REMARKS ON FINITE INVARIANT MEASURES FOR ONE-PARAMETER GROUP OF MEASURABLE TRANSFORMATIONS

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(Received Sept. 12, 1968; revised Nov. 8, 1968)

1. Introduction

Let (Ω, B, m) be a probability measure space and (T_t) be a measurable non-singular one-parameter group (defined below) of bi-measurable transformations of Ω onto itself. What is a necessary and sufficient condition for the existence of a finite (T_t)-invariant measure μ which is equivalent to m? The purpose of this paper is to show that some necessary and sufficient conditions for the existence of finite invariant measures for a one-parameter group reduce to those for a measurable transformation (Theorem 2) and measurability of a one-parameter group means its continuity in a sense (Theorem 1).

2. Definitions and remarks on them

Let (Ω, B, m) be a σ-finite measure space, i.e., Ω be an abstract space, B be a σ-algebra of subsets of Ω and m be a σ-finite measure on B. Let (Ω, B, μ) be another σ-finite measure space. The measure μ is said to be equivalent to m (notation; $\mu \sim m$) if μ is absolutely continuous with respect to m (notation; $\mu \ll m$) and conversely m is absolutely continuous with respect to μ.

A bi-measurable transformation T of a σ-finite measure space (Ω, B, m) onto itself is said to be non-singular if $m(T^{-1}E)=m(TE)=0$ for any measurable set E with $m(E)=0$. Let (T_t) $(-\infty<t<\infty)$ be a one-parameter group of bi-measurable transformations of the measure space (Ω, B, m) onto itself, i.e., for each t, T_t be a bi-measurable transformation of Ω onto itself and $T_{t+s}\omega = T_t \circ T_s\omega$ $(-\infty<t, s<\infty,\ \omega \in \Omega)$. (T_t) is said to be measurable if $f(T_t\omega)$ is a $L\times B$-measurable function for any B-measurable function f on Ω. (L is the σ-algebra of Lebesgue measurable sets of the real line). (T_t) is said to be non-singular (with respect to m) if we have $m(T_tE)=0$ for all t from $m(E)=0$. (T_t) is said to be a σ-finite flow on (Ω, B, m) if m is (T_t)-invariant and (Ω, B, m) is σ-finite. In particular, if $m(\Omega)=1$, (T_t) is said to be a flow on (Ω, B, m).

We can now state our problem. Let (T_t) be a measurable one-parameter group of bi-measurable transformations of a σ-finite measure space (Ω, B, m) onto itself. What is a necessary and sufficient condition for the existence of a finite (T_t)-invariant measure μ with $\mu \gg m$? As we can find a probability measure space (Ω, B, μ) with $\mu \sim m$, we may and do assume that (Ω, B, m) is a probability measure space. If there exists a finite invariant measure μ with $\mu \gg m$, $m(T_t E)$ is a continuous function of t for each measurable set E (Theorem 1). Let $\tilde{m}$ be a measure defined by

$$\tilde{m}(E)=\sum_{n=1}^{\infty}\frac{1}{2^n}m(T_{\alpha(n)}E)\,,$$

where a set $\{\alpha(n)\}$ $(n=1, 2, \cdots)$ is a countable dense set of the real line. It is easy to verify the following:

$$\mu \gg \tilde{m} \gg m\,,$$

(T_t) is non-singular with respect to $\tilde{m}$.

From this we may and do assume that (T_t) is non-singular with respect to m. If there exists a finite invariant measure μ with $\mu \gg m$, there exists a finite invariant measure $\tilde{\mu}$ with $\tilde{\mu} \sim m$. This is shown as follows. By Radon-Nikodym's theorem, there exists a non-negative measurable function f such that

$$m(E)=\int_E f(\omega)d\mu(\omega) \qquad (E \in B)\,.$$

Put $A=\{\omega; f(\omega)>0\}$. We have $\mu(T_t A \Delta A)=0$, where $A \Delta B = A \cup B - A \cap B$. We define a new measure $\tilde{\mu}$ by

$$\tilde{\mu}(E)=\mu(E \cap A)\,.$$

It is easy to verify that $\tilde{\mu} \sim m$ and $\tilde{\mu}$ is (T_t)-invariant. The above consideration (which is found in [1] in a discrete parameter case) shows that it is natural to formulate the problem of invariant measures in the following way: Let (Ω, B, m) be a probability measure space and (T_t) be a measurable non-singular one-parameter group of bi-measurable transformations of Ω onto itself. What is a necessary and sufficient condition for the existence of a finite (T_t)-invariant measure μ with $\mu \sim m$?

