DISTRIBUTION OF PRODUCT AND QUOTIENT OF BESSEL FUNCTION VARIATES*

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Summary

In this paper, the technique of Mellin transforms is employed to obtain the distribution of the product and quotient of two independent Bessel function random variables. Two different types of Bessel function variates are considered. The results are then specialized to yield a wide variety of classical distributions of importance in applications.

1. Introduction and general remarks

A number of papers have been devoted in recent years to the properties of Bessel function distributions and their various applications (see, e.g., [6], [8], [12] and [13]). These distributions include, as special cases, the familiar Gamma, Chi-squared, Non-central Chi-squared, Chi, Non-central Chi, Rayleigh, Folded Gaussian and various other distributions which occur, for example, as radial distributions in engineering [12] and population distribution problems [16]. Several distributions encountered in "randomization" such as the randomized gamma distribution, and in the first passage problems in stochastic processes are special cases of the Bessel function distributions (see [6], pp. 58–60).

Our purpose in this paper is to derive the distributions of the product and quotient of two independent Bessel function random variables. We consider two types of Bessel function variates; they are introduced in section 2. The method we have employed is that of Mellin transforms**. A systematic account of the applications of Mellin transforms to some problems in statistics was first given by Epstein [3], and subsequently, several authors have employed this technique to solve the

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** Dr. P. C. Consul informs the authors that in his paper "On the distribution of the ratio of the measures of divergence between two multivariate populations," *Mathematische Nachrichten*, 32 (1966), 149-155, he obtained, using a different method, an alternative expression for the distribution of the quotient of two independent Bessel function variates.

distribution problem for the product and quotient of independent random variables (see e.g., [2], [9], [14a], [14] and [17]). A comprehensive bibliography till 1963 on the problems connected with the distribution of products and quotients is presented in [2]. The most recent papers ([9], [11] and [14]) solve the problem for classes of distributions including normal and gamma distributions. The results from the theory of Mellin transforms relevant to the present paper, as well as the transforms of the Bessel function distributions are given in section 2.

In sections 3 and 4 we derive the main results of this paper—the distributions of the product and quotient of two independent Bessel function variates, respectively.

Finally, in section 5 some important particular cases of our general results are presented.

2. The Bessel function distributions and their Mellin transforms

A random variable is said to have a *type I Bessel function distribution* if its p.d.f. is given by

$$f(x;\beta,\theta,\lambda)=Cx^{(\lambda-1)/2}e^{-\theta x}I_{\lambda-1}(\beta\sqrt{x}),\qquad (x\geqq 0) \tag{2.1}$$

where

$$C=(2/\beta)^{\lambda-1}\theta^{\lambda}e^{-\beta^2/4\theta},\qquad \theta>0,\ \lambda>0,\ \beta\geqq 0,$$

and $I_\nu(z)$ is the modified Bessel function of the first kind given by

$$I_\nu(z)=\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\nu+1)}\left(\frac{z}{2}\right)^{2m+\nu}. \tag{2.2}$$

We say that a random variable has a *type II Bessel function distribution* if its p.d.f. is given by

$$g(x;\beta,\theta,\lambda)=Dx^{\lambda}e^{-\theta x^2/2}I_{\lambda-1}(\beta x),\qquad (x\geqq 0) \tag{2.3}$$

where

$$D=(1/\beta)^{\lambda-1}\theta^{\lambda}e^{-\beta^2/2\theta},\qquad \theta>0,\ \lambda>0,\ \beta\geqq 0$$

and $I_\nu(z)$ is as in (2.2).

(Note that one could obtain (2.3) from (2.2) by a simple change of variables. However, for convenience in the derivation of our special cases we treat the two types of Bessel function variates separately. In this paper all the results are derived in detail for the type I distributions only, and the analogous results for type II are stated without proofs.)

The Mellin transform of a positive random variable ξ with continuous p.d.f. $f(x)$ is given by (see, e.g., [3])

$$M(f(x)\,|\,s)=E(\xi^{s-1})=\int_0^\infty x^{s-1}f(x)dx\,. \tag{2.4}$$

Under suitable restrictions (detailed conditions are described in [15]) on $M(f(x)\,|\,s)$ considered as a function of the complex variable s, there is an inversion integral

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}M(f(x)\,|\,s)dx \tag{2.5}$$

for all x where the path of integration is any line parallel to the imaginary axis and lying within the strip of analyticity of $M(f(x)\,|\,s)$.

