FINITE AND INFINITE INVARIANT MEASURES FOR A MEASURABLE TRANSFORMATION

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1. Introduction

Let T be a one-to-one non-singular bi-measurable transformation of a probability measure space (Ω, B, m) onto itself. We consider the existence of measures which are invariant under the transformation T and equivalent to the measure m. Roughly speaking we show the following propositions. There exists a finite invariant measure if and only if the transformation is compact (Theorem 1). There exists a finite invariant measure if and only if the transformation is almost periodic (Theorem 2). Theorem 3 is a generalization of Theorem 1 to a σ-finite case. There exists a σ-finite invariant measure if and only if the transformation is σ-compact. Theorem 4 is a stronger version of Theorem 3. The notion "Compact and σ-compact of the transformation" is introduced here (Definitions 1, 2). The last theorem is a decomposition theorem. The whole space Ω is divided into two invariant sets such that in one of them we can construct a σ-finite invariant measure and in any part of the other we cannot construct any σ-finite invariant measure which is equivalent to m and not identically zero (Theorem 5). The proofs are based on the existence of Banach limit and so-called induced transformations.

2. Notations, definitions and lemma

Let (Ω, B, m) be a measure space, i.e., Ω be an abstract space, B be a σ-algebra of subsets of Ω and m be a measure defined on B. We consider only a σ-finite measure space here. Let (Ω, B, μ) and (Ω, B, ν) be two measure spaces. The measure μ and ν are said to be equivalent (notation; $\mu \sim \nu$), if they are mutually absolutely continuous. If μ and ν are equivalent on a measurable set A, then we denote it by $\mu \sim \nu(A)$. The measure ν is said to be uniformly absolutely continuous with respect to the measure μ, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(E) < \varepsilon$ for any measurable set E with $\mu(E) < \delta$ [3].

A one-to-one transformation of a probability measure space (Ω, B, m)

onto itself is said to be bi-measurable if both T and T^{-1} are measurable. The bi-measurable transformation T is said to be non-singular (with respect to m), if $m(TE)=m(T^{-1}E)=0$ for any measurable set E with $m(E)=0$. We call a non-singular bi-measurable transformation of (Ω, B, m) onto itself an automorphism on (Ω, B, m) for brevity. Let T be an automorphism on (Ω, B, m). The measure m is said to be (T-)invariant, if $m(T^{-1}E)=m(TE)=m(E)$ for any measurable set E. A measurable set A is said to be (T-)equivalent to a measurable set B (notation; $A\sim B$), if there exist two countable decompositions $\{A_n\}$ and $\{B_n\}$ $(n=1, 2, \cdots)$ of A and B respectively and a sequence $\{i(n)\}$ $(n=1, 2, \cdots)$ of integers such that $T^{i(n)}A_n=B_n$ $(n=1, 2, \cdots)$ (A_n can be empty for some natural number n.) ([1], [2]).

DEFINITION 1. From now on we fix a probability measure space (Ω, B, m) and consider an automorphism T on Ω. A measurable set A is said to be (T-)compact, if it satisfies the following: For any $\varepsilon>0$, there exists $\delta>0$ such that if $m(B)<\delta$ and $B\subset A$, then $m(B')<\varepsilon$ for any measurable set B' with $B'\sim B$ and $B\subset A$. A measurable set A is said to be strongly (T-)compact, if it satisfies the following: For any $\varepsilon>0$, there exists $\delta>0$ such that if $m(B)<\delta$ and $B\subset A$, then $\sum_{i=1}^{\infty} m(T^{n_i}B_i)<\varepsilon$ for any countable decomposition $\{B_i\}$ $(i=1, 2, \cdots)$ of B and any sequence of integers $\{n_i\}$ $(i=1, 2, \cdots)$ with $\bigcup_{i=1}^{\infty} T^{n_i}B_i\subset A$.

