ON THE ASYMPTOTIC THEORY OF RANK ORDER TESTS FOR EXPERIMENTS INVOLVING PAIRED COMPARISONS*

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(Received April 25, 1968; revised July 18, 1968)

Summary

The problem of testing the hypothesis of no difference among several treatments for the case when the comparisons between the treatments is possible only in pairs has been considered by Durbin [3], Bradley and Terry [1], Elteren and Noether [4], and Mehra and Puri [6], among others. Following the lines of Sen and Puri [9], a new approach to the asymptotic theory of rank order tests for this problem is developed. This avoids the unnecessarily complicated and lengthy conditional approach of Mehra and Puri [6] and also simplifies the proofs considerably.

1. Preliminary notions

Let us consider t ($\geqq 2$) treatments in an experiment involving paired comparisons, and suppose that for the pair (i, j) of treatments $(1\leqq i\leqq j\leqq t)$, the N_{ij} encounters yield the random variables X_{ijl}, $l=1, \cdots, N_{ij}$, which are independent and identically distributed according to an absolutely continuous cumulative distribution function (cdf) $F^{(i,j)}(x)$, for $1\leqq i<j\leqq t$. The null hypothesis to be tested states that

$$(1.1) \quad H_0: F^{(i,j)}(x)+F^{(i,j)}(-x)=1 \quad \text{and} \quad F^{(i,j)}(x)=F(x) \qquad \text{for all } i\neq j\,,$$

that is, each $F^{(i,j)}(x)$ is symmetric with respect to the origin and that furthermore all the $t(t-1)/2$ cdf's are identical.

Under the null hypothesis, all the X's are independent and have the common cdf $F(x)$. Let $Z_{N,r}^{(i,j)}=1$ (or -1) if the rth smallest of N $(=\sum_{i<j} N_{ij})$ ordered absolute observations $(|X_{ijl}|,\ l=1, \cdots, N_{ij},\ 1\leqq i\leqq j\leqq t)$ is from the (i, j)th pair and the corresponding X_{ijl} is positive (or negative) and otherwise let $Z_{N,r}^{(i,j)}=0$. The proposed test statistic may then be defined as

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* Prepared under contract Nonr-285(38) with the Office of Naval Research, and Research Grant GM-12868 with the National Institutes of Health, Public Health Service.

$$\mathcal{S}_N=(1/tA^2)\sum_{i=1}^{t}\left(\sum_{j=1\,(\neq i)}^{t} N_{ij}^{1/2}T_{N,ij}\right)^2, \tag{1.2}$$

where A is some (positive) constant,

$$T_{N,ij}=\sum_{r=1}^{N} E_{N,r}Z_{N,r}^{(i,j)}/N_{ij}\,, \qquad 1\leqq i<j\leqq t\,, \tag{1.3}$$

and $E_{N,r}=J_N(r/(N+1))$, $r=1,\cdots,N$ are explicit functions of the ranks satisfying the assumptions 1, 2 and 3 of section 2. For example, $E_{N,r}$ may be the expected value of the rth order statistic in a sample of size N from the chi-distribution with one degree of freedom or from the uniform distribution over (0, 1). The test is based on $\mathcal{S}_N$, rejecting the null hypothesis (1.1) for significantly large values of it. The object of the present paper is to develop the asymptotic distribution theory of $T_{N,ij}$, $1\leqq i<j\leqq t$ and of $\mathcal{S}_N$. This generalizes the result of Sen and Puri [9] to the multisample analog of the one sample case (but to the univariate problem). This also supplies new proofs of the theorems to follow, which are not only simpler and shorter than the corresponding ones by Mehra and Puri [6] but also based on a direct approach instead of the unnecessary complicated conditional approach of the latter paper (cf. Part III). When $t=2$ and $E_{N,r}=r/(N+1)$, then the $\mathcal{S}_N$ test coincides with the symmetrical two-tail version of the one sample Wilcoxon paired-comparison test. Similarly, with $t=2$ and taking $E_{N,r}$ as the expected value of the rth order statistic in a sample of size N from the chi-distribution with one degree of freedom, we get the two sided version of the normal scores one sample paired comparison test. Thus, depending on what $E_{N,r}$ we take, the $\mathcal{S}_N$ tests may be regarded as the multisample analog of some one sample paired-comparison tests.

2. Joint asymptotic normality

Before proving the main theorem of this section we introduce a few notations and assumptions.