The following two basic properties of Mellin transforms are of special relevance to the distribution theory [3]:

(i) If ξ_1 and ξ_2 are independent positive random variables with Mellin transforms $M(f_1(x)\,|\,s)$ and $M(f_2(x)\,|\,s)$, respectively, then the Mellin transform of the product $\xi_1\xi_2$ is given by $M(f_1(x)\,|\,s)M(f_2(x)\,|\,s)$.

(ii) If $M(f(x)\,|\,s)$ is the Mellin transform of a random variable ζ, then the Mellin transform of $1/\zeta$ is given by $M(f(x)\,|\,2-s)$.

We now derive the Mellin transforms of the types I and II Bessel function densities given by (2.1) and (2.3), respectively. For the type I we have

$$M(f(x)\,|\,s)=C\int_0^\infty x^{s-1}x^{(\lambda-1)/2}e^{-\theta x}I_{\lambda-1}(\beta\sqrt{x}\,)dx\,. \tag{2.6}$$

To evaluate the integral substitute the series expression (2.2) for $I_{\lambda-1}\cdot(\beta\sqrt{x}\,)$ and integrate term by term. Termwise integration is justified because the series can be easily seen to be uniformly convergent [7]. We thus obtain

$$M(f(x)\,|\,s)=C\sum_{m=0}^{\infty}\frac{(\beta/2)^{2m+\lambda-1}}{m!\Gamma(m+\lambda)}\int_0^\infty e^{-\theta x}x^{s+m+\lambda-2}dx\,. \tag{2.7}$$

The integral on the right-hand side equals $\Gamma(s+m+\lambda-1)/\theta^{s+m+\lambda-1}$. Substituting the value of C, we get, after some algebraic manipulations:

$$M(f(x)\,|\,s)=\frac{e^{-\beta^2/(4\theta)}}{\theta^{s-1}}\sum_{m=0}^{\infty}\frac{[\beta/(2\sqrt{\theta}\,)]^{2m}\Gamma(s+m+\lambda-1)}{m!\Gamma(m+\lambda)}\,. \tag{2.8}$$

Analogously, the following expression for the Mellin transform of the type II Bessel function density (2.3) is easily obtained:

$$M(g(x)\,|\,s)=e^{-\beta^2/(2\theta)}\sum_{m=0}^{\infty}\frac{[\beta/(\sqrt{2\theta})]^{2m}\Gamma(m+\lambda+s/2-1/2)}{m!\Gamma(m+\lambda)}\left(\frac{2}{\theta}\right)^{(s-1)/2}\,. \tag{2.9}$$

3. Distribution of the product

Let ξ_1 and ξ_2 be independent Bessel function random variables of type I with p.d.f.'s $f(x; \beta_1, \theta_1, \lambda_1)$ and $f(x; \beta_2, \theta_2, \lambda_2)$, respectively. Their Mellin transforms are given by (2.8) with the proper substitution of the values of the parameters β, θ and λ. Let $\eta=\xi_1\xi_2$, and denote by $p_\eta(y)$ the density of η. The Mellin transform of $p_\eta(y)$ is then given by the product of the Mellin transforms of ξ_1 and ξ_2:

$$(3.1)\quad M(p_\eta(y)\mid s)=\frac{\exp\{-[\beta_1^2/(4\theta_1)+\beta_2^2/(4\theta_2)]\}}{(\theta_1\theta_2)^{s-1}}\sum_{k=0}^{\infty}\frac{[\beta_1/(2\sqrt{\theta_1})]^{2k}\Gamma(s+k+\lambda_1-1)}{k!\Gamma(k+\lambda_1)} \cdot\sum_{m=0}^{\infty}\frac{[\beta_2/(2\sqrt{\theta_2})]^{2m}\Gamma(s+m+\lambda_2-1)}{m!\Gamma(m+\lambda_2)}.$$