Remark. Let $\Omega\sim\Omega'$ and denote this equivalence by α, i.e., $\alpha=(\{\Omega_n\}, \{i(n)\})$ $(n=1, 2, \cdots)$. We define a measure m_α by

$$m_\alpha(E)=\sum_{n=1}^{\infty} m(T^{i(n)}(E\cap\Omega_n)) \qquad \text{for any } E\in B\,.$$

We can show that the whole space Ω is T-compact if and only if $\{m_\alpha\}$ is weakly sequentially compact.

DEFINITION 2. The automorphism T is said to be (strongly) compact, if the whole space Ω is (strongly) T-compact. A measurable set A is said to be (strongly) σ-compact (with respect to T), if there exists a countable decomposition $\{A_n\}$ $(n=1, 2, \cdots)$ of A such that every set A_n $(n=1, 2, \cdots)$ is (strongly) T-compact. The automorphism T is said to be (strongly) σ-compact, if the whole space Ω is (strongly) σ-compact with respect to T.

Remark. If a measurable set A is compact, then it is bounded in the sense of Hopf [1]. If a measurable set A is σ-compact, then it is σ-bounded in the sense of Halmos [2].

Here we explain so-called induced transformations ([2], [4]). Let A

be a measurable set contained in the conservative part of Ω. We define an integral-valued function $p(\omega; A)$ on Ω by

$$p(\omega; A)=\min\{n; T^n\omega \in A, n>0\}\cdot \chi_A(\omega),$$

where we put $p(\omega; A)=\infty$, if $\omega \in A$ and $\{n; T^n\omega \in A, n>0\}=\phi$. Put $A'=A-\bigcup_{n=-\infty}^{\infty} T^n\{\omega; p(\omega; A)=\infty\}$. Then $m(A')=m(A)$ and $p(\omega; A')=p(\omega; A)$ for any $\omega \in A'$. We define a transformation S of A onto itself by

$$S\omega=\begin{cases} T^{p(\omega; A)}\omega & \text{if } \omega \in A', \\ \omega & \text{if } \omega \in A-A'. \end{cases}$$

The transformation S is a one-to-one non-singular bi-measurable transformation of A onto itself and called an induced automorphism on A.

We give a lemma which asserts existence of a Banach limit.

LEMMA 1. *For a sequence* $\{x_n\}$ $(n=1, 2, \cdots)$ *of real numbers there exists its Banach limit (notation;* $\operatorname{LIM}_{n\to\infty}(x_n)$*) of the following properties.*

(1) *For any sequence* $\{x_n\}$ $(n=1, 2, \cdots)$ *we have*

$$\liminf_{n\to\infty}(x_n) \leqq \operatorname{LIM}_{n\to\infty}(x_n) \leqq \limsup_{n\to\infty}(x_n).$$

(2) *For any two real numbers* α, β *and two sequences* $\{x_n\}$ *and* $\{y_n\}$ $(n=1, 2, \cdots)$ *we have*

$$\operatorname{LIM}_{n\to\infty}(\alpha x_n+\beta y_n)=\alpha\cdot\operatorname{LIM}_{n\to\infty}(x_n)+\beta\cdot\operatorname{LIM}_{n\to\infty}(y_n).$$

3. Finite invariant measures

In this section we consider an automorphism T on a fixed probability measure space (Ω, B, m). We prove two existence theorems of a finite T-invariant measure μ with $\mu \sim m$.

THEOREM 1. *There exists a finite T-invariant measure* μ *with* $\mu\sim m$ *if and only if T is compact.*

PROOF. Necessity: Let μ be a finite invariant measure on (Ω, B) with $\mu\sim m$. Then μ and m are mutually uniformly absolutely continuous. Let ε be an arbitrary positive number. For this ε, there exists $\eta>0$ such that if $\mu(B)<\eta$, then $m(B)<\varepsilon$. For this η, there exists $\delta>0$ such that if $m(A)<\delta$, then $\mu(A)<\eta$. Therefore if $m(A)<\delta$ and $B\sim A$, then, as $\mu(B)=\mu(A)$, we conclude $m(B)<\varepsilon$, which means compactness of T.

