SOME NON-ORTHOGONAL UNSATURATED MAIN EFFECT AND RESOLUTION *V* PLANS DERIVED FROM A ONE-RESTRICTIONAL LATTICE*

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(Received July 18, 1966; revised June 19, 1967)

1. Introduction and summary

Kishen [7], Mood [8], and Raghavarao [10] have given methods to construct non-orthogonal main effect plans and Webb [12], Patel [9], Addelman [1] and Banerjee [4] have constructed non-orthogonal resolution *V* plans. The plans given by Raghavarao [10] and Webb [12] are called saturated plans, since no degree of freedom is available for estimation of the error variance. The plans given by Patel [9] and Banerjee [4] are not fully resolution *V* type plans, since not all two-factor interactions can be estimated. Banerjee [3] and Addelman [1] presented plans for the $k/2^m$ replicate of a 2^n factorial, which for proper k, allows estimation of two-factor interactions, when higher order interactions are assumed to be absent. A summary of the techniques employed by the above authors is given by Addelman [2].

One-restrictional lattice designs for 2^n treatments in blocks of 2 plots each have been given by Kempthorne [5], Kempthorne and Federer [6] and by Raktoe [11]. In this paper we show how to utilize a single replicate of a one-restrictional lattice to construct saturated non-orthogonal fractional plans for the 2^n factorial. The 2^{n-1} blocks are ordered according to an ordering of levels of two factor interactions. From this ordered array, specific instructions are given for constructing saturated non-orthogonal main effect plans and for constructing saturated non-orthogonal resolution *V* plans. Some unsaturated main effect plans with solutions for estimating the parameters are given. A procedure for constructing non-orthogonal resolution *V* plans is also presented.

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* Research supported in part by NIH Research Grant GM-05900.

Paper No. BU-130 in the Biometrics Unit, and No. 518 in the Department of Plant Breeding and Biometry, Cornell University, Ithaca, New York.

2. The special replicate of the one-restrictional lattice

Consider the 2^n factorial and associate with it the one-restrictional lattice design $2^n = 2^{n-1} \times 2$, i.e. 2^n treatment combinations in 2^{n-1} blocks of 2 plots each. Next, consider the replicate obtained by confounding the $2^{n-1}-1$ even-number interactions (geometrically the $(n-2)$-flat of the projective geometry $PG(n-1, 2)$) with the blocks. In setting up this replicate we choose $(n-1)$ two-factor interactions to generate the confounding scheme and it is obvious that this can always be done. Also, we will order the treatments in the blocks in a fixed manner as illustrated below for the one-restrictional lattice of 2^5 treatments in 2^4 blocks of 2 plots each. The ordering of the blocks follows from the ordering of the levels of the two-factor interactions.

2^5 treatments in 2^4 blocks of 2 plots each

Block No.	Treatments in blocks		Levels of effects confounded with blocks (AB)	(AC)	(AD)	(AE)
1	00000	11111	0	0	0	0
2	01111	10000	1	1	1	1
3	10111	01000	1	0	0	0
4	11011	00100	0	1	0	0
5	11101	00010	0	0	1	0
6	11110	00001	0	0	0	1
7	01100	10011	1	1	0	0
8	01010	10101	1	0	1	0
9	01001	10110	1	0	0	1
10	00110	11001	0	1	1	0
11	00101	11010	0	1	0	1
12	00011	11100	0	0	1	1
13	10001	01110	1	1	1	0
14	10010	01101	1	1	0	1
15	10100	01011	1	0	1	1
16	11000	00111	0	1	1	1

In the following discussion block effects will be ignored, since the blocks are used solely to obtain the above type of structure.

On inspection of the treatments within a block it is apparent that they are mirror images of each other. Also, if we denote the vectors of observations corresponding to the treatments in columns 1 and 2 by Y_1 and Y_2 respectively and if we denote the effect parameters by the vector $\beta' = (\mu, \beta_1'/2, \beta_2'/2, \cdots, \beta_n/2)$, where μ is the mean effect, β_1 is the column vector of main effects, β_2 is the column vector of two-factor interactions, etc., and β_n is the n-factor interaction and finally if we assume the usual linear model, then

$$E\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & X_1 & X_2 & X_3 \cdots & X_n \\ 1 & -X_1 & X_2 & -X_3 \cdots & (-1)^n X_n \end{pmatrix} \beta$$

where $\mathbf{1}$ is a 2^{n-1} column vector of ones, X_1 is a $2^{n-1} \times n$ matrix, X_2 is a $2^{n-1} \times \binom{n}{2}$ matrix, etc., X_n is a 2^{n-1} column vector. The elements of all the X matrices consist of plus and minus ones. In other words, if we consider the observations within the same block (or pairs) we see that even-number effects have the same sign and odd-number effects have opposite signs. This property of our lattice replicate will be utilized in sections 4 and 5.

