ON NONPARAMETRIC T-METHOD OF MULTIPLE COMPARISONS FOR RANDOMIZED BLOCKS*

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Summary

Some nonparametric generalizations of Tukey's [9] T-method of multiple comparisons are considered for randomized blocks and the allied efficiency results are studied. For this, the distribution theory of aligned rank order statistics developed in [6], [7] is extended for multiple comparisons along the lines of [5] which deals with one-way layouts.

1. Introduction

Consider n randomized blocks (p plots with one observation per cell). According to the usual linear model, the chance variable X_{ij} associated with the yield of the plot in the ith block receiving the jth treatment is expressed as

$$X_{ij}=\mu+\beta_1+\tau_j+\varepsilon_{ij}\ , \qquad \boldsymbol{\tau}=(\tau_1,\cdots,\tau_p)\perp \boldsymbol{J}_p=(1,\cdots,1)\ , \tag{1.1}$$

for $j=1,\cdots,p$; $i=1,\cdots,n$, where μ is the *mean effect*, $\beta_1,\cdots,\beta_n$ are the *block effects* (parameters under fixed effects model or random variables under mixed effects model), $\boldsymbol{\tau}$ is the *treatment effect* vector (parameter of interest), and ε_{ij}'s are the error components. It is assumed that $\boldsymbol{\varepsilon}_i=(\varepsilon_{i1},\cdots,\varepsilon_{ip})$ has a continuous joint cumulative distribution function (cdf) $F(\boldsymbol{\varepsilon})$ which is symmetric in its p arguments. This is less restrictive than that of the assumption that all the no error components are independent and identically distributed (iid). Often, the test for $H_0:\boldsymbol{\tau}=\mathbf{0}$ is of relatively minor importance, and one may be more interested in simultaneous inference on the set of estimable contrasts in $\boldsymbol{\tau}$, viz.,

$$\boldsymbol{\Phi}=\{\phi=\boldsymbol{c}\cdot\boldsymbol{\tau}\ ;\ \boldsymbol{c}\perp\boldsymbol{J}_p\}\ . \tag{1.2}$$

When $F(\boldsymbol{\varepsilon})$ is a totally symmetric multivariate normal cdf, Tukey's [9]

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T-method provides simultaneous tests and confidence bounds for any number of elements of $\boldsymbol{\Phi}$; for other procedures, not to be discussed here, see Wilks ([10], pp. 290–295). The development on nonparametric procedures (valid for any arbitrarily totally symmetric continuous cdf) is rather spotty and piece-meal. The existing procedures include (i) treatments versus control sign test (cf. Steel [8]), (ii) all comparison sign test (cf. Nemenyi [3]), (iii) simultaneous rank sum test based on within block rankings (cf. Nemenyi [3]), (iv) treatments versus control multiple comparisons signed rank test (cf. Nemenyi [3] and Hollander [2]), and (v) a class of aligned rank order multiple comparisons tests (cf. Sen [7]), among others. The treatments vs. control procedures depend on the choice of a control when a natural choice may not exist, and thus are subject to some arbitrariness. The procedures (i), (ii) and (iii) only utilize the inter-block comparisons and sacrifice the information contained in inter-block comparisons. For this reason they are usually less efficient, particularly the sign tests. The procedure (v) is free from both the above drawbacks, but it is subject to cumbrous inversion procedure for obtaining simultaneous confidence bound to members of $\boldsymbol{\Phi}$. However, all the above procedures are subject to the following criticism. These procedures afford simultaneous tests or confidence bounds for either the $p-1$ treatment-control differences or the $p(p-1)/2$ paired differences among the p treatments. For contrasts other than paired differences these procedures are not valid. For example, $\tau_1+\tau_2-2\tau_3=0$ does not necessarily mean that $\tau_1=\tau_2=\tau_3$, where as $\tau_1-\tau_2=0$ implies so. Consequently, the rank procedures resting actually upon the assumption that $\boldsymbol{\tau}=\mathbf{0}$ for paired differences can not be applied in the general case as there $\boldsymbol{\tau}$ is not necessarily $\mathbf{0}$. The same problem is also true for one criterion analysis of variance problem, and in [5], the difficulty has been obviated by an inversion technique yielding estimates of contrasts based on rank tests. This approach is extended here to randomized blocks. To utilize the information contained in inter-block comparisons, ranking after alignment (cf. [7]) is used. For this purpose, the theory developed in [6], [7] is extended along the lines of [5]. Allied efficiency results are also studied.