Sufficiency: Put $\mu(A)=\operatorname{LIM}_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} m(T^kA)\right)$. We show that μ is a finite invariant measure with $\mu\sim m$. We obtain the equality $\mu(TA)=\mu(A)$ from

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} m(T^{k+1}A)-\frac{1}{n}\sum_{k=0}^{n-1} m(T^{k}A)\right)=0\,.$$

We show that $\lim_{n\to\infty}\mu(A_n)=0$ for any decreasing sequence of sets $\{A_n\}$ $(A_n\supset A_{n+1},\ n=1,2,\cdots)$ with $\bigcap_{n=1}^{\infty}A_n=\phi$. Let ε be an arbitrary positive number. From the assumption there exists a natural number q such that $m(T^nA_p)<\varepsilon$ $(p\geqq q,\ n=0,\pm 1,\pm 2,\cdots)$. Therefore we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} m(T^kA_p)\leqq\varepsilon$$

for any natural number p with $p\geqq q$, which proves that μ is a measure. It is easy to see that μ is absolutely continuous with respect to m. If $\mu(A)=0$, then $\mu\left(\bigcup_{n=-\infty}^{\infty}T^nA\right)=0$. From this we have $m(A)=0$.

We show two lemmas which are simple but useful to extend invariant measures.

LEMMA 2. *Let S be an induced automorphism on a measurable set A of positive measure which is contained in the conservative part of Ω. Let μ be an S-invariant finite measure on A with $\mu\sim m(A)$. Then there exists a unique T-invariant σ-finite measure ν on the minimum invariant set $[A]$ including A such that ν is equal to μ on A and $\nu\sim m([A])$. In particular if $\int p(\omega;A)d\mu(\omega)<\infty$, then ν is finite on $[A]$.*

PROOF. Put $A_q=\{\omega;\,p(\omega;A)=q\}$ $(q=1,2,\cdots)$. The minimum invariant set $[A]$ including A is represented as the form

$$\left(\bigcup_{q=1}^{\infty}\bigcup_{k=0}^{q-1}T^kA_q\right)\cup N,$$

where N is a measurable set with $m(N)=0$. We define a measure ν on $[A]$ by

$$\nu(B)=\sum_{q=1}^{\infty}\sum_{k=0}^{q-1}\mu(T^{-k}B\cap A_q)$$

for any measurable set B with $B\subset[A]$. If $\int p(\omega;A)d\mu(\omega)<\infty$, then $\nu([A])=\int p(\omega;A)d\mu(\omega)<\infty$. We can verify that ν satisfies all the conditions [2].

LEMMA 3. *If there exists a countable decomposition $\{\Omega_n\}$ $(\Omega_n\in B,$*

$n=1, 2, \cdots$) of Ω and a sequence $\{\mu_n\}$ ($n=1, 2, \cdots$) of (σ-) finite T-invariant measures such that μ_n is defined on the minimum T-invariant set $[\Omega_n]$ including Ω_n with $\mu_n \sim m([\Omega_n])$, then there exists a (σ-) finite T-invariant measure μ with $\mu \sim m$.

PROOF. Put $A_n=[\Omega_n]-\bigcup_{k=1}^{n-1}[\Omega_k]$ ($n=1, 2, \cdots$). It is obvious that $\Omega=\bigcup_{n=1}^{\infty} A_n$, $A_n \cap A_m=\phi$ ($n \neq m$) and $TA_n=T^{-1}A_n=A_n$ ($n=1, 2, \cdots$). We define a σ-finite measure μ on (Ω, B) by

$$\mu(E)=\sum_{n=1}^{\infty} \mu_n(E \cap A_n) \qquad \text{for any measurable set } E.$$

As A_n ($n=1, 2, \cdots$) is an invariant set, μ is an invariant measure. In particular if $\mu_n([\Omega_n])<\infty$ ($n=1, 2, \cdots$), then we can choose a sequence of positive number $\{\alpha_n\}$ ($\alpha_n>0$, $n=1, 2, \cdots$) such that $\sum_{n=1}^{\infty} \alpha_n \mu_n(A_n)<\infty$. Putting $\mu(E)=\sum_{n=1}^{\infty} \alpha_n \mu_n(E \cap A_n)$, we obtain a finite T-invariant measure μ with $\mu \sim m$.