The above expression can be easily rewritten in the following form:

$$(3.2)\quad M(p_\eta(y)\mid s)=\frac{\exp\{-[\beta_1^2/(4\theta_1)+\beta_2^2/(4\theta_2)]\}}{(\theta_1\theta_2)^{s-1}} \cdot\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{[\beta_1/(2\sqrt{\theta_1})]^{2i}[\beta_2/(2\sqrt{\theta_2})]^{2(j-i)}}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)} \cdot\Gamma(s+i+\lambda_1-1)\Gamma(s+j-i+\lambda_2-1).$$

We now get $p_\eta(y)$ by inverting the above using (2.5). The essential step in the inversion process is the evaluation of the following integral:

$$(3.3)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^{-s}\Gamma(s+i+\lambda_1-1)\Gamma(s+j-i+\lambda_2-1)ds}{(\theta_1\theta_2)^{s-1}}$$

where $c>0$. Letting $z=y\theta_1\theta_2$, the expression (3.3) reduces to

$$(3.4)\quad \frac{\theta_1\theta_2}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}\Gamma(s+i+\lambda_1-1)\Gamma(s+j-i+\lambda_2-1)ds.$$

The above integral can now be evaluated using formula 17 ([5], p. 349), and is seen to be equal to

$$(3.5)\quad 2\theta_1\theta_2 z^{(\lambda_1+\lambda_2+j-2)/2}K_{(\lambda_1-\lambda_2+2i-j)/2}(2\sqrt{z})$$

where $K_\nu(x)$ (tabulated in [1]) is the modified Bessel function of the third kind [4] given by

$$(3.6)\quad K_\nu(x)=\frac{1}{2}\pi(\sin\nu\pi)^{-1}[I_{-\nu}(x)-I_\nu(x)]$$

in which $I_\nu(x)$ is given by (2.2). Returning to the original variable y we obtain from (3.5)

$$(3.7)\qquad 2(\theta_1\theta_2)^{(\lambda_1+\lambda_2+j)/2}y^{(\lambda_1+\lambda_2+j-2)/2}K_{(\lambda_1-\lambda_2+2i-j)/2}(2\sqrt{\theta_1\theta_2 y})\,.$$

The inversion process can now be completed to yield

$$(3.8)\quad p_\eta(y)=2\{\exp\{-[\beta_1^2/(4\theta_1)+\beta_2^2/(4\theta_2)]\}\}(\theta_1\theta_2)^{(\lambda_1+\lambda_2)/2}y^{(\lambda_1+\lambda_2-2)/2}$$
$$\cdot\sum_{j=0}^{\infty}(\theta_1 y/\theta_2)^{j/2}(\beta_2/2)^{2j}\sum_{i=0}^{j}\frac{[(\beta_1\sqrt{\theta_2})/(\beta_2\sqrt{\theta_1})]^{2i}K_{(\lambda_1-\lambda_2+2i-j)}(2\sqrt{\theta_1\theta_2 y})}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)}\,,$$
$$(y\geqq 0)\,.$$

The result corresponding to (3.8) for the type II Bessel function variates is derived following the same procedure, and we state it here without the proof:

Let ξ_1' and ξ_2' be independent type II Bessel function variates with respective densities $g(x;\beta_1,\theta_1,\lambda_1)$ and $g(x;\beta_2,\theta_2,\lambda_2)$ as in (2.3). Then the density $p_{\eta'}(y)$ of $\eta'=\xi_1'\xi_2'$ is given by

$$(3.9)\qquad p_{\eta'}(y)=\frac{\exp\{-[\beta_1^2/(2\theta_1)+\beta_2^2/(2\theta_2)]\}(\theta_1\theta_2)^{(\lambda_1+\lambda_2)/2}y^{\lambda_1+\lambda_2-1}}{2^{\lambda_1+\lambda_2-2}}$$
$$\cdot\sum_{j=0}^{\infty}(\beta_2/\sqrt{2})^{2j}(\theta_1/\theta_2)^{j/2}y^j$$
$$\cdot\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}(\theta_2/\theta_1)^iK_{(\lambda_1-\lambda_2+2i-j)/2}(y\sqrt{\theta_1\theta_2})}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)}\,,\qquad (y\geqq 0)\,.$$