3. Saturated non-orthogonal main effect and resolution V plans

The saturated main effect plans given by Raghavarao [10] are at once obtainable from the replicate of the one-restrictional lattice presented above. Since in this case the interest lies in estimating the mean μ and the n main effects, we simply take the first $(n+1)$ observations in the first column. These $(n+1)$ observations will yield the plan with the well known information matrix $nI+J$ for the case n is odd and the equally well known information matrix $(n-1)I+2J$ for the case $(n+1) \equiv 2 \pmod 4$, where I is the identity matrix and J is a matrix of ones of dimension $n+1$.

For the cases $n=3$, 4 and 5 we have the following plans:

$n=3$	$n=4$	$n=5$
000	0000	00000
011	0111	01111
101	1011	10111
110	1101	11011
	1110	11101
		11110

The saturated non-orthogonal resolution V plans given by Webb [12] are also immediately obtained from the replicate of the one-restrictional lattice for the cases $n \geqq 5$. Since we have to estimate $1+n+\binom{n}{2} = (n^2+n+2)/2$ parameters we simply take the first $(n^2+n+2)/2$ observations from the first column. For the cases $n=3$ and $n=4$ we use the plans:

$n=3$	
000	
011	100
101	010
110	001

$n=4$	
0000	
0111	
1011	
1101	
1110	
0110	1001
0101	1010
1100	0011

Note that the main effect plan is always included in the resolution V plan. Also, for the case $n=5$ we have the orthogonal plan, namely $(\text{ABCDE})_0$. In this case we have also the equivalent plan $(\text{ABCDE})_1$, given by the second column of treatments.

4. Unsaturated non-orthogonal main effect plans

In considering unsaturated non-orthogonal main effect plans it is logical to select blocks or pairs of treatments as candidates, because then we will be able to utilize the properties mentioned in section 2. Doing so leads immediately to consideration of differences and sums of pairs of observations. The forming of differences and sums, in k pairs of observations is equivalent to multiplying the observational system of the left by the $2k\times 2k$ matrix C:

$$C=\left[\begin{array}{cccc} 1\;-1 & & 0 & \\ & 1\;-1 & & \\ 0 & & \ddots & \\ & & & 1\;-1 \\ \cdots & \cdots & \cdots & \cdots \\ 1\;\;1 & & 0 & \\ & 1\;\;1 & & \\ 0 & & \ddots & \\ & & & 1\;\;1 \end{array}\right].$$

Since $|C|\neq 0$, we know that C is invertible and hence least squares estimates will be invariant under C.

In a main effect plan there are n main effects. If we now take n pairs, namely the n basis points and their mirror images (i.e. the 2nd up to and inclusive of the $(n+1)$st pair in our replicate) and if we form differences between the pairs to estimate the main effects and the sum to estimate the mean, then this procedure will lead us to estimates of the main effects which will be orthogonal to the estimate of the mean.

In terms of the usual model we have the following systems:

$$\mathrm{E}[Y_1-Y_2]=(-2I+J)\beta_1$$

$$\mathrm{E}[Y_1+Y_2]=2\mathbf{1}\mu$$

where I is the $n\times n$ identity matrix and J is the $n\times n$ matrix of ones and $\mathbf{1}$ is an n column vector of ones. The information matrix and its inverse for the main effects are respectively:

$$X'X=4I+(n-4)J$$

$$[X'X]^{-1}=\frac{1}{4}I-\frac{(n-4)}{4(n^2-4n+4)}J^*.$$

The information matrix for μ is $4\,\mathbf{1}'\mathbf{1}$ and hence the solution is $\hat{\mu}=\bar{Y}$.

It should be noted that for the case $n=4$ we have the orthogonal plan ($n-4=0$, so that J vanishes) consisting of the well-known set of treatments given by $(\mathrm{ABCD})_1$:

$n=4$	
0111	1000
1011	0100
1101	0010
1110	0001

In this case the equivalent orthogonal plan is formed by $(\mathrm{ABCD})_0$.