Let $c=t(t-1)/2$ denote the number of all possible pairs and label the pair (i, j) by $\alpha=(i-1)t+j-\binom{i+1}{2}$, $1\leqq i<j\leqq t$. Then $X_{\alpha 1},\cdots,X_{\alpha N}$ are the observations corresponding to the αth pair, $\alpha=1,\cdots,c$. Let $N=\sum_{\alpha=1}^{c} N_\alpha$ and $\rho_\alpha^{(N)}=N_\alpha/N$ and assume that for all N, the inequalities $0<\rho_0\leqq\rho_\alpha^{(N)}\leqq 1-\rho_0<1$ hold for some fixed $\rho_0\leqq 1/c$ and $\alpha=1,\cdots,c$.

Denote

$$F_{N_\alpha}^{(\alpha)}(x)=N_\alpha^{-1} \quad (\text{number of } X_{\alpha l}\leqq x,\ l=1,\cdots,N_\alpha)\,, \tag{2.1}$$

$$H_{N_\alpha}^{(\alpha)}(x)=N_\alpha^{-1} \quad (\text{number of } |X_{\alpha l}|\leqq x,\ l=1,\cdots,N_\alpha) \tag{2.2}$$

$$=F_{N_\alpha}^{(\alpha)}(x)-F_{N_\alpha}^{(\alpha)}(-x-)\,, \qquad x\geqq 0\,,$$

$$H^{(\alpha)}(x)=F^{(\alpha)}(x)-F^{(\alpha)}(-x-)\,, \qquad x\geqq 0\,, \tag{2.3}$$

$$H_N(x)=\sum_{\alpha=1}^{c}\rho_N^{(\alpha)}H_{N_\alpha}^{(\alpha)}(x)\,, \qquad H(x)=\sum_{\alpha=1}^{c}\rho_N^{(\alpha)}H^{(\alpha)}(x)\,, \tag{2.4}$$

$$F_N(x)=\sum_{\alpha=1}^{c}\rho_N^{(\alpha)}F_{N_\alpha}^{(\alpha)}(x)\,, \qquad F(x)=\sum_{\alpha=1}^{c}\rho_N^{(\alpha)}F^{(\alpha)}(x)\,. \tag{2.5}$$

Then, $T_{N,\alpha}$ defined in (1.3) may be rewritten as

$$T_{N,\alpha}=\int_{x=0}^{\infty}J_N\left\{\frac{N}{N+1}H_N(x)\right\}d\{F_{N_\alpha}^{(\alpha)}(x)+F_{N_\alpha}^{(\alpha)}(-x-)\}\,, \qquad \alpha=1,\cdots,c\,. \tag{2.6}$$

Regarding the function J_N, we make the following assumptions (also considered in Sen and Puri [9]).

ASSUMPTION 1. $\lim_{N\to\infty}J_N(u)=J(u)$ exists for $0<u<1$, and is not a constant; $J(0)=0$.

ASSUMPTION 2. $$\int_0^\infty\left[J_N\left\{\frac{N}{N+1}H_N(x)\right\}-J\left\{\frac{N}{N+1}H_N(x)\right\}\right]\cdot d\{F_{N_\alpha}^{(\alpha)}(x)+F_{N_\alpha}^{(\alpha)}(-x)\}=o_p(N^{-1/2})\,.$$

ASSUMPTION 3. $J(u)$ is absolutely continuous for $0<u<1$ and $|J^{(i)}(u)|=|d^{(i)}J(u)/du^{(i)}|\leqq K[u(1-u)]^{\delta-i-1/2}$, $i=0,1$ for some K and some $\delta>0$.

THEOREM 2.1. *Under assumptions* 1 *to* 3, *the random vector* $[\sqrt{N_1}\cdot(T_{N,1}-\mu_{N,1}),\cdots,\sqrt{N_c}(T_{N,c}-\mu_{N,c})]$ *where*

$$\mu_{N,\alpha}=\int_{x=0}^{\infty}J\{H(x)\}d\{F^{(\alpha)}(x)+F^{(\alpha)}(-x)\}\,, \qquad \alpha=1,\cdots,c \tag{2.7}$$

has a limiting normal distribution with mean vector zero and dispersion matrix $(\sqrt{N_\alpha N_{\alpha'}}\,\sigma_{N,\alpha\alpha'})$ *given by* (5.12) *and* (5.13) *respectively.*

