PROCEDURES FOR A BEST POPULATION PROBLEM WHEN THE CRITERION OF BESTNESS INVOLVES A FIXED TOLERANCE REGION*)

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Summary

In this paper we supply tables of constants necessary to use the procedures developed in [4]. We also present a new procedure for one of the cases not discussed in that paper, as well as a proof that it is parameter-free at level P^*.

1. Introduction

The framework of a "best" population problem consists (usually) of the following ingredients:

(1) there is a collection $\Pi=(\pi_1, \cdots, \pi_k)$ of k populations or processes, defined over the *same* sample space (which in this paper is the real line);

(2) the population π_i is distributed with probability density function $f(x \mid \theta_i)$, where θ_i may be vector-valued;

(3) interest focuses on a specific criterion $h_i=g(\theta_i)$, where the functional form of g is known—for example, $g(\theta_i)$ might be the population mean or the reciprocal of the variance of the ith population, $i=1, \cdots, k$;

(4) we wish to find (select, pick, estimate, etc.) that population which has the largest value amongst the h_i, $i=1, \cdots, k$.

Using the above notation, we can state the following definition:

DEFINITION 1.1. A collection of populations $\Pi=(\pi_1, \cdots, \pi_k)$ contains a best population with respect to the criterion $h_i=g(\theta_i)$ if and only if there exists an ordering of the h_i such that

$$h_{[k]}>h_{[k-1]}\geqq h_{[k-2]}\geqq \cdots \geqq h_{[1]} . \tag{1.1}$$

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We then say that the population corresponding to $h_{[k]}$ is the *best* population and we designate it by $\pi_{[k]}$.

Now most of the literature on best population problems aims at finding statistical procedures which will select a subset of Π in such a way that the best population is included in the subset with probability at least as large as a predetermined number, say P^*. We have the following definition:

DEFINITION 1.2. Let a sample of n_i independent observations be taken independently from each population (or process) Π_i of a collection $\Pi=(\pi_1, \cdots, \pi_k)$. If a statistical procedure used to select a subset of Π is such that the best population (see (1.1)) is included in the subset, we say that a *correct selection* (*CS*) has been made. Further, if the procedure makes a correct selection such that $P(CS) \geqq P^*$, where P^* is a preassigned number, with the procedure (and of course, P^*) independent of $(\theta_1, \cdots, \theta_k)$, we then say that the procedure is *parameter-free at level* P^*.

To re-emphasize in light of the (above) Definition 1.2, we are interested in this paper in certain parameter-free procedures which will retain a subset of Π in such a way that the best population will be in this retained subset with probability of at least P^*, and where the notion of "bestness" arises from the following considerations.

It very often happens that interest focuses on a specific interval $A=(a_1, a_2)$. For example, in the assembling of "stable" amplifiers, certain electronic tubes used in the amplifier must have transconductances that lie within the specified limits a_1 and a_2, $a_1<a_2$; or, in the manufacture of a certain type of thread, the quality of the thread is judged to be "high" if the tensile strength is greater than a known number, say a_1; that is, the tensile strength should have value lying in $A=(a_1, \infty)$ etc. (For other engineering applications of this sort, see [1], [3] and [5].)

Obviously, it is quite important for the manufacturer to know what percentage of items he is producing meet the required specifications—he is probably in the situation where he knows that he must produce at least $100\alpha\%$ of the items that do satisfy the requirements or specifications to make a profit. Now if the manufacturer can choose among k different processes to produce the items, he will wish to use that process which gives the largest number of items falling in the specified interval A, so that he is interested in which process gives largest value to the coverage of the interval A, that is, $h_i=g(\theta_i)$ is now taken to be

$$C_i=C_i(A)=\int_A f(x \mid \theta_i)dx\,, \qquad i=1, \cdots, k\,. \tag{1.2}$$

(We are including here the case that one basic process may be capable of k independent modifications.) Hence, we are interested in finding the population that has largest value amongst the C_i, and specifically, to construct selection procedures which are parameter-free at level P^*, say.

2. Normal populations

In this section we examine the problem of section 1 for a collection of normal distributions, when $A=(-\infty, a]$, with the constant "a" known and specified beforehand. We will assume that all $n_i=n$, $i=1, \cdots, k$.

Suppose then we consider a collection of populations $\Pi=(\pi_1, \cdots, \pi_k)$, with π_i distributed as $N(\mu_i, \sigma_i^2)$, $i=1, \cdots, k$ and where there exists a best population which has largest value among the k coverages

$$
\begin{aligned}
C_i(a) &= \int_{-\infty}^{a} (2\pi\sigma_i^2)^{-1/2} \exp\{-(x-\mu_i)^2/2\sigma_i^2\} dx \\
&= \int_{-\infty}^{(a-\mu_i)/\sigma_i} \frac{1}{\sqrt{2\pi}} \exp\{-z^2/2\} dz \\
&= \Phi((a-\mu_i)/\sigma_i)\,, \qquad i=1, \cdots, k\,.
\end{aligned} \tag{2.1}
$$

Since $\Phi(t)$ is a monotone increasing function of t, the problem of selecting the best population is the problem of the selection of that population with the largest value of $(a-\mu_i)/\sigma_i$, or least value of $(\mu_i-a)/\sigma_i$.