2. The main results

We shall first consider the problem of paired differences. To eliminate the nuisance parameters, we consider as in [7] the aligned observation

$$(2.1)\quad Y_{ij}=X_{ij}-X_{i.}=\tau_j+e_{ij}\,;\qquad e_{ij}=\varepsilon_{ij}-\varepsilon_{i.}\,;\qquad j=1,\cdots,p,\ i=1,\cdots,n\,,$$

where $X_{i.}$ and $\varepsilon_{i.}$ are respectively the block averages of X_{ij}'s and ε_{ij}'s.

Since $F(\boldsymbol{e})$ is assumed to be totally symmetric, it follows that the cdf $G(\boldsymbol{e})$ of $\boldsymbol{e}_i=(e_{i1}, \cdots, e_{ip})$ is also symmetric in its p arguments, though it is necessarily a singular distribution of rank at most equal to $p-1$. Let then

$$(2.2) \qquad \boldsymbol{Y}_n^{(j)}=(Y_{1j}, \cdots, Y_{nj}) \qquad j=1, \cdots, p ,$$

$$(2.3) \qquad \boldsymbol{E}_n=(E_{n,1}, \cdots, E_{n,2n}) , \qquad E_{n,i}=J_n(i/(2n+1)), \qquad 1\leqq i\leqq 2n ,$$

where J_n satisfies the regularity conditions of Chernoff and Savage [1], as further modified in section 4 of [6]. Let

$$(2.4) \qquad \bar{E}_n=(1/2n) \sum_{i=1}^{2n} E_{n,i} \quad \text{and} \quad A_n^2=(1/(2n-1)) \sum_{i=1}^{2n} (E_{n,i}-\bar{E}_n)^2 .$$

For $\boldsymbol{Y}_n^{(j)}$ and $\boldsymbol{Y}_n^{(k)}$ consider the usual Chernoff-Savage [1] two-sample rank order statistics (though based on matched samples)

$$(2.5) \qquad h(\boldsymbol{Y}_n^{(j)}, \boldsymbol{Y}_n^{(k)})=(1/n) \sum_{i=1}^{2n} E_{n,i}Z_{n,i}^{(j,k)} ,$$

where $Z_{n,i}^{(j,k)}$ is 1 or 0 according as the ith smallest observation in the combined (j, k) set is from the jth set or not, $i=1, \cdots, 2n$. The proposed procedure rests on the following statistic

$$(2.6) \qquad S_n=\underset{j\neq k}{\text{Max}} \, [2\{(p-1)n/p\}^{1/2} \, | \, h(\boldsymbol{Y}_n^{(j)}, \boldsymbol{Y}_n^{(k)})-\bar{E}_n \, | / A_n] .$$

For the study of the distribution theory of S_n, we let $J(u)=\lim_{n\to\infty} J_n(u)$, assume it to exist for all $0<u<1$ and further that $J(u)$ as well as $J_n(u)$ to be $\uparrow$ in $u : 0<u<1$. Let then

$$(2.7) \qquad \mu^0=\int_0^1 J(u)du \quad \text{and} \quad A^2=\int_0^1 J^2(u)du-\left(\int_0^1 J(u)du\right)^2 ,$$

so that $A^2>0$. Also, if in (2.5), the vector $\boldsymbol{Y}_n^{(j)}$ be replaced by $\boldsymbol{Y}_n^{(j)}+a\boldsymbol{J}_n$, the resulting statistic is denoted by $h(\boldsymbol{Y}_n^{(j)}+a\boldsymbol{J}_n, \boldsymbol{Y}_n^{(k)})$, which is thus $\uparrow$ in a. The univariate marginal cdf corresponding to the totally symmetric cdf $G(\boldsymbol{e})$ is denoted by $G^*(e)$. Let then $c(u)$ be 1 or 0 according as u is $\geqq 0$ or not, and

$$(2.8) \qquad Z_{ij}=\int_{-\infty}^{\infty} [c(X-Y_{ij})-G^*(x)]J'(G(x))dG^*(x) \qquad \text{for } j=1, \cdots, p;\ i=1, \cdots, n ,$$

$$(2.9) \qquad \bar{Z}_j=(1/n) \sum_{i=1}^{n} Z_{ij} \qquad \text{for } j=1, \cdots, n .$$

LEMMA 2.1 *Under* $H_0 : \boldsymbol{\tau}=\mathbf{0}$,

$$n^{1/2}\,|\{h(Y_n^{(j)}, Y_n^{(k)})-\bar{E}_n-1/2(\bar{Z}_k-\bar{Z}_j)\}|=o_p(1) \quad \textit{for all } j\neq k=1, \cdots, p\,.$$

The proof follows precisely along the same line as in Theorem 5.1 of Sen [6], and hence, for intended brevity, is omitted.

