A NOTE ON EXPONENTIAL BOUNDS FOR BINOMIAL PROBABILITIES

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In [1] Kambo and Kotz gave bounds on deviations from the mean of binomial probabilities in order to strengthen results of Okamoto [2]. The bounds are based on the following inequality:

$$(1)\qquad L(p, c)=(p+c)\log\left(1+\frac{c}{p}\right)+(1-p-c)\log\left(1-\frac{c}{1-p}\right)\geqq 2c^2+\frac{4}{3}c^4\,,$$

where $0<p<1$ and $c\geqq 0$. Putting $c=\frac{1}{3}$, $p=\frac{1}{3}$ one sees immediately that (1) does not hold in the stated generality. An inspection of the proof of that inequality shows that the argument of symmetry does not go through. On the other hand, for most of the values (p, c) for which (1) holds, the inequality can be sharpened considerably. It may be remarked that values $c\geqq 1-p$ are of no interest for the considered problem and that the right-hand side of (1) should be independent of p.

THEOREM. *Let* $0\leqq c<1-p$.

(a) *If* $0<p<1$, *then*

$$(2)\qquad L(p, c)\geqq 2c^2+\frac{4}{9}c^4\,.$$

(b) *If* $p\geqq\frac{1}{2}$ *or if* $p+c<\frac{1}{2}$, *then*

$$(3)\qquad L(p, c)\geqq \log\left[(1+2c)^{1/2+c}(1-2c)^{1/2-c}\right]\,.$$

PROOF. Let $X=1-2(p+c)$, $Y=1-2p$. In [1] it is shown that

$$L(p, c)=\sum_{r=1}^{\infty}\frac{1}{2r(2r-1)}\left[X^{2r}-2rXY^{2r-1}+(2r-1)Y^{2r}\right]$$

and that all terms in the last series are not negative.

(a) Taking the first two terms of the series, we obtain

$$L(p, c) \geqq 2c^2 + \frac{2}{3}c^2[3(1-2p)^2 - 4c(1-2p) + 2c^2] . \tag{4}$$

The right-hand side of (4) achieves its minimum at $p=\frac{3-2c}{6}$, from which we get (2).

(b) Under the assumptions of part (b) it follows that $z=\frac{Y}{X}>0$. We will show that

$$f_r(z) = 1 - 2rz^{2r-1} + (2r-1)z^{2r} - (1-z)^{2r} \geqq 0 \quad \text{for } z \geqq 0, \text{ and } r = 1, 2, \cdots . \tag{5}$$

In the case that $r=1$, the inequality is trite. For the first derivatives $f_r'(z)$, $r=2, 3, \cdots$, we get $f_r'(z)=0$ for

$$z_1 = 1 , \quad z_2 = [1+(2r-1)^{1/(2r-2)}]^{-1} \quad \text{and} \quad z_3 = [1-(2r-1)^{1/(2r-2)}]^{-1} ,$$

i.e., $f_r(z)$ has in $(0, \infty)$ extrema only in z_1 and z_2, $0<z_2<1$. From the second derivative $f_r''(z)$ it is seen that $f_r(z)$ has a minimum in z_1. Since $f_r(0)=0$ we obtain (5). If we replace z again by Y/X we get

$$X^{2r} - 2rXY^{2r-1} + (2r-1)Y^{2r} - (X-Y)^{2r} \geqq 0 .$$

Hence

$$L(p, c) \geqq \sum_{r=1}^{\infty} \frac{1}{2r(2r-1)}(X-Y)^{2r} = \sum_{r=1}^{\infty} \frac{(2c)^{2r}}{2r-1} - \sum_{r=1}^{\infty} \frac{(2c)^{2r}}{2r}$$
$$= c \log\left(\frac{1+2c}{1-2c}\right) - \frac{1}{2} \log \frac{1}{1-4c^2}$$

which implies (3).

Remark. From the theorem follows that the exponential bounds of Theorem 1, part b) and Theorem 2, part a) in [1] may be replaced with the help of (3). So we have for instance;

$$P\left(\frac{N}{n} - p \geqq c\right) \leqq (1-2c)^{n(c-1/2)}(1+2c)^{-n(c+1/2)}$$
$$\text{if} \quad p \geqq \frac{1}{2} \quad \text{or} \quad p+c<\frac{1}{2} .$$

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REFERENCES

[1] N. S. Kambo and S. Kotz, "On exponential bounds for binomial probabilities," *Ann. Inst. Statist. Math.*, 18 (1966), 277-287.

[2] M. Okamoto, "Some inequalities relating to the partial sum of binomial probabilities," *Ann. Inst. Statist. Math.*, 10 (1958), 29-35.