4. Distribution of the quotient

Let ξ_1 and ξ_2 be independent type I Bessel function variates with respective densities $f(x;\beta_1,\theta_1,\lambda_1)$ and $f(x;\beta_2,\theta_2,\lambda_2)$ as in (2.1). Let $q_\zeta(y)$ denote the density of $\zeta=\xi_1/\xi_2$. We now derive an expression for $q_\zeta(y)$ using the fact that

$$(4.1)\qquad M(q_\zeta(y)\,|\,s)=M(f(y;\beta_1,\theta_1,\lambda_1)\,|\,s)M(f(y;\beta_2,\theta_2,\lambda_2)\,|\,2-s)\,.$$

The above identity combined with expression (2.8) yields

$$(4.2)\qquad M(q_\zeta(y)\,|\,s)=\exp\left\{-\left[\frac{\beta_1^2}{4\theta_1}+\frac{\beta_2^2}{4\theta_2}\right]\right\}\sum_{k=0}^{\infty}\frac{[\beta_1/(2\sqrt{\theta_1})]^{2k}\Gamma(s+k+\lambda_1-1)}{k!\Gamma(k+\lambda_1)\theta_1^{s-1}}$$
$$\cdot\sum_{m=0}^{\infty}\frac{[\beta_2/(2\sqrt{\theta_2})]^{2m}\Gamma(-s+m+\lambda_2+1)}{m!\Gamma(m+\lambda_2)\theta_2^{1-s}}$$
$$=\exp\left\{-\left[\frac{\beta_1^2}{4\theta_1}+\frac{\beta_2^2}{4\theta_2}\right]\right\}\sum_{j=0}^{\infty}[\beta_2/(2\sqrt{\theta_2})]^{2j}$$
$$\cdot\sum_{i=0}^{j}\left(\frac{\beta_1\sqrt{\theta_2}}{\beta_2\sqrt{\theta_1}}\right)^{2i}\frac{1}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)}$$
$$\cdot\frac{\Gamma(s+i+\lambda_1-1)\Gamma(-s+j-i+\lambda_2+1)}{\theta_1^{s-1}\theta_2^{1-s}}\,.$$

It is easily seen that the inversion of (4.2) reduces to the evaluation of the following integral:

$$(4.3)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\frac{\Gamma(s+i+\lambda_1-1)\Gamma(-s+j-i+\lambda_2+1)}{\theta_1^{s-1}\theta_2^{1-s}}ds$$
$$=\frac{\theta_1}{\theta_2}\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{y\theta_1}{\theta_2}\right)^{-s}\Gamma(s+i+\lambda_1-1)\Gamma(-s+j-i+\lambda_2+1)ds\right].$$

The last integral in (4.3) can now be evaluated by using the formula (15) in [5], (p. 349) to yield

$$(4.4)\quad \frac{\theta_1}{\theta_2}\Gamma(\lambda_1+\lambda_2+j)\left(\frac{y\theta_1}{\theta_2}\right)^{i+\lambda_1-1}\left(1+\frac{y\theta_1}{\theta_2}\right)^{-\lambda_1-\lambda_2-j}.$$

The inversion of $M(q_\zeta(y)\,|\,s)$ can now be completed using (4.4), and we obtain

$$(4.5)\quad q_\zeta(y)=\frac{\exp\{-[\beta_1^2/(4\theta_1)+\beta_2^2/(4\theta_2)]\}y^{\lambda_1-1}\theta_1^{\lambda_1}\theta_2^{\lambda_2}}{(\theta_1 y+\theta_2)^{\lambda_1+\lambda_2}}$$
$$\cdot\sum_{j=0}^{\infty}\left(\frac{\beta_2}{2}\right)^{2j}\frac{\Gamma(\lambda_1+\lambda_2+j)}{(\theta_1 y+\theta_2)^j}\sum_{i=0}^{j}\left(\frac{\beta_1}{\beta_2}\right)^{2i}\frac{y^i}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)},$$
$$(y\geqq 0).$$