THEOREM 2. *A necessary and sufficient condition for the existence of a finite T-invariant measure μ with $\mu \sim m$ is the following:*
For any $\varepsilon>0$, there exists a countable decomposition $\{\Omega_n\}$ ($n=1, 2, \cdots$) of Ω satisfying (1) *and* (2).

(1) *For each n, for any measurable set A and any integer k such that $A \subset \Omega_n$, $T^k A \subset \Omega_n$ we have*

$$m(A)/(1+\varepsilon) \leqq m(T^k A) \leqq (1+\varepsilon) m(A)\,.$$

(2) $\int p(\omega; \Omega_n) dm(\omega)<\infty \qquad (n=1, 2, \cdots)\,,$

where $p(\omega; A)=\min\{n; T^n\omega \in A,\ n>0\} \cdot \chi_A(\omega)$.

PROOF. Necessity: We assume that there exists a T-invariant finite measure μ on (Ω, B) with $\mu \sim m$. By Radon-Nikodym's theorem, there exists a non-negative measurable function $f(\omega)$ such that

$$(*) \qquad m(E)=\int_E f(\omega) d\mu(\omega) \qquad (E \in B)\,.$$

We put

$$\Omega_{2n}=\{\omega; (1+\varepsilon)^n \leqq f(\omega)<(1+\varepsilon)^{n+1}\} \qquad (n=1, 2, \cdots)\,,$$

$$\Omega_{2n+1}=\{\omega; (1+\varepsilon)^{-n} \leqq f(\omega)<(1+\varepsilon)^{-n+1}\} \qquad (n=0, 1, 2, \cdots)\,.$$

We have $m\left(\Omega-\bigcup_{n=1}^{\infty}\Omega_n\right)=0$, since $0<f(\omega)<\infty$ (a.e. μ). Let A be any measurable set with $A\subset\Omega_{2n}$. Let k be an arbitrary integer with $T^kA\subset\Omega_{2n}$. Then by (*) we have

$$(1+\varepsilon)^n\mu(T^kA)\leqq m(T^kA)\leqq(1+\varepsilon)^{n+1}\mu(T^kA)\,,$$

$$(1+\varepsilon)^n\mu(A)\leqq m(A)\leqq(1+\varepsilon)^{n+1}\mu(A)\,.$$

From this, we obtain the inequality,

$$m(A)/(1+\varepsilon)\leqq m(T^kA)\leqq(1+\varepsilon)m(A)\,.$$

We obtain the same inequality when n is odd. We put $\Omega_\infty=\Omega-\bigcup_{n=1}^{\infty}\Omega_n$. As it is trivial that Ω_∞ satisfies (1) and (2), we need only to prove (2) for Ω_n $(n=1, 2, \cdots)$. We have

$$\int p(\omega\,;\Omega_n)dm(\omega)\leqq(1+\varepsilon)^{(n+2)/2}\int p(\omega\,;\Omega_n)d\mu(\omega)\quad(n:\text{ even})\,,$$

$$\int p(\omega\,;\Omega_n)dm(\omega)\leqq\int p(\omega\,;\Omega_n)d\mu(\omega)\quad(n:\text{ odd})\,.$$

Therefore it is sufficient for the proof to show that $\int p(\omega\,;A)d\mu(\omega)<\infty$ for any measurable set A. Put $A_q=\{\omega\,;p(\omega\,;A)=q\}$ $(q=1, 2, \cdots)$. Noting that $T^kA_q\cap T^{k'}A_{q'}=\phi$ $((k, q)\neq(k', q'),\ 0\leqq k\leqq q-1,\ 0\leqq k'\leqq q'-1)$, we have

$$\begin{aligned}\int p(\omega\,;A)d\mu(\omega)&=\sum_{q=1}^{\infty}q\mu(A_q)=\sum_{q=1}^{\infty}\sum_{k=0}^{q-1}\mu(T^kA_q)\\&=\mu\left(\bigcup_{q=1}^{\infty}\bigcup_{k=0}^{q-1}T^kA_q\right)\leqq\mu(\Omega)<\infty\,.\end{aligned}$$