PROOF. Proceeding exactly as in Sen and Puri [9], we write $T_{N,\alpha}$ as

$$T_{N,\alpha}=\mu_{N,\alpha}+B_{1N,\alpha}+B_{2N,\alpha}+\sum_{i=1}^{4}C_{iN,\alpha}\,, \tag{2.8}$$

where $\mu_{N,\alpha}$ is given by (2.7) and

$$B_{1N,\alpha}=\int_0^\infty J\{H(x)\}d\{(F_{N_\alpha}^{(\alpha)}(x)-F^{(\alpha)}(x))+(F_{N_\alpha}^{(\alpha)}(-x-)-F^{(\alpha)}(-x))\}\,, \tag{2.9}$$

$$B_{2N,\alpha}=\int_0^\infty\{H_N(x)-H(x)\}J'\{H(x)\}d\{F^{(\alpha)}(x)+F^{(\alpha)}(-x)\}\,, \tag{2.10}$$

$$(2.11)\quad C_{1N,\alpha}=-\frac{1}{N+1}\int_0^\infty H_N(x)J'\{H(x)\}d\{F_{N_\alpha}^{(\alpha)}(x)+F_{N_\alpha}^{(\alpha)}(-x-)\}\,,$$

$$(2.12)\quad C_{2N,\alpha}=\int_0^\infty \{H_N(x)-H(x)\}J'\{H(x)\}d\{(F_{N_\alpha}^{(\alpha)}(x)-F^{(\alpha)}(x)) +(F_{N_\alpha}^{(\alpha)}(-x-)-F^{(\alpha)}(-x))\}\,,$$

$$(2.13)\quad C_{3N,\alpha}=\int_0^\infty \left[J\left\{\frac{N}{N+1}H_N(x)\right\}-J\{H(x)\}-\left\{\frac{N}{N+1}H_N(x)-H(x)\right\} \cdot J'\{H(x)\}\right]\cdot d\{F_{N_\alpha}^{(\alpha)}(x)+F_{N_\alpha}^{(\alpha)}(-x-)\}\,,$$

$$(2.14)\quad C_{4N,\alpha}=\int_0^\infty\left[J_N\left\{\frac{N}{N+1}H_N(x)\right\}-J\left\{\frac{N}{N+1}H_N(x)\right\}\right] \cdot d\{F_{N_\alpha}^{(\alpha)}(x)+F_{N_\alpha}^{(\alpha)}(-x-)\}\,.$$

The term $\mu_{N,\alpha}$ is finite by assumption 3, and the C-terms are all $o_p(N^{-1/2})$, the proof being exactly the same as in Sen and Puri [9]. The difference $\sqrt{N_\alpha}(T_{N,\alpha}-\mu_{N,\alpha})-\sqrt{N_\alpha}(B_{1N,\alpha}+B_{2N,\alpha})$ tends to zero in probability and so the vectors $[\sqrt{N_\alpha}(T_{N,\alpha}-\mu_{N,\alpha}),\ \alpha=1,\cdots,c]$ and $[\sqrt{N_\alpha}(B_{1N,\alpha}+B_{2N,\alpha}),\ \alpha=1,\cdots,c]$ possess the same limiting distribution, if they have one at all. Thus to prove the theorem, it suffices to show that for any real δ_α, $\alpha=1,\cdots,c$, not all zero, $\sum_{\alpha=1}^{c}\delta_\alpha N_\alpha^{1/2}(B_{1N,\alpha}+B_{2N,\alpha})$ has normal distribution in the limit. Now denoting $c(u)=1$ or 0 according as $u\geqq 0$ or not,

$$(2.15)\qquad B_0(X_{\alpha l})=J[H(|X_{\alpha l}|)]\{c(X_{\alpha l})-(1-c(X_{\alpha l}))\}-\mu_{N,\alpha}\,,$$

and

$$(2.16)\quad C_0(X_{il})=\rho_N^{(i)}\int_0^\infty [c(x-|X_{il}|)-H^{(i)}(x)]J'[H(x)]d[F^{(\alpha)}(x)+F^{(\alpha)}(-x)]\,,$$

we can express $B_{1N,\alpha}+B_{2N,\alpha}$ as

$$(2.17)\qquad B_{1N,\alpha}+B_{2N,\alpha}=\frac{1}{N_\alpha}\sum_{l=1}^{N_\alpha}B_0(X_{\alpha l})+\sum_{i=1}^{c}\left\{\frac{1}{N_i}\sum_{l=1}^{N_i}C_0(X_{il})\right\}.$$