The result corresponding to (4.5) for the type II Bessel function variates is as follows: If ξ_1' and ξ_2' are independent type II Bessel function variates with respective densities $g(x;\beta_1,\theta_1,\lambda_1)$ and $g(x;\beta_2,\theta_2,\lambda_2)$ as in (2.3), then the density $q_{\zeta'}(y)$ of $\zeta'=\xi_1'/\xi_2'$ is given by

$$(4.6)\quad q_{\zeta'}(y)=2\exp\left\{-\left[\frac{\beta_1^2}{2\theta_1}+\frac{\beta_2^2}{2\theta_2}\right]\right\}\frac{\theta_1^{\lambda_1}\theta_2^{\lambda_2}y^{2\lambda_1-1}}{(\theta_1 y^2+\theta_2)^{\lambda_1+\lambda_2}}\sum_{j=0}^{\infty}\frac{(\beta_2/\sqrt{2})^{2j}\Gamma(\lambda_1+\lambda_2+j)}{(\theta_1 y^2+\theta_2)^j}$$
$$\cdot\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}y^{2i}}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)},\qquad (y\geqq 0).$$

5. Some special cases

We shall now reduce the expressions (3.8), (3.9), (4.5) and (4.6) to some interesting special cases. In what follows, wherever the quotient of two variates is involved, the indices 1 and 2 correspond to the numerator and denominator, respectively.

A. *Particular cases of Bessel function variates of type I*

(i) *Chi-squared distribution*

Letting $\beta=0$, $\lambda=n/2$ and $\theta=1/2$ in (2.1), we obtain the familiar chi-squared density with n degrees of freedom:

$$(5.1) \qquad f(x;n)=\frac{1}{2^{n/2}\Gamma(n/2)}e^{-x/2}x^{(n-2)/2}, \qquad (x\geqq 0).$$

Expression (3.8) with $\beta_1=\beta_2=0$, $\lambda_1=n_1/2$, $\lambda_2=n_2/2$, $\theta_1=\theta_2=1/2$ yields the following density of the product η of two independent chi-squared variates with n_1 and n_2 degrees of freedom, respectively:

$$(5.2) \qquad p_\eta(y)=\frac{y^{(n_1/4+n_2/4)-1}K_{n_1/4-n_2/4}(\sqrt{y})}{2^{(n_1/2+n_2/2)-1}\Gamma(n_1/2)\Gamma(n_2/2)}, \qquad (y\geqq 0).$$

Similarly, from expression (4.5) we obtain the density of the quotient ζ of two independent chi-squared variates with respective degrees of freedom n_1 and n_2:

$$(5.3) \qquad q_\zeta(y)=\frac{\Gamma(n_1/2+n_2/2)}{\Gamma(n_1/2)\Gamma(n_2/2)}\frac{y^{(n_1/2)-1}}{(1+y)^{(n_1+n_2)/2}}, \qquad (y\geqq 0)$$

which is the familiar F-distribution.

(ii) *Non-central chi-squared distribution*

In (2.1), putting $\lambda=n/2$ and $\theta=1/2$ we obtain the non-central chi-squared density with n degrees of freedom and non-centrality parameter β:

$$(5.4) \qquad f(x;\beta,n)=\frac{1}{2\beta^{(n/2)-1}}e^{-(\beta^2+x)/2}x^{(n/2-1)/2}I_{(n/2)-1}(\beta\sqrt{x}), \qquad (x\geqq 0).$$

Substituting $\lambda_1=n_1/2$, $\lambda_2=n_2/2$, $\theta_1=\theta_2=1/2$ in (3.8), we obtain the p.d.f. of the product of a chi-squared (β_1, n_1) variate and a chi-squared (β_2, n_2) variate:

$$(5.5) \qquad p_\eta(y)=\frac{\exp\{-(\beta_1^2/2+\beta_2^2/2)\}}{2^{(n_2/2+n_2/2)-1}}y^{(n_1/4+n_2/4)-1}\sum_{j=0}^{\infty}(\beta_2/2)^{2j}y^{j/2}$$
$$\cdot\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}K_{(n_1/2-n_2/2+2i-j)/2}(\sqrt{y})}{i!(j-i)!\Gamma(n_1/2+i)\Gamma(n_2/2+j-i)}, \qquad (y\geqq 0).$$