Sufficiency: Let ε be a fixed positive number. Then there exists a countable decomposition $\{\Omega_n\}$ $(n=1, 2, \cdots)$ of Ω satisfying both conditions (1) and (2). Let us fix this decomposition. Let us denote an induced automorphism on Ω_n by S_n $(n=1, 2, \cdots)$. It is easy to see that we can construct such an automorphism S_n, as $\int p(\omega\,;\Omega_n)dm(\omega)<\infty$ $(n=1, 2, \cdots)$. If we can construct an S_n-invariant finite measure μ_n on Ω_n such that $\int p(\omega\,;\Omega_n)d\mu_n(\omega)<\infty$ and $\mu_n\sim m(\Omega_n)$ $(n=1, 2, \cdots)$, then by Lemmas 2 and 3, we can construct a T-invariant finite measure on Ω. Therefore we shall construct such a measure μ_n on Ω_n with the properties. We denote S_n and Ω_n by S and Ω respectively. We have

$$m(A)/(1+\varepsilon)\leqq m(S^kA)\leqq(1+\varepsilon)m(A)$$

for any measurable set A with $A\subset\Omega$. Let us define a set function μ by

$$\mu(A)=\underset{n\to\infty}{\mathrm{LIM}}\left(\frac{1}{n}\sum_{k=0}^{n-1} m(S^k A)\right).$$

We have $m(A)/(1+\varepsilon)\leqq\mu(A)\leqq(1+\varepsilon)m(A)$. It is easy to verify that μ is an S-invariant finite measure on Ω, $\int p(\omega;\Omega)d\mu(\omega)<\infty$ and $\mu\sim m$.

Remark. As we can see easily from the proof, the condition that for any $\varepsilon>0$ there exists a countable decomposition $\{\Omega_n\}$ $(n=1, 2, \cdots)$ of Ω satisfying the condition (1), is a necessary and sufficient one for the existence of a σ-finite T-invariant measure μ with $\mu\sim m$. The author heard that this result was obtained by L. K. Arnold in his doctoral thesis of Brown University (1966).

4. σ-finite invariant measure

Let T be an automorphism on a fixed probability measure space (Ω, B, m). In this section we give two existence theorems of a σ-finite T-invariant measure μ with $\mu\sim m$.

THEOREM 3. *There exists a σ-finite T-invariant measure μ with $\mu\sim m$ if and only if T is σ-compact.*

PROOF. Necessity: Let μ be a σ-finite invariant measure with $\mu\sim m$. Then there exists a countable decomposition $\{\Omega_n\}$ $(n=1, 2, \cdots)$ of Ω such that $\mu(\Omega_n)<\infty$ $(n=1, 2, \cdots)$. Considering both measures μ and m as defined on Ω_n, they are both finite and equivalent. By the same way as the first half of the proof of Theorem 1, we obtain the necessity.

Sufficiency: Let Ω_0 be the dissipative part of Ω. Then there exists a wandering set A such that

$$\Omega_0=\bigcup_{n=-\infty}^{\infty} T^n A, \qquad T^n A\cap T^m A=\phi \quad (n\neq m).$$

Put

$$\mu(E)=\sum_{n=-\infty}^{\infty} m(T^{-n}E\cap A) \qquad \text{for any measurable set } E \text{ with } E\subset\Omega_0.$$

It is easy to see that μ is a σ-finite invariant measure with $\mu\sim m(\Omega_0)$. Put $\Omega_n^*=\Omega_n\cap(\Omega-\Omega_0)$ $(n=1, 2, \cdots)$. It is obvious that every set Ω_n^* is T-compact. We define an induced automorphism S_n $(n=1, 2, \cdots)$ on Ω_n^*. Every set Ω_n^* is S_n-compact. By Theorem 1 there exists a finite S_n-invariant measure μ_n on Ω_n^* with $\mu_n\sim m(\Omega_n^*)$. By Lemmas 2 and 3, we obtain a σ-finite T-invariant measure μ with $\mu\sim m(\Omega-\Omega_0)$.