Let us also denote by $G^{**}(x, y)$ the bivariate joint cdf of any two variates corresponding to the totally symmetric cdf $G(\boldsymbol{e})$, and let

$$\rho_J \cdot A^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[G^{**}(x, y)-G^*(x)G^*(y)] \cdot J'(G^*(x))J'(G^*(y))dG^*(x)dG^*(y)\,. \tag{2.10}$$

LEMMA 2.2. *Under* $H_0:\boldsymbol{\tau}=\mathbf{0}$, $n^{1/2}(\bar{Z}_1, \cdots, \bar{Z}_p)$ *has asymptotically a multinormal distribution with a null mean vector and a dispersion matrix with elements* $A^2(\delta_{jk}+(1-\delta_{jk})\rho_J)$ *where* δ_{jk} *is the usual Kronecker delta.*

PROOF. Proceeding as in Theorem 5.1 of Sen [6], it follows that under $H_0:\boldsymbol{\tau}=\mathbf{0}$, Z_{ij} has mean zero, variance A^2, and the covariance of Z_{ij} and Z_{ik} $(j\neq k)$ is equal to $A^2\rho_J$. The rest of the proof follows from (2.9) and the (vector valued) central limit theorem for iid random variables.

LEMMA 2.3. $\rho_J \geqq -1/(p-1)$.

The proof follows simply by noting that $(Z_{i1}, \cdots, Z_{ip})$ are interchangeable random variables (under H_0), and hence their common correlation coefficient can not be less than $-1/(p-1)$.

Let $R_{p,\alpha}$ be the upper $100\alpha\,\%$ point of the distribution of the sample range of a sample of size p from a standard normal distribution. Then, proceeding as in Wilks ([10], pp. 290–295), and using the preceding three lemmas, we arrive at the following.

THEOREM 2.4. *Under* $H_0:\boldsymbol{\tau}=\mathbf{0}$,

$$\lim_{n\to\infty} P\{S_n \geqq R_{p,\alpha}\} \leqq \alpha\,.$$

For small sample sizes, we can use the intra-block permutation invariance (cf. [7]) to develop strictly (but only conditionally) distribution free test based on S_n by reference to the $(p!)^n$ equally likely (conditionally) realizations of $\boldsymbol{Y}_n^{(j)}$, $j=1, \cdots, p$. However, the labour involved increases prohibitively as n increases.

The simultaneous test for all paired differences may now be formulated as follows: Compute the values of $h(\boldsymbol{Y}_n^{(j)}, \boldsymbol{Y}_n^{(k)})$ for all $j<k=1, \cdots, p$. Regard those $\tau_j-\tau_k$ to be different from zero for which

$$2\{(p-1)n/p\}^{1/2}\,|\,h(\boldsymbol{Y}_n^{(j)}, \boldsymbol{Y}_n^{(k)})-\bar{E}_n\,| \geqq A_n\cdot R_{p,\alpha} \qquad (j\neq k=1, \cdots, p)\,. \tag{2.11}$$

By virtue of Theorem 2.4, this is an asymptotically size α multiple comparison test for all possible paired differences. The problem of attaching a simultaneous confidence bound to all possible paired differences can then be solved exactly as in Sen ([5], pp. 324–326), with the only change that $W_{n,\alpha}$ in [5] has to be replaced by $\{p/(p-1)\}^{1/2}\cdot R_{p,\alpha}$ and c by p. For brevity, the details are therefore omitted. As such, the expression for the asymptotic efficiency derived in (2.32) of [5] also remains true in the situation considered in this paper, with the only change that the equality sign has to be replaced considered in this paper, with the only change that equality sign has to be replaced by $\geqq$. The reason for this is that in Theorem 2.4. the probability is less than or equal to α, whereas in Theorem 2.1 of [5] or in the parametric case (cf. Wilks [10], pp. 290–295) the equality sign holds.

The general case of multiple comparisons for contrasts other than paired differences again follows as in section 2.3 of Sen [5], with the changes suggested earlier with the paired differences. The justification of the extension of (2.36) of [5] to two-way layouts again follows from the results of Puri and Sen [4]. As such, the same efficiency expressions also hold for the randomized blocks considered here. For brevity, these are therefore not reproduced again.

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