The same substitutions in (4.5) yield the p.d.f. of the quotient of the two non-central chi-squared variates:

$$(5.6) \qquad q_\zeta(y)=\frac{\exp\{-(\beta_1^2/2+\beta_2^2/2)\}y^{(n_1/2)-1}}{(1+y)^{(n_1+n_2)/2}}\sum_{j=0}^{\infty}\frac{(\beta_2/\sqrt{2})^{2j}\Gamma[(n_1/2+n_2/2)+j]}{(1+y)^j}$$
$$\cdot\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}y^i}{i!(j-i)!\Gamma(n_1/2+i)\Gamma(n_2/2+j-i)}, \qquad (y\geqq 0).$$

(iii) *Randomized gamma distribution*

The randomized gamma density may be obtained from (2.1) by letting $\beta=2\sqrt{\mu}$, $\theta=1$ and $\lambda=\rho+1$ [6]:

$$(5.7) \qquad f(x;\mu,\rho)=e^{-(\mu+x)}\sqrt{\left(\frac{x}{\mu}\right)^{\rho}}\,I_{\rho}(2\sqrt{\mu x})\,, \qquad (x\geqq 0)\,.$$

The distributions of the product and the quotient of two independent randomized gamma variates with respective parameters μ_1, ρ_1 and μ_2, ρ_2 may now be obtained by making the corresponding substitutions in (3.8) and (4.5) respectively. For the product we have

$$(5.8) \qquad \rho_{\eta}(y)=2e^{-(\mu_1+\mu_2)}y^{(\rho_1+\rho_2)/2}\sum_{j=0}^{\infty}y^{j/2}\mu_2^j \cdot\sum_{i=0}^{j}\frac{(\mu_2/\mu_1)^i K_{(\rho_1-\rho_2+2i-j)/2}(2\sqrt{y})}{i!(j-i)!\Gamma(i+\rho_1+1)\Gamma(j-i+\rho_2+1)}\,, \qquad (y\geqq 0)$$

and for the quotient

$$(5.9) \qquad q_{\zeta}(y)=\frac{e^{-(\mu_1+\mu_2)}y^{\rho_1}}{(1+y)^{\rho_1+\rho_2+2}}\sum_{j=0}^{\infty}\frac{\mu_2^j\Gamma(\rho_1+\rho_2+2+j)}{(1+y)^j} \cdot\sum_{i=0}^{j}\frac{(\mu_1/\mu_2)^i y^i}{i!(j-i)!\Gamma(i+\rho_1+1)\Gamma(j-i+\rho_2+1)}\,, \qquad (y\geqq 0)\,.$$

B. *Particular cases of Bessel function variates of type II*

(i) *Chi-distribution*

In expression (2.3) let $\beta=0$, $\theta=\theta'/\sigma^2$ and $\lambda=\theta'$ (then write θ for θ'), which gives,

$$(5.10) \qquad g(x;\theta,\sigma^2)=\frac{2}{\Gamma(\theta)}\left(\frac{\theta}{2\sigma^2}\right)^{\theta}x^{2\theta-1}e^{-\theta x^2/(2\sigma^2)}\,, \qquad (x\geqq 0)$$

where (5.10) can be transformed into the chi-distribution with n degrees of freedom via the substitution $\theta=n/2$ and $\sigma=1/2$.

If ξ_1' has p.d.f. $g(x;\theta_1,\sigma_1^2)$ and ξ_2' has p.d.f. $g(x;\theta_2,\sigma_2^2)$, and if ξ_1' and ξ_2' are independent, then the density of $\eta'=\xi_1'\xi_2'$ is given by

$$(5.11) \quad p_{\eta'}(y)=\frac{2[\sqrt{\theta_1\theta_2}/(\sigma_1\sigma_2)]^{\theta_1+\theta_2}(y/2)^{\theta_1+\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)}K_{(\theta_1-\theta_2)/2}[y\sqrt{\theta_1\theta_2}/(\sigma_1\sigma_2)]\,, \qquad (y\geqq 0)\,.$$