Hence

$$\begin{aligned}
&\sum_{\alpha=1}^{c}\delta_\alpha(B_{1N,\alpha}+B_{2N,\alpha})\\
&=\sum_{\alpha=1}^{c}\delta_\alpha\left[\frac{1}{N_\alpha}\sum_{l=1}^{N_\alpha}B_0(X_{\alpha l})+\sum_{i=1}^{c}\left\{\frac{1}{N_i}\sum_{l=1}^{N_i}C_0(X_{il})\right\}\right]\\
&=\sum_{i=1}^{c}\left[\frac{1}{N_i}\sum_{l=1}^{N_i}C_0(X_{il})\sum_{\alpha=1}^{c}\delta_\alpha\right]+\sum_{\alpha=1}^{c}\delta_\alpha\left[\frac{1}{N_\alpha}\sum_{l=1}^{N_\alpha}B_0(X_{\alpha l})\right]\\
&=\sum_{i=1}^{c}\left[\frac{1}{N_i}\sum_{l=1}^{N_i}C_0(X_{il})\sum_{\alpha=1}^{c}\delta_\alpha\right]+\sum_{i=1}^{c}\left[\delta_i\frac{1}{N_i}\sum_{l=1}^{N_i}B_0(X_{il})\right],
\end{aligned}$$

$$=\sum_{i=1}^{c}\left[\frac{1}{N_i}\sum_{l=1}^{N_i}D_0(X_{il})\right], \tag{2.18}$$

where

$$D_0(X_{il})=C_0(X_{il})\sum_{\alpha=1}^{c}\delta_\alpha+\delta_i B_0(X_{il}) . \tag{2.19}$$

The c-summations given by (2.18) involve independent samples of identically distributed random variables having finite $2+\delta'$ moments. Thus by the Central Limit Theorem, each sum properly normalized has normal distribution in the limit and hence the sum of c-summations will have normal distribution in the limit. The theorem follows. (The variance covariance terms of $[\sqrt{N_\alpha}(B_{1N,\alpha}+B_{2N,\alpha}),\ \alpha=1,\cdots,c]$ are computed in the Appendix).

3. The limiting distribution of $\mathcal{S}_N$ under shift alternatives

From this section onward, we concern ourselves with a sequence of admissible alternative hypotheses $\{H_N\}$ which specify that for each $\alpha=1,\cdots,c$,

$$H_N:\ F^{(\alpha)}(x)=F(x+\mu_\alpha/N^{1/2}),\ F(x)+F(-x)=1 .$$

Then we have the following theorem.

THEOREM 3.1. *If* (i) $\rho_N^{(\alpha)}\rightarrow\rho^{(\alpha)}$ *with* $0<\rho^{(\alpha)}<1$, $\alpha=1,\cdots,c$, *and* (ii) *the conditions of Theorem* 2.1 *are satisfied, then under the sequence* $\{H_N\}$ *of alternative hypotheses, the* $c=t(t-1)/2$ *random variables* $\{N_\alpha^{1/2}T_{N,\alpha},\ \alpha=1,\cdots,c\}$ *have asymptotically independent normal distributions with means* η_α, $\alpha=1,\cdots,c$, *and a common variance* A^2, *where*

$$\eta^{(\alpha)}=2\mu_\alpha\sqrt{\rho^{(\alpha)}}\left(\int_0^\infty\frac{d}{dx}J\{F(x)-F(-x)\}dF(x)\right),$$

and

$$A^2=\int_0^1 J^2(x)dx . \tag{3.1}$$

The proof of this theorem is an immediate consequence of Theorem 2.1 and the facts that

$$\lim_{N\rightarrow\infty}N_\alpha\sigma_{N,\alpha\alpha}=A^2 ,\qquad \lim_{N\rightarrow\infty}(N_\alpha N_{\alpha'})^{1/2}\sigma_{N,\alpha\alpha'}=0 ,$$

and

$$\lim_{N\rightarrow\infty}N_\alpha^{1/2}\mu_{N,\alpha}=\eta^{(\alpha)}\,*.$$

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* Here we have assumed the regularity conditions which allow differentiation under the integral sign.

