_{j=1}^{m}\sigma_j}>0.$$

LEMMA 3. *If n dimensional random variable $(\xi_1,\xi_2,\cdots,\xi_n)$ with $\beta=E\sum_{j=1}^{n}\xi_j^2<\infty$ has a bounded continuous density function $p(x_1,x_2,\cdots,x_n)$, then*

$$H_\varepsilon((\xi_1,\cdots,\xi_n))\geqq\frac{n}{2}\log\frac{1}{\varepsilon^2}+\frac{n}{2}\log n-n\log\sqrt{2\pi e}+h((\xi_1,\cdots,\xi_n)),$$

$$H_\varepsilon((\xi_1,\cdots,\xi_n))\leqq\frac{n}{2}\log\frac{1}{\varepsilon^2}+\frac{n}{2}\log n-n\log\sqrt{2\pi e\left(1-\frac{\varepsilon^2}{\beta}\right)}$$
$$+h((\xi_1,\cdots,\xi_n))+o(1)$$

as $\varepsilon\downarrow 0$, where $h((\xi_1,\cdots,\xi_n))=-\int_{R^n}p(x_1,\cdots,x_n)\log p(x_1,\cdots,x_n)dx_1\cdots dx_n$ is the differential entropy of $(\xi_1,\cdots,\xi_n)$.

For the proof of lemma 3, [2] and [4] should be referred. We omit the proof and note only that the latter inequality is a little stronger than the inequality in [4] which is proved under more general conditions about probability density function. We might rather derive these inequalities following ideas of [2].

Lemmas 1, 2 and 3 enable us to evaluate $H_\varepsilon(\{\xi(t)\})$ of a stochastic process from above and below with the help of the formulas of ε-entropy for finite dimensional random variables, i.e. the inequalities in lemma 3.

2. Throughout the rest of the paper let $\xi(t)$ $(0\leqq t\leqq T)$ be a diffusion process introduced in Summary. We may consider such $\xi(t)$ to be constructed by the stochastic integral equation:

$$\xi(t)=\int_0^t a(\xi(u))dB(u) \qquad (B(u) \text{ is a Brownian motion}).$$

Its covariance function

$$r(t, s)=\int_0^{t\wedge s} Ea(\xi(u))^2du \qquad (t\wedge s=\min\{t, s\})$$

is obviously continuous in t and s. We trace the argument of section 1. For this case, for the eigenvalues of the integral equation (4) we have

LEMMA 4.

$$\sigma_j=Cj^{-2}+o(j^{-2}), \qquad C=\frac{1}{\pi^2}\left\{\int_0^T \sqrt{Ea(\xi(u))^2}\,du\right\}^2.$$

PROOF. In this case (4) is equivalent to the boundary value problem of a second order differential equation:

$$\begin{cases} -\dfrac{d}{dt}\left\{\dfrac{1}{Ea(\xi(u))^2}\dfrac{d\varphi}{dt}\right\}=\dfrac{1}{\sigma}\varphi \\ \varphi(0)=\varphi'(T)=0. \end{cases}$$

The statement of the lemma follows from the result of ([1], p. 361).

Define random variables

$$\xi_j=\int_0^T \xi(u)\varphi_j(u)du \qquad (j=1, 2, \cdots),$$

where $\{\varphi_j(t)\}$ are continuous eigen functions corresponding to the integral equation (4). In order to investigate the characteristic function of $(\xi_1, \xi_2, \cdots, \xi_n)$:

$$\psi(\lambda_1, \lambda_2, \cdots, \lambda_n)=E\left\{\exp\left(i\sum_{j=1}^n \lambda_j\xi_j\right)\right\},$$

we introduce a process

$$\zeta(t)=\exp\left\{i\sum_{j=1}^n \lambda_j\int_t^T \xi(u)\varphi_j(u)du\right\} \qquad (0\leq t\leq T).$$