3. Measurability and continuity

THEOREM 1. *Let (Ω, B, m) be a probability measure space and (T_t) be a measurable one-parameter group of bi-measurable transformations of (Ω, B, m) onto itself. If there exists a σ-finite (T_t)-invariant measure μ*

on (Ω, B) with $\mu \gg m$, then the following two continuity conditions hold.

(1) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t-t'| < \delta$, $|m(T_tE)-m(T_{t'}E)| < \varepsilon$ for any $E \in B$.*

(2) *For any set $E \in B$ and any t, we have*

$$\lim_{h\to 0} m(T_{t+h}E \Delta T_tE)=0 .$$

COROLLARY. *If (T_t) is a measurable flow on (Ω, B, m), it is continuous, i.e.,*

$$\lim_{t\to 0} m(T_tE\Delta E)=0 .$$

We need two lemmas to prove the theorem.

LEMMA 1. *Let $f(t)$ be a real-valued function on the real line such that $|f|^p$ is Lebesgue-integrable for some p $(1 \leqq p < \infty)$ on any finite interval $[a, b)$. Then for any finite interval $[a, b)$ we have*

$$\lim_{h\to 0} \int_a^b |f(t+h)-f(t)|^p dt=0 .$$

The lemma is well-known and therefore its proof is omitted.

LEMMA 2. *Let (T_t) be a σ-finite flow on (Ω, B, μ) and $f(\omega)$ be a real-valued function on Ω such that $|f|^p$ is integrable for some p $(1 \leqq p < \infty)$. Then we have*

$$\lim_{h\to 0} \int |f(T_{t+h}\omega)-f(T_t\omega)|^p d\mu(\omega)=0 .$$

Remark. When Ω is separable or $f(T_t\omega)$ is separately-valued, this lemma is proved in [5] or [6] respectively.

PROOF. We have

$$\int\int_a^b |f(T_t\omega)|^p dt d\mu=\int_a^b \int |f(T_t\omega)|^p d\mu dt=(b-a)\int |f|^p d\mu < \infty .$$

From this

$$\int_a^b |f(T_t\omega)|^p dt < \infty \text{ (a.e. } \mu) .$$

By Lebesgue's convergence theorem and Lemma 1, we have,

$$\lim_{h\to 0} \int\int_a^b |f(T_{t+h}\omega)-f(T_t\omega)|^p dt d\mu$$
$$=\int \lim_{h\to 0} \int_a^b |f(T_{t+h}\omega)-f(T_t\omega)|^p dt d\mu=0 .$$

Noting that μ is an invariant measure, we have

$$\lim_{h\to 0}\int |f(T_{t+h}\omega)-f(T_t\omega)|^p d\mu=0 .$$

PROOF OF THE THEOREM 1. As $m\ll\mu$, by Radon-Nikodym's theorem, there exists a non-negative measurable function $f(\omega)$ such that

$$m(E)=\int_E f(\omega)d\mu(\omega) \qquad (E\in B) .$$

As μ is (T_t)-invariant we have,

$$|m(T_{t+h}E)-m(T_tE)|\leqq\int |f(T_{t+h}\omega)-f(T_t\omega)|d\mu(\omega) .$$

By Lemma 2 ($p=1$), we have

$$\lim_{h\to 0}|m(T_{t+h}E)-m(T_tE)|=0 .$$

It is obvious that convergence is uniform with respect to t and E. We have proved the first continuity condition. We first prove (2) for any measurable set E with $\mu(E)<\infty$. We have

$$m(T_{t+h}E\Delta T_tE)=\int |\chi_E(T_{t+h}^{-1}\omega)-\chi_E(T_t^{-1}\omega)|f(\omega)d\mu(\omega) .$$

Let ε be an arbitrary positive number. We can find a function $g(\omega)$ such that

$$\int |g(\omega)|^2 d\mu(\omega)<\infty \quad\text{and}\quad \int |g(\omega)-f(\omega)|d\mu(\omega)<\varepsilon .$$

We have

$$m(T_{t+h}E\Delta T_tE)\leqq\left(\int |\chi_E(T_h\omega)-\chi_E(\omega)|^2 d\mu\right)^{1/2} \cdot\left(\int |g|^2 d\mu\right)^{1/2}+2\int |g-f|d\mu .$$

By Lemma 2 ($p=2$) we have

$$\limsup_{h\to 0} m(T_{t+h}E\Delta T_tE)\leqq 2\varepsilon .$$

Hence, as ε is arbitrary, we have the conclusion. We prove the general case. Let ε be an arbitrary positive number. There exists an increasing sequence $\{E_n\}$ $(n=1, 2, \cdots)$ of sets such that $E=\bigcup_{n=1}^{\infty}E_n$ and $\mu(E_n)<\infty$ $(n=1, 2, \cdots)$. Put $F_n=E-E_n$ $(n=1, 2, \cdots)$. For any t and any ε, there exists a natural number p such that $m(T_tF_n)<\varepsilon$ $(n\geqq p)$.