It follows therefore that the random vector $W=(W_1, \cdots, W_t)$ where $W_i=\sum_{j\neq 1}(N_{ij}^{1/2}T_{N,ij}-\eta^{(ij)})/At^{1/2}$ has a limiting normal distribution with mean vector zero and a covariance matrix whose (j,j')th term is $(\delta_{jj'}-1/t)$ where $\delta_{jj'}$ is the Kronecker delta. Consequently, we have the following theorem:

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, *the statistic* $\mathcal{S}_N$ *defined in* (1.2) *has asymptotically as* $N\to\infty$, *the non-central chi-square distribution with* $t-1$ *degrees of freedom and non-centrality parameter* $\Delta_{\mathcal{S}}$ *where*

$$\Delta_{\mathcal{S}}=\frac{4}{A^2t}\left(\int_0^\infty \frac{d}{dx}J\{F(x)-F(-x)\}dF(x)\right)^2\sum_{i=1}^{t}\left[\sum_{j\neq i}\rho^{ij}(\theta_i-\theta_j)\right]. \tag{3.2}$$

For the special case, when all N_{ij} *are equal, that is, when the design is balanced,*

$$\Delta_{\mathcal{S}}=\frac{8}{A^2(t-1)}\left(\int_0^\infty \frac{d}{dx}J\{F(x)-F(-x)\}\right)^2\sum_{i=1}^{t}(\theta_i-\bar\theta)^2\,, \tag{3.3}$$

where

$$\bar\theta=\sum_{i=1}^{t}\theta_i/t\,.$$

4. Asymptotic relative efficiency

In this section we discuss briefly the asymptotic relative efficiency (ARE) of the $\mathcal{S}_N$ test relative to the parametric competitor, that is, the analysis of variance $\mathcal{F}$-test, and the Durbin-Bradley-Terry D test. Van Elteren and Noether [4] have shown that for the case of balanced incomplete block designs, Durbin's D test [3] and the analysis of variance $\mathcal{F}$-test are asymptotically distributed as non-central chi-square with $t-1$ degrees of freedom and non-centrality parameter $\Delta_{\mathcal{F}}$ and Δ_D respectively, where

$$\Delta_{\mathcal{F}}=2\sum_{i=1}^{t}(\theta_i-\bar\theta)^2/\sigma^2(t-1)\,, \tag{4.1}$$

where σ^2 is the variance of $F(x)$, and

$$\Delta_D=8f^2(0)\sum_{i=1}^{t}(\theta_i-\bar\theta)^2/(t-1) \tag{4.2}$$

(f is the density of F). Furthermore, it was also shown by Van Elteren and Noether [4] that the Bradley-Terry test [1] is asymptotically equivalent to Durbin's paired comparison test [3]. Hence, using a theorem of Hannan [5], the Pitman efficiency of the $\mathcal{S}$-test relative to the $\mathcal{T}$-test and the D-test are given by

$$(4.3) \qquad e_{\mathcal{S},\mathcal{T}}=\Delta_{\mathcal{S}}/\Delta_{\mathcal{T}}=4\sigma^2\left(\int_0^\infty \frac{d}{dx}[J\{F(x)-F(-x)\}]dF(x)\right)^2\Big/A^2,$$

and

$$(4.4) \qquad e_{\mathcal{S},D}=\Delta_{\mathcal{S}}/\Delta_D=2\left(\int_0^\infty \frac{d}{dx}[J\{F(x)-F(-x)\}]dF(x)\right)^2\Big/A^2f^2(0),$$

respectively. The above efficiencies depend upon the score function J, and the underlying distribution function F. In what follows we shall consider some special cases.

Special Cases

(a) Let J be the inverse of the chi-distribution with one degree of freedom. Then the $\mathcal{S}$-test reduces to the $\mathcal{S}(\Phi)$-test. In this case

$$(4.5) \qquad e_{\mathcal{S}(\Phi),\mathcal{T}}=\sigma^2\left(\int_{-\infty}^\infty \frac{f^2(x)dx}{\phi\{\Phi^{-1}[F(x)]\}}\right)^2,$$

where ϕ is the standard normal density function whose cdf is Φ. This efficiency is the same as that of the one or two sample normal scores test relative to the Student's t-test, and is known to be $\geqq 1$ for all F, and is 1 if and only if F is normal. (Puri [7], [8]).

$$(4.6) \qquad e_{\mathcal{S}(\Phi),D}=\left(\int_{-\infty}^\infty \frac{f^2(x)dx}{\phi\{\Phi^{-1}[F(x)]\}}\right)^2\Big/4f^2(0).$$

This efficiency is the same as that of the one or two sample normal scores test relative to the sign test. This equals $\pi/2$, ∞ or $2/\pi$ according as F is normal, uniform over $(0, 1)$ or double exponential respectively.