If we put $v(t, x)=E_{t,x}[\zeta(t)]$, denoting by $E_{t,x}$ the expectation with respect to the conditional probability $P_{t,x}(\cdot)=P(\cdot\,|\,\xi(t)=x)$, there holds a relation

$$\psi(\lambda_1, \lambda_2, \cdots, \lambda_n)=v(0, 0).$$

On the other hand, since for $u\geq t$

$$\xi(u)=\xi(t)+\int_t^u a(\xi(s))dB(s)$$

holds with probability one, we have almost surely

$$\zeta(t)=\exp\left\{i\xi(t)\sum_{j=1}^{n}\lambda_j f_j(t)\right\}\cdot\tilde{\zeta}(t)\,,$$
$$(8)\qquad \tilde{\zeta}(t)=\exp\left\{i\sum_{j=1}^{n}\lambda_j\int_0^T a(\xi(s))f_j(s)dB(s)\right\},$$
$$f_j(s)=\int_s^T \varphi_j(u)du\,.$$

We have used a fact that we can change the order of ordinary integral and stochastic integral in this case.

Taking expectations $E_{t,x}$ of both sides of (8), we obtain

$$v(t,x)=\exp\left\{ix\sum_{j=1}^{n}\lambda_j f_j(t)\right\}E_{t,x}[\tilde{\zeta}(t)]\,.$$

LEMMA 5. *$\tilde{v}(t,x)=E_{t,x}[\tilde{\zeta}(t)]$ is written as follows:*

$$(9)\qquad \tilde{v}(t,x)=E_{t,x}\left[\exp\left\{\int_0^T A(s,(\xi(s))ds\right\}\right],$$

where
$$A(t,x)=-\frac{1}{2}a(x)^2\sum_{j=1}^{n}\lambda_j^2 f_j^2(t)\,.$$

PROOF. We apply Ito's formula of stochastic integral to $\tilde{\zeta}(t)$ to get

$$\tilde{\zeta}(t)=\tilde{\zeta}(T)+\int_t^T A(s,\xi(s))\tilde{\zeta}(s)ds\,.$$

Taking expectations of both sides and using Markov property, we know $\tilde{v}(t,x)$ satisfies the following equation:

$$(10)\qquad \tilde{v}(t,x)=1+E_{t,x}\int_t^T A(s,\xi(s))\tilde{v}(s,\xi(s))ds\,.$$

Because of boundedness of $|A(t,x)|$ it is easily shown by usual iteration method that (10) has a unique solution (9).

On the basis of lemma 5 we obtain the following estimate about the characteristic function of $(\xi_1,\cdots,\xi_n)$:

LEMMA 6.
$$|\phi(\lambda_1,\lambda_2,\cdots,\lambda_n)|\leqq\exp\left\{-\frac{1}{2}\cdot\frac{k}{K}\sum_{j=1}^{n}\sigma_j\lambda_j^2\right\}.$$

PROOF. As $\phi(\lambda_1,\lambda_2,\cdots,\lambda_n)=v(0,0)=\tilde{v}(0,0)$, we see from lemma 5,

$$(11)\qquad |\phi(\lambda_1,\lambda_2,\cdots,\lambda_n)|\leqq E_{0,0}\left|\exp\left\{\int_0^T A(s,\xi(s))ds\right\}\right|$$
$$=E\exp\left\{-\frac{1}{2}\int_0^T a(\xi(s))^2\sum_{j=1}^{n}\lambda_j^2 f_j^2(s)ds\right\}$$

$$\leqq \exp\left\{-\frac{k}{2}\sum_{j=1}^{n}\lambda_j^2\int_0^T f_j(s)^2 ds\right\}.$$