Expression (5.11) is derived from (3.9) by performing corresponding substitutions. For the density of $\zeta'=\xi_1'/\xi_2'$ we have from (4.6)

$$(5.12) \qquad q_{\zeta'}(y)=\frac{2\Gamma(\theta_1+\theta_2)(\theta_1\sigma_2^2)^{\theta_1}(\theta_2\sigma_1^2)^{\theta_2}y^{2\theta_1-1}}{\Gamma(\theta_1)\Gamma(\theta_2)(\theta_1\sigma_2^2y^2+\theta_2\sigma_1^2)^{\theta_1+\theta_2}}\,, \qquad (y\geqq 0)\,.$$

The Maxwell-Boltzmann distribution and the Rayleigh distribution which are particularly useful as radial distributions in engineering and physical problems are special cases of (5.10).

(ii) *Non-central chi-distribution with two degrees of freedom*

In expression (2.3), let $\beta=\beta'/\sigma^2$, $\theta=1/\sigma^2$, and $\lambda=1$ which yields

$$(5.13)\qquad g(x;\beta,\sigma^2)=\frac{x}{\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}(\beta^2+x^2)\right\}I_0\left(\frac{\beta x}{\sigma^2}\right),\qquad (x\geqq 0).$$

From (3.9) the density of the product of two independent non-central chi-variates with respective parameters (β_1, σ_1^2) and (β_2, σ_2^2) is given by

$$(5.14)\qquad p_{\eta'}(y)=\exp\left\{-\left[\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_2^2}{2\sigma_2^2}\right]\right\}\frac{y}{\sigma_1^2\sigma_2^2}\sum_{j=0}^{\infty}[\beta_2/(\sqrt{2}\,\sigma_2^2)]^{2j}(y^{\sigma_2/\sigma_1})^j \cdot\sum_{i=0}^{j}\frac{[\beta_1\sigma_2/(\beta_2\sigma_1)]^{2i}}{(i!)^2((j-i)!)^2}K_{i-(j/2)}[y/(\sigma_1\sigma_2)],\qquad (y\geqq 0)$$

and from (4.6) the density of their quotient is given by

$$(5.15)\qquad q_{\zeta'}(y)=\frac{2\sigma_1^2\sigma_2^2\exp\{-[\beta_1^2/(2\sigma_1^2)+\beta_2^2/(2\sigma_2^2)]\}y}{(\sigma_2^2y^2+\sigma_1^2)^2}\sum_{j=0}^{\infty}\frac{[\beta_2\sigma_1/(\sqrt{2}\,\sigma_2)]^{2j}(j+1)!}{(\sigma_2^2y^2+\sigma_1^2)^j} \cdot\sum_{i=0}^{j}\frac{(\beta_1 y/\beta_2)^{2i}(\sigma_2/\sigma_1)^{4i}}{(i!)^2((j-i)!)^2},\qquad (y\geqq 0).$$

(iii) *Folded Gaussian distribution*

Substituting $\beta=0$, $\theta=1/\sigma^2$ and $\lambda=1/2$ in (2.3) we get

$$(5.16)\qquad g(x;\sigma^2)=\frac{2}{\sigma\sqrt{2\pi}}e^{-x^2/(2\sigma^2)},\qquad (x\geqq 0)$$

which is the folded Gaussian distribution. The density of the product η' of a folded Gaussian variate with parameter σ_1^2 by an independent folded Gaussian variate with parameter σ_2^2 is given by

$$(5.17)\qquad p_{\eta'}(y)=\frac{2}{\pi\sigma_1\sigma_2}K_0[y/(\sigma_1\sigma_2)],\qquad (y\geqq 0)$$

and the density of their quotient ζ' by

$$(5.18)\qquad q_{\zeta'}(y)=\frac{2\sigma_1\sigma_2}{\pi(\sigma_1^2+\sigma_2^2y^2)},\qquad (y\geqq 0).$$

A concluding remark. Using the methods described in this paper, we have also obtained the distributions of the product and the quotient of two independent generalized Gamma variates introduced by Stacy (see, e.g., [10]). These results are not presented here, since a derivation (using characteristic functions) of the distribution of the quotient is given in [10], and a derivation for the product by the same author appeared very recently in the *Annals of Mathematical Statistics*, 39 (1968), 1751–1752 (see also [14a]).

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