(b) Let J be the inverse of the rectangular distribution over $(0, 1)$. Then the $\mathcal{S}$-test reduces to the $\mathcal{S}(W)$ test. In this case

$$(4.7) \qquad e_{\mathcal{S}(W),\mathcal{T}}=12\sigma^2\left(\int_{-\infty}^\infty f^2(x)dx\right)^2,$$

and

$$(4.8) \qquad e_{\mathcal{S}(W),D}=3\left(\int_{-\infty}^\infty f^2(x)dx\right)^2\Big/f^2(0).$$

The efficiency (4.7) is the same as that of the one or two sample Wilcoxon test relative to the Student's t-test and is known to satisfy $e_{\mathcal{S}(W),\mathcal{T}}\geqq 0.864$ for all F, $e_{\mathcal{S}(W),\mathcal{T}}=3/\pi\sim 0.955$ when F is normal, and $e_{\mathcal{S}(W),\mathcal{T}}>1$ for many non-normal distributions. (For the Gamma distribution with parameter $p=1$, $e_{\mathcal{S}(W),\mathcal{T}}=3$.) The efficiency (4.8) is the same as that of the one sample Wilcoxon test relative to the sign test. It is 3/2 if F is normal, and 3/4 if F is double exponential.

From (4.5) and (4.7), it follows that the asymptotic relative efficiency of the $\mathcal{S}(W)$ test relative to the $\mathcal{S}(\Phi)$ test is given by

$$e_{\mathcal{S}(W),\,\mathcal{S}(\Phi)}=12\left(\frac{\int_{-\infty}^{\infty} f^2(x)dx}{\int_{-\infty}^{\infty} \frac{f^2(x)dx}{\phi\{\Phi^{-1}[F(x)]\}}}\right)^2 . \tag{4.9}$$

The efficiency (4.9) is the same as that of the one or two sample Wilcoxon test relative to the normal scores test. This is $3/\pi$, 0 or 1.18 according as F is normal, uniform over (0, 1) or double exponential. In fact $e_{\mathcal{S}(W),\,\mathcal{S}(\Phi)}\leqq 6/\pi$ for all F.

Finally, from (4.3) and (4.4)

$$e_{D,\mathcal{T}}=4\sigma^2 f^2(0) . \tag{4.10}$$

This efficiency is the same as that of the one sample sign test relative to the Student's t-test. This is $2/\pi$ if f is normal, and is always $\geqq 1/3$, provided F possesses a unimodal density.

Appendix

5. Dispersion matrix of $(\sqrt{N_\alpha}(B_{1N,\alpha}+B_{2N,\alpha}),\ \alpha=1, \cdots, c)$

To obtain the dispersion matrix of $(\sqrt{N_\alpha}(B_{1N,\alpha}+B_{2N,\alpha}),\ \alpha=1, \cdots, c)$ we use representation (2.9) and (2.10) of $B_{1N,\alpha}$ and $B_{2N,\alpha}$ respectively. Integrating $B_{1N,\alpha}$ by parts, we obtain

$$B_{1N,\alpha}=D_{1N,\alpha}+D_{2N,\alpha} \tag{5.1}$$

where

$$D_{1N,\alpha}=-\int_0^{\infty}(F_{N_\alpha}^{(\alpha)}(x)-F^{(\alpha)}(x))J'\{H(x)\}dH(x) , \tag{5.2}$$

$$D_{2N,\alpha}=-\int_0^{\infty}(F_{N_\alpha}^{(\alpha)}(-x-)-F^{(\alpha)}(-x))J'\{H(x)\}dH(x) . \tag{5.3}$$

Since $E(B_{1N,\alpha}+B_{2N,\alpha})=0$, we find that

$$\begin{aligned}\sigma_{N,\alpha\alpha}&=\operatorname{var}(B_{1N,\alpha}+B_{2N,\alpha})=E(D_{1N,\alpha}+D_{2N,\alpha}+B_{2N,\alpha})^2\\&=E(D_{1N,\alpha})^2+E(D_{2N,\alpha})^2+E(B_{2N,\alpha})^2+2E(D_{1N,\alpha}D_{2N,\alpha})\\&\quad+2E(D_{1N,\alpha}B_{2N,\alpha})+2E(D_{2N,\alpha}B_{2N,\alpha}) .\end{aligned} \tag{5.4}$$