The problem splits itself into various cases. To restate, we address ourselves to the problem of picking a subset of the k populations, based on k independent samples of n independent observations each, in such a way that $P(CS) \geqq P^*$. The first five cases stated below are shown to be parameter-free at level P^* in [4], and the reader is referred to that paper for details of proofs. In this paper, we supply tables of the necessary constants needed to implement the procedures. The procedure for Case 6 is new, and accordingly we include the proof that it is parameter-free.

Case 1. μ's unknown and variable; σ_i^2 known, $\sigma_i^2 \equiv \sigma^2$, $i=1, \cdots, k$.

The parameter-free level P^* procedure for this case is

Procedure 1. Retain population π_i in the subset if

$$
\bar{X}_i < \bar{X}_{(1)} + d_1 \tag{2.2}
$$

where $\bar{X}_{(1)}$ is the smallest of the k sample means $\bar{X}_i$, $i=1, \cdots, k$, and $d_1 = d_1(P^*, k, n)$ is chosen to make (2.2) of level P^*.

Now if we set $d_1' = \sqrt{n}\, d_1/\sigma$, it is shown in [4] that

(2.3) $$P^*=\int_{-\infty}^{\infty}[1-\Phi(z_1-d_1')]^{k-1}\phi(z_1)dz_1$$

where $\phi(t)$ and $\Phi(t)$ are the density and cumulative distribution function, respectively, of a normal distribution, mean 0 and variance 1. Values of d_1' for the cases $P^*=.75$, .90, .95 and .99 are given in Table 2.1 for $k=2$, 3 and 4.

Table 2.1*. Values of the constant d_1', where $d_1'=\sqrt{n}\,d_1/\sigma$, and where d_1 is the constant needed to make the procedure 2.2 of level P^*

	$k=2$				$k=3$				$k=4$			
P^*	.75	.90	.95	.99	.75	.90	.95	.99	.75	.90	.95	.99
	.9539	1.8124	2.3262	3.2900	1.4338	2.2302	2.7101	3.6173	1.6822	2.4156	2.9162	3.7970

* This table was calculated by Ernest Gloyd, Department of Statistics, University of Wisconsin, January, 1968. It has been brought to our attention that these calculations form a subset of a table contained in [2], calculated by R. E. Bechhoffer (1954).

Case 2. μ's unknown and variable; σ^2's known and variable.

For this situation we use the following parameter-free procedure of level P^*, namely

Procedure 2. Retain population π_i in the subset if

$$Z_i<Z_{(1)}+d_2$$

where $Z_i=(\bar{X}_i-a)/\sigma_i$, $Z_{(1)}=\min_{i=1}^{k} Z_i$ and $d_2=d_1'/\sqrt{n}$, where d_1' is defined by (2.3).

Case 3. μ's unknown and variable; σ_i^2 unknown, $\sigma_i^2=\sigma^2$.

It is clear from (2.1) that for this case we again wish to retain in the selected subset of Π that population with the smallest μ. However, as we do not know the common value of σ^2, we will need to estimate it, and for this we make use of a pooled estimator W^2 of σ^2 given by

(2.5) $$W^2=\frac{(n-1)V_1^2+\cdots+(n-1)V_k^2}{k(n-1)}=\frac{1}{k}\sum_{i=1}^{k}V_i^2$$

with $V_i^2=(n-1)^{-1}\sum_{j=1}^{n}(X_{ij}-\bar{X}_i)^2$, $i=1,\cdots,k$. The parameter-free level P^* procedure for this case is

Procedure 3. Retain population π_i if

(2.6) $$\bar{X}_i\leqq\bar{X}_{(1)}+d_3W$$

Table 2.2. Values of the constant d_3 needed to make procedure (2.6) parameter-free at level P^*