For this ε, there exists $\delta'>0$ such that $|m(T_{t+h}A)-m(T_tA)|<\varepsilon$ for any h with $|h|<\delta'$ and any set $A\in B$ (Theorem 1, (1)). As $\mu(E_p)<\infty$, there exists $\delta''>0$ such that $m(T_{t+h}E_p\Delta T_tE_p)<\varepsilon$ for any h with $|h|<\delta''$. Put $\delta=\min(\delta', \delta'')$. If $|h|<\delta$, then we have

$$m(T_{t+h}E\Delta T_tE)\leqq m(T_{t+h}E_p\Delta T_tE_p)+m(T_{t+h}F_p)+m(T_tF_p)<4\varepsilon .$$

Remark. If $\mu(\Omega)<\infty$, then for any $\varepsilon>0$ and any $E\in B$, there exists $\delta>0$ such that if $|t-t'|<\delta$, then $m(T_tE\Delta T_{t'}E)<\varepsilon$.

4. Theorem concerning the existence theorem of invariant measures

THEOREM 2. *Let (Ω, B, m) be a probability measure space, (T_t) be a measurable non-singular one-parameter group of bi-measurable transformations of Ω onto itself and T be a non-singular bi-measurable transformation of Ω onto itself. Let (C) be a necessary and sufficient condition for the existence of a T-invariant finite measure μ with $\mu\sim m$. Then a necessary and sufficient condition for the existence of a (T_t)-invariant finite measure μ with $\mu\sim m$ is either* (1) *or* (2).

(1) *There exists s $(s\neq 0)$ such that T_s satisfies* (C).

(2) *For all t, T_t satisfies* (C).

PROOF. It is obvious that we need only to prove that (1) is sufficient. Let μ be a T_s-invariant finite measure with $\mu\sim m$. We define a new measure $\tilde{\mu}$ by

$$\tilde{\mu}(E)=\frac{1}{s}\int_0^s \mu(T_tE)dt .$$

We can readily verify that $\tilde{\mu}$ is (T_t)-invariant and $\tilde{\mu}\sim m$.

COROLLARY. *Let (Ω, B, m), (T_t) and T be the same with those of Theorem 2. Then the following seven conditions are equivalent.*

(1) *There exists a finite (T_t)-invariant measure μ with $\mu\sim m$.*

(2) *There exists s $(s\neq 0)$ such that a limit $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} m(T_s^kE)$ exists for any measurable set E* [2].

(3) *For any real number t and any measurable set E, there exists a limit $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} m(T_t^kE)$* [2].

(4) *There exists s $(s\neq 0)$ such that for any $\varepsilon>0$, there exists $\delta>0$ such that if $m(E)<\delta$, $m(T_s^nE)<\varepsilon$ $(n=0, \pm 1, \pm 2, \cdots)$* [3].

(5) *For any $\varepsilon>0$, there exists $\delta>0$ such that if $m(E)<\delta$, $m(T_tE)<\varepsilon$ for all t* [3].

(6) *There exists s $(s\neq 0)$ such that if $\sum_{i=1}^{\infty} m(T_s^{n_i}E)<\infty$ for some*

sequence of natural numbers $\{n_i\}$ $(n_1<n_2<\cdots)$, $m(E)=0$.

(7) *If* $m(E)>0$, $\sum_{i=1}^{\infty} m(T_t^{n_i}E)=\infty$ *for any* t *and any sequence of natural numbers* $\{n_i\}$ $(n_1<n_2<\cdots)$.

The following is of some interest.

(8) *If* T^k *is bounded for some* k $(k\neq 0)$, T *is bounded, i.e., if* T *is not bounded,* T^k *is not bounded for any* k $(k\neq 0)$. (T *is said to be bounded, if* $m(A-B)=0$ *for any two measurable sets* A, B *such that* $A=\bigcup_{n=-\infty}^{\infty} A_n$, $A_n\cap A_m=\phi$ $(n\neq m)$, $B=\bigcup_{n=-\infty}^{\infty} T^nA_n$, $T^nA_n\cap T^mA_m=\phi$ $(n\neq m)$ *and* $A\supset B$.) [4].

PROOF. These statements are trivial consequences of Theorem 2 and the invariant measure problem for a transformation.

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