On the other hand multiply by $\varphi_j(t)$ the both sides of the following integral equation, which is actually equation (4),

$$\sigma_j\varphi_j(t)=\int_0^t r(s)\varphi_j(s)ds+r(t)\int_t^T \varphi_j(s)ds ,$$

where $r(s)=\int_0^s Ea(\xi(u))^2du$, and integrate from 0 to T to get the relation

$$\begin{aligned}\sigma_j&=2\int_0^T r(t)\varphi_j(t)\left\{\int_t^T \varphi_j(s)ds\right\}dt\\ &=-\int_0^T r(t)\frac{d}{dt}\left\{\int_t^T \varphi_j(s)ds\right\}^2 dt\\ &=\int_0^T r'(t)\left\{\int_t^T \varphi_j(s)ds\right\}^2 dt=\int_0^T Ea(\xi(t))^2 f_j(t)^2 dt .\end{aligned}$$

Hence

$$\int_0^T f_j(t)^2dt \geqq \frac{\sigma_j}{K} .$$

This inequality together with (11) proves lemma 6.

We need a lemma estimating the differential entropy from above and below:

LEMMA 7. *n dimensional random variable $(\xi_1, \xi_2, \cdots, \xi_n)$ has a continuous bounded probability density function $p(x_1, x_2, \cdots, x_n)$ and its differential entropy $h((\xi_1, \xi_2, \cdots, \xi_n))$ is bounded from above and below as follows:*

$$h((\xi_1, \xi_2, \cdots, \xi_n)) \leqq \log \prod_{j=1}^{n} (2\pi e\sigma_j)^{1/2} ,$$

$$h((\xi_1, \xi_2, \cdots, \xi_n)) \geqq -\log \prod_{j=1}^{n}\left(\frac{K}{2\pi k\sigma_j}\right)^{1/2} .$$

PROOF. As the characteristic function $\phi(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a L^1-function of $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ from lemma 6, there exists a density function $p(x_1, x_2, \cdots, x_n)$ which is represented by its inverse Fourier transformation:

$$p(x_1, x_2, \cdots, x_n)=\frac{1}{(2\pi)^n}\int_{R^n}\exp\left(-i\sum_{j=1}^{n}\lambda_j x_j\right)\phi(\lambda_1, \lambda_2, \cdots, \lambda_n)d\lambda_1\cdots d\lambda_n .$$

Obviously $p(x_1, x_2, \cdots, x_n)$ is continuous and bounded:

$$M=\sup p(x_1, x_2, \cdots, x_n)$$

$$\leqq \frac{1}{(2\pi)^n}\int |\psi(\lambda_1, \lambda_2, \cdots, \lambda_n)| d\lambda_1 d\lambda_2 \cdots d\lambda_n$$

$$= \prod_{j=1}^{n}\left(\frac{K}{2\pi k\sigma_j}\right)^{1/2}.$$

The first inequality of the lemma follows at once from the fact that differential entropy takes its maximum value in case of Gaussian random variable when $E\xi_j=0$, and covariances $E\xi_i\xi_j=\sigma_j\delta_{ij}$ $(i, j=1, 2, \cdots, n)$. The second inequality follows from a trivial inequality $h((\xi_1, \xi_2, \cdots, \xi_n)) \geqq -\log M'$ for every $M' \geqq M$.

3. Now we proceed to the proof of the asymptotic inequalities of the theorem. As is verified in 2, lemma 1, 2 and 3 are applicable to the diffusion process $\xi(t)$ $(0\leqq t\leqq T)$. We combine inequalities shown in lemmas to get the desired evaluations.

(i) Evaluation from above;

Take a sufficiently small $\varepsilon^2 \leqq \sum_{j=1}^{\infty}\sigma_j$. For each m such that $\varepsilon^2 - \sum_{j=m+1}^{\infty}\sigma_j = \theta^2 > 0$ we have from lemmas 1, 2, 3, 7 and 4

$$H_\varepsilon(\{\xi(t)\}) \leqq H_\theta((\xi_1, \cdots, \xi_m))$$

$$\leqq \frac{m}{2}\log\frac{1}{\theta^2} + \frac{m}{2}\log m - m\log\sqrt{2\pi e\left(1-\varepsilon^2\Big/\sum_{j=1}^{m}\sigma_j\right)}$$

$$+h((\xi_1, \cdots, \xi_m)) + o(1)$$

$$\leqq \frac{1}{2}\log\prod_{j=1}^{m}\frac{\sigma_j}{\theta^2/m} - m\log\sqrt{1-\varepsilon^2\Big/\sum_{j=1}^{m}\sigma_j} + o(1).$$