Denote

$$A^{(\alpha)}(u, v)=F^{(\alpha)}(u)\{1-F^{(\alpha)}(v)\},$$

$$B(u, v)=J'\{H(u)\}J'\{H(v)\}dH(u)dH(v),$$

$$C^{(\alpha)}(u, v)=J'\{H(u)\}J'\{H(v)\}dH(u)d\{F^{(\alpha)}(v)+F^{(\alpha)}(-v)\}.$$

$$\lambda^{(\alpha)}(u, v)=J'\{H(u)\}J'\{H(v)\}d\{F^{(\alpha)}(u)+F^{(\alpha)}(-u)\} \cdot d\{F^{(\alpha)}(u)+F^{(\alpha)}(-v)\}.$$

Then

$$E(D_{1N,\alpha})^2=E\int_{x=0}^{\infty}\int_{y=0}^{\infty}(F_{N_\alpha}^{(\alpha)}(x)-F^{(\alpha)}(x))(F_{N_\alpha}^{(\alpha)}(y)-F^{(\alpha)}(y)) \cdot J'\{H(x)\}J'\{H(y)\}dH(x)dH(y) \tag{5.5}$$

$$=2E\iint_{0<x<y<\infty}(F_{N_\alpha}^{(\alpha)}(x)-F^{(\alpha)}(x))(F_{N_\alpha}^{(\alpha)}(y)-F^{(\alpha)}(y)) \cdot J'\{H(x)\}J'\{H(y)\}dH(x)dH(y)$$

$$=\frac{2}{N_\alpha}\iint_{0<x<y<\infty}A^{(\alpha)}(x, y)B(x, y). \tag{5.6}$$

Note that the application of Fubini's Theorem permits the interchange of integral and expectation. Similarly

$$E(D_{2N,\alpha})^2=\frac{2}{N_\alpha}\iint_{0<x<y<\infty}A^{(\alpha)}(-y, -x)B(x, y), \tag{5.7}$$

$$E(B_{2N,\alpha})^2=\frac{2}{N}\left[\sum_{i=1}^{c}\rho_N^{(i)}\left\{\iint_{0<x<y<\infty}[A^{(i)}(x, y)+A^{(i)}(-y, x)]\lambda^{(\alpha)}(x, y)\right\} -\int_{x=0}^{\infty}\int_{y=0}^{\infty}A^{(i)}(-y, x)\lambda^{(\alpha)}(x, y)\right], \tag{5.8}$$

$$E(D_{1N,\alpha}D_{2N,\alpha})=\frac{1}{N_\alpha}\int_{x=0}\int_{y=0}A^{(\alpha)}(-y, x)B(x, y). \tag{5.9}$$

To compute $E(D_{1N,\alpha}B_{2N,\alpha})$, we replace $H_N(x)-H(x)$ in (2.4) by

$$\sum_{i=1}^{c}\rho_N^{(i)}(F_{N_i}^{(i)}(x)-F^{(i)}(x))-(F_{N_i}^{(i)}(-x-)-F^{(i)}(-x))$$

and obtain

$$E(D_{1N,\alpha}B_{2N,\alpha})=-\frac{1}{N}\left[\iint_{0<x<y<\infty}A^{(\alpha)}(x, y)C^{(\alpha)}(x, y) +\iint_{0<y<x<\infty}A^{(\alpha)}(y, x)C^{(\alpha)}(x, y)\right] \tag{5.10}$$

$$+\frac{1}{N}\int_{x=0}^{\infty}\int_{y=0}^{\infty} A^{(\alpha)}(-y, x)C^{(\alpha)}(x, y) .$$

Similarly

$$E(D_{2N,\alpha}B_{2N,\alpha})=\frac{1}{N}\Big[\iint\limits_{0<x<y<\infty} A^{(\alpha)}(-y, x)C^{(\alpha)}(y, x) + \iint\limits_{0<y<x<\infty} A^{(\alpha)}(-x, -y)C^{(\alpha)}(y, x)\Big] - \frac{1}{N}\int_{x=0}^{\infty}\int_{y=0}^{\infty} A^{(\alpha)}(-y, x)C^{(\alpha)}(y, x) . \tag{5.11}$$