THEOREM 4. *There exists a σ-finite T-invariant measure if and only if T is strongly σ-compact.*

PROOF. Sufficiency: It is obvious that T is σ-compact. The result follows from Theorem 3.

Necessity: Let μ be a σ-finite invariant measure with $\mu \sim m$. Then by Radon-Nikodym's theorem, there exists a non-negative measurable function on Ω such that

$$m(E)=\int_E f(\omega)d\mu(\omega) \qquad \text{for any set } E \in B.$$

Let a be any real number with $a>1$. Put

$$\Omega_{2n}=\{\omega;\, a^n \leqq f(\omega) < a^{n+1}\} \qquad (n=1, 2, \cdots),$$

$$\Omega_{2n+1}=\{\omega;\, a^{-n} \leqq f(\omega) < a^{-n+1}\} \qquad (n=0, 1, 2, \cdots).$$

Then we easily obtain the result by the same way as the proof of Theorem 2.

5. Decomposition theorem

THEOREM 5. *The whole space has following unique decompositions,*

$$\Omega=\Omega_c+\bar{\Omega}_c, \qquad \Omega_c=\Omega_{c\sigma}+\bar{\Omega}_{c\sigma}, \qquad \Omega_{c\sigma}=\Omega_{c\sigma c}+\bar{\Omega}_{c\sigma c},$$

Ω_c : *conservative part,* $\bar{\Omega}_c$: *dissipative part,*

$\Omega_{c\sigma}$: *σ-compact part,* $\Omega_{c\sigma c}$: *compact part.*

The whole space is divided into two invariant sets $\Omega^*(=\bar{\Omega}_c+\Omega_{c\sigma})$, $\Omega^{**}(=\Omega-\Omega^*)$. We have a σ-finite invariant measure μ on Ω^* with $\mu \sim m(\Omega^*)$. Let A be an arbitrary set with $A \subset \Omega^{**}$ and $m(A)>0$. We have no σ-finite invariant measure on A with $\mu \sim m(A)$.

PROOF. It is well known that $\Omega=\Omega_c+\bar{\Omega}_c$. If there exists no σ-compact set of positive measure in Ω_c, then there is nothing to prove. For we put $\Omega_{c\sigma}=\phi$ and $\bar{\Omega}_{c\sigma}=\Omega_c$. Let us assume that there exists a σ-compact set of positive measure in Ω_c. Then there exists a compact set A of positive measure. Let S be an induced automorphism on A. Since A is T-compact, it is also S-compact. By Theorem 1 there exists an S-invariant finite measure μ on A with $\mu \sim m(A)$. We extend the measure to the T-invariant set $[A]$ by Lemma 2. We have thus proved that if there exists a σ-compact set of positive measure, then there exists a σ-compact invariant set of positive measure. Let F be a family of all the σ-compact invariant sets of Ω. We put $\alpha=\sup_{A \in F} m(A)$. It is easy to see that we obtain a σ-compact invariant set A^* with $m(A^*)=\alpha$. We put $\Omega_{c\sigma}=A^*$, $\bar{\Omega}_{c\sigma}=\Omega_c-\Omega_{c\sigma}$. Let B be a measurable set of positive measure in $\bar{\Omega}_{c\sigma}$. Then B is not σ-compact. If there exists no compact invariant set of positive measure in $\Omega_{c\sigma}$, then we put $\Omega_{c\sigma c}=\phi$, $\bar{\Omega}_{c\sigma c}=\Omega_{c\sigma}$. If there exists a compact invariant set of positive measure, we also obtain the maximal compact set A^*. We put $\Omega_{c\sigma c}=A^*$, $\bar{\Omega}_{c\sigma c}=\Omega_{c\sigma}-\Omega_{c\sigma c}$. It is obvious that each decomposition is unique with exception to a set of measure 0. The last statement follows from the above.

Remark. The theorem remains true if we exchange σ-compact for strongly σ-compact. The σ-compact part of Ω and the strongly σ-compact coincide.

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