If we choose m and θ so that $\frac{\theta^2}{m} \sim \sigma_m$ *), the relations $\theta^2 = \varepsilon^2 - \sum_{j=m+1}^{\infty}\sigma_j$ and $\sigma_j = Cj^{-2} + o(j^{-2})$ (lemma 4) and $\sum_{j=m+1}^{\infty}\frac{1}{j^2} \sim \frac{1}{m^2}$ determine $m = \left[\frac{2C}{\varepsilon^2}\right]$ **). Finally we have by using Stirling's formula $m! = \sqrt{2\pi}\, e^{-m} m^{m+1/2}$,

$$H_\varepsilon(\{\xi(t)\}) \leqq m + o(m) = 2C\frac{1}{\varepsilon^2} + o\left(\frac{1}{\varepsilon^2}\right).$$

(ii) Evaluation from below;

Take a sufficiently small $\varepsilon > 0$. For every n we have from lemmas 1, 2, 3, 7 and 4

$$H_\varepsilon(\{\xi(t)\}) \geqq H_\varepsilon((\xi_1, \cdots, \xi_n))$$

*) $f_m \sim g_m$ means $\lim f_m/g_m = 1$.

**) $[x]$ is the integer part of x.

$$\geqq \frac{n}{2} \log \frac{1}{\varepsilon^2} + \frac{n}{2} \log n - n \log \sqrt{2\pi e} + h((\xi_1, \cdots, \xi_n))$$

$$\geqq \frac{n}{2} \log \frac{1}{\varepsilon^2} + \frac{n}{2} \log n - n \log \sqrt{2\pi e} - \log \prod_{j=1}^{n} \left(\frac{K}{2\pi k \sigma_j} \right)^{1/2}$$

and by using Stirling's formula

$$= \frac{n}{2} \log \frac{kC}{\varepsilon^2 eK} - \frac{n}{2} \log n + n + o(n) .$$

If we choose $n = \left[\frac{kC}{eK} \frac{1}{\varepsilon^2} \right]$, the first and second terms of the last line cancel each other to be $o(n)$ and we have

$$H_\varepsilon(\{\xi(t)\}) \geqq n + o(n) = \frac{kC}{eK} \frac{1}{\varepsilon^2} + o\left(\frac{1}{\varepsilon^2} \right) .$$

Thus completes the proof.

Remark 1. We can extend our theorem also for a temporally inhomogeneous diffusion process with slight modification of the above discussion. Let $\xi(t)$ $(0 \leqq t \leqq T)$ be a diffusion process constructed by the stochastic integral equation:

$$\xi(t) = \int_0^t b(u, \xi(u)) du + \int_0^t a(u, \xi(u)) dB(u) \qquad (0 \leqq t \leqq T)$$

where we assume

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leqq L |x - y| ,$$

$$b(t, x)^2 \leqq L(1 + x^2) ,$$

$$0 < k \leqq a(t, x)^2 \leqq K .$$

Then we have

$$\frac{k}{eK} \frac{1}{\pi^2} \left\{ \int_0^T \sqrt{E a(u, \xi(u))^2}\, du \right\}^2 \frac{1}{\varepsilon^2} \lesssim H_\varepsilon(\{\xi(t)\}) \lesssim \frac{2}{\pi^2} \left\{ \int_0^T \sqrt{E a(u, \xi(u))^2}\, du \right\}^2 \frac{1}{\varepsilon^2} .$$

Remark 2. Our evaluation (i) in the proof of the theorem is quite analogous to that of infinite dimensional Gaussian random variable [5].

THE INSTITUTE OF STATISTICAL MATHEMATICS*)

*) Now at Tokyo Education University

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