Hence, using the results (5.6) to (5.11) in (5.4), we obtain

$$\begin{aligned} N_\alpha\sigma_{N,\alpha\alpha}=2\Big[&\iint\limits_{0<x<y<\infty} [A^{(\alpha)}(x, y)+A^{(\alpha)}(-y, x)]B(x, y)\\ &- \iint\limits_{0<x<y<\infty} \rho_N^{(\alpha)}[A^{(\alpha)}(x, y)C^{(\alpha)}(x, y)-A^{(\alpha)}(-y, x)C^{(\alpha)}(y, x)]\\ &- \iint\limits_{0<y<x<\infty} \rho_N^{(\alpha)}[A^{(\alpha)}(y, x)C^{(\alpha)}(x, y)-A^{(\alpha)}(-x, -y)C^{(\alpha)}(y, x)]\\ &+\int_{x=0}^{\infty}\int_{y=0}^{\infty} [B(x, y)+\rho_N^{(\alpha)}C^{(\alpha)}(x, y)-\rho_N^{(\alpha)}C^{(\alpha)}(y, x)]A^{(\alpha)}(-y, x)]\\ &+\rho_N^{(\alpha)}\sum_{i=1}^{c}\rho_N^{(i)}\Big\{\iint\limits_{-\infty<x<y<\infty} [A^{(i)}(x, y)+A^{(i)}(-y, x)]\lambda^{(\alpha)}(x, y)\Big\}\\ &-\int_{x=0}^{\infty}\int_{y=0}^{\infty} A^{(i)}(-y, x)\lambda^{(\alpha)}(x, y)\Big] . \end{aligned} \tag{5.12}$$

To compute the covariance terms, first note that because of the independence of $F_{N_\alpha}^{(\alpha)}(x)$ and $F_{N_{\alpha'}}^{(\alpha')}(x)$ for $\alpha\neq\alpha'$,

$$\begin{aligned} E(D_{1N,\alpha}D_{1N,\alpha'})&=E(D_{1N,\alpha}D_{2N,\alpha'})=E(D_{2N,\alpha}D_{1N,\alpha'})\\ &=E(D_{2N,\alpha}D_{2N,\alpha'})=0 . \end{aligned}$$

Hence, for $\alpha\neq\alpha'$

$$\begin{aligned} \sigma_{N,\alpha\alpha'}&=\text{cov}\,(B_{1N,\alpha}+B_{2N,\alpha},\ B_{1N,\alpha'}+B_{2N,\alpha'})\\ &=E(D_{1N,\alpha}+D_{2N,\alpha}+B_{1N,\alpha})(D_{1N,\alpha'}+D_{2N,\alpha'}+B_{2N,\alpha'})\\ &=E(D_{1N,\alpha}B_{2N,\alpha'})+E(D_{2N,\alpha}B_{2N,\alpha'})\\ &\quad+E(B_{2N,\alpha}D_{1N,\alpha'})+E(B_{2N,\alpha}D_{2N,\alpha'})+E(B_{2N,\alpha}B_{2N,\alpha'}) . \end{aligned}$$

Routine computations yield, for $\alpha\neq\alpha'$

$$(5.13)\quad N\sigma_{N,\alpha\alpha'}=\iint\limits_{0<x<y<\infty}[-A^{(\alpha)}(x,y)C^{(\alpha')}(x,y)+A^{(\alpha)}(-y,-x)C^{(\alpha')}(y,x)$$
$$+A^{(\alpha')}(y,x)C^{(\alpha)}(x,y)+A^{(\alpha')}(-y,-x)C^{(\alpha)}(y,x)]$$
$$+\iint\limits_{0<y<x<\infty}[-A^{(\alpha)}(y,x)C^{(\alpha')}(x,y)+A^{(\alpha)}(-x,-y)C^{(\alpha')}(y,x)$$
$$+A^{(\alpha')}(y,x)C^{(\alpha)}(x,y)+A^{(\alpha')}(-x,-y)C^{(\alpha)}(y,x)]$$
$$+\int_{x=0}^{\infty}\int_{y=0}^{\infty}[A^{(\alpha)}(-y,x)C^{(\alpha')}(x,y)-A^{(\alpha)}(-y,x)C^{(\alpha')}(y,x)$$
$$+A^{(\alpha')}(-y,x)C^{(\alpha)}(x,y)-A^{(\alpha')}(-y,x)C^{(\alpha)}(y,x)]$$
$$+\sum_{i=1}^{c}\rho_N^{(i)}\iint\limits_{0<x<y<\infty}H^{(i)}(x)\{1-H^{(i)}(y)\}J'\{H(x)\}J'\{H(y)\}$$
$$\cdot\, d\{F^{(\alpha)}(x)+F^{(\alpha)}(-x)\}d\{F^{(\alpha)}(y)+F^{(\alpha)}(-\bar{y})\}\,.$$

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