It should be stressed here that in the "basis" plan the estimate of the mean is orthogonal to estimates of the main effects. The "basis" plan presented above has the advantage of being so simple that in fact no construction problem exists. Also, the analysis boils down to a simple routine. The plan is unsaturated, since there are $(n-1)$ degrees of freedom available for the estimation of the error variance.

Banerjee [4] has considered the first $(n+1)$ pairs in our replicate of the one-restrictional lattice and he has shown that his "index number" plan leads to estimates of the main effects, which are orthogonal to the estimates of the mean and n two-factor interactions if $(n^2-3n)/2$ of these are assumed to be "absent". The "index number plan" leads to the following information matrix for the main effects:

$$X'X=4I+(n-3)J.$$

Banerjee's "index number" plan is not a strict main effect plan and also not a resolution V plan, but it is somewhere in between. From the

* Many matrices in the paper are of the form $T=aA+bB$ where $A=J_n/n$, $B=I_n-J_n/n$, and the inverse of T is $T^{-1}=A/a+B/b$.

viewpoint of main effects only our "basis" plan is superior to the "index number" plan, since in our case we need 2 observations less.

5. Non-orthogonal unsaturated resolution V plans

In this type of plan we have to estimate $1+n+\binom{n}{2}=(n^2+n+2)/2$ parameters, but working with differences and sums as in section 4 it is clear that we only need $1+\binom{n}{2}=(n^2-n+2)/2$ pairs of observations to produce a non-orthogonal unsaturated resolution V plan. This plan consists of taking the origin plus its image (i.e. the first pair) and the observations having exactly two factors at the high level plus their mirror images (i.e. from the $(n+2)$nd up to and inclusive of the $(n^2-n+2)/2$th pair). This rule holds only for the cases $n \geqq 5$. Thus for example, for $n=5$ we have the following plan, given by $(5^2-5+2)/2=11$ pairs:

$n=5$	
00000	11111
01100	10011
01010	10101
01001	10110
00110	11001
00101	11010
00011	11100
10001	01110
10010	01101
10100	01011
11000	00111

For the case $n=3$ we need all the treatments and for the case $n=4$ we adopt the following plan:

$n=4$	
0110	1001
0101	1010
1100	0011
0111	1000
1011	0100
1101	0010
1110	0001

Utilizing the differences between pairs we arrive at the following information matrix for the main effects for all cases where $n \geqq 4$:

$$X'X=(n^2-n+4)I/2-J$$

where I is an $n\times n$ identity matrix and J is an $n\times n$ matrix of 1's. Forming the sums of pairs leads us to the information matrix and its inverse of the mean and two-factor interactions, namely:

$$X'X=\begin{bmatrix} 2(n^2-n+2) & -2\,\mathbf{1}' \\ -2\,\mathbf{1} & \dfrac{(n^2-n+4)}{2}I-J \end{bmatrix}$$

$$[X'X]^{-1}=\begin{bmatrix} a & b\,\mathbf{1}' \\ b\,\mathbf{1} & cI+dJ \end{bmatrix}$$

where

$$a=\frac{1}{2(n^2+n+2)}+\frac{1}{(n^2-n+2)^2}\left\{\frac{n(n-1)}{(n^2-n+4)}+\frac{[n(n+1)]^2}{2(n^2-n+2)^2(n^2-n+4)}\right\}$$

$$b=\frac{1}{(n^2-n+2)}\left\{\frac{2(n^2-n+2)^2+n(n-1)}{(n^2-n+2)^2(n^2-n+4)}\right\}$$

$$c=\frac{2}{(n^2-n+4)}$$

$$d=\frac{2}{(n^2-n+2)^2(n^2-n+4)}\,.$$

The matrix I is an $n(n-1)/2\times n(n-1)/2$ identity matrix, $\mathbf{1}$ is $n(n-1)/2$ column vector of 1's and J is an $n(n-1)/2\times n(n-1)/2$ matrix of 1's.

We should note that the estimates for the main effects are orthogonal to the estimates of the mean and the two-factor interactions. Also, there are $(n^2-3n+2)/2$ degrees of freedom available for the estimation of the error variance.

Acknowledgement

The comments of a referee were helpful in making the results more concise.

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