	k=2				k=3				k=4			
n \ P^*	.75	.90	.95	.99	.75	.90	.95	.99	.75	.90	.95	.99
2	.9661	2.1498	3.2282	7.2682	1.3178	2.2736	3.0336	5.2595	1.4708	2.3058	2.9242	4.5285
4	.5359	1.0535	1.3956	2.1519	.7788	1.2416	1.5409	2.1708	.8991	1.3343	1.6118	2.1813
6	.4174	.8076	1.0539	1.5575	.6148	.9694	1.1917	1.6372	.7144	1.0518	1.2619	1.6784
8	.3543	.6815	.8845	1.2880	.5249	.8241	1.0095	1.3741	.6117	.8977	1.0742	1.4193
10	.3134	.6010	.7779	1.1244	.4658	.7298	.8921	1.2087	.5436	.7966	.9519	1.2533
12	.2841	.5438	.7027	1.0112	.4231	.6620	.8083	1.0919	.4943	.7235	.8638	1.1349
14	.2618	.5004	.6459	.9268	.3904	.6102	.7445	1.0038	.4564	.6675	.7965	1.0450
16	.2440	.4661	.6010	.8606	.3643	.5690	.6938	.9341	.4260	.6229	.7429	.9736
18	.2294	.4379	.5644	.8069	.3428	.5351	.6523	.8772	.4011	.5861	.6988	.9151
20	.2172	.4143	.5338	.7622	.3247	.5067	.6174	.8297	.3800	.5552	.6618	.8661
22	.2067	.3942	.5077	.7242	.3092	.4824	.5876	.7891	.3620	.5287	.6301	.8242
24	.1976	.3768	.4851	.6914	.2958	.4613	.5618	.7540	.3463	.5057	.6026	.7878
26	.1897	.3615	.4652	.6627	.2839	.4427	.5391	.7232	.3325	.4854	.5784	.7560
28	.1826	.3479	.4476	.6373	.2734	.4262	.5189	.6959	.3202	.4675	.5569	.7276
30	.1762	.3357	.4319	.6146	.2640	.4114	.5009	.6714	.3092	.4513	.5376	.7023
32	.1705	.3248	.4177	.5942	.2554	.3981	.4846	.6494	.2992	.4368	.5202	.6794
34	.1653	.3148	.4049	.5757	.2477	.3860	.4698	.6294	.2902	.4235	.5044	.6586
36	.1605	.3057	.3931	.5588	.2406	.3749	.4563	.6112	.2819	.4114	.4899	.6397
38	.1562	.2974	.3824	.5434	.2341	.3647	.4438	.5944	.2743	.4003	.4767	.6222
40	.1521	.2897	.3724	.5291	.2281	.3554	.4324	.5790	.2673	.3900	.4644	.6062
42	.1484	.2825	.3632	.5159	.2225	.3467	.4218	.5647	.2608	.3805	.4531	.5913
44	.1449	.2759	.3547	.5037	.2174	.3386	.4119	.5514	.2547	.3717	.4425	.5774
46	.1417	.2697	.3467	.4923	.2125	.3310	.4027	.5390	.2491	.3634	.4326	.5645
48	.1387	.2639	.3392	.4816	.2080	.3240	.3941	.5274	.2438	.3557	.4234	.5524
50	.1358	.2585	.3322	.4716	.2038	.3173	.3860	.5166	.2388	.3484	.4148	.5411
52	.1332	.2534	.3257	.4622	.1998	.3111	.3784	.5063	.2342	.3416	.4066	.5304
54	.1306	.2486	.3195	.4533	.1960	.3052	.3713	.4967	.2298	.3351	.3989	.5204
56	.1283	.2440	.3136	.4449	.1924	.2997	.3645	.4876	.2256	.3290	.3917	.5109
58	.1260	.2397	.3081	.4370	.1891	.2944	.3581	.4790	.2216	.3233	.3848	.5018
60	.1239	.2356	.3028	.4295	.1859	.2894	.3520	.4708	.2179	.3178	.3783	.4933
65	.1189	.2263	.2907	.4123	.1785	.2779	.3380	.4520	.2093	.3052	.3633	.4737
70	.1146	.2179	.2800	.3970	.1720	.2677	.3256	.4353	.2016	.2940	.3500	.4563
75	.1107	.2105	.2704	.3833	.1661	.2586	.3144	.4204	.1948	.2840	.3380	.4407
80	.1071	.2037	.2617	.3709	.1608	.2503	.3043	.4069	.1885	.2749	.3272	.4265
85	.1039	.1976	.2538	.3596	.1560	.2428	.2952	.3946	.1829	.2667	.3174	.4137
90	.1009	.1919	.2465	.3494	.1515	.2359	.2868	.3833	.1777	.2591	.3084	.4019
95	.0982	.1868	.2399	.3399	.1475	.2295	.2791	.3730	.1729	.2522	.3001	.3911
100	.0957	.1820	.2338	.3312	.1437	.2237	.2720	.3635	.1685	.2458	.2924	.3811

where d_3 is a constant satisfying

$$P^* = \int_{-\infty}^{\infty} [1 - T(t - \sqrt{n}\, d_3)]^{k-1} dT(t)\,, \tag{2.7}$$

with $T(t)$ denoting the cumulative distribution function of a Student-t variable with $k(n-1)$ degrees of freedom. Table 2.2 gives some values of d_3 for selected P^* and n, with $k=2, 3$ and 4.

Case 4. μ's known with $\mu_i \equiv \mu$, $i=1, \cdots, k$; σ^2's unknown and variable.

This case splits itself into the two following cases.

Case 4.1. $\mu > a$. The conditions defining this case together with (2.1) imply that we are looking for that population with the largest of the σ_i. The parameter-free level P^* procedure is as follows:

Procedure 4.1. Retain π_i in the subset if

$$V_i'^2 \geqq d_{4,1} V_{(k)}'^2 \tag{2.8}$$

where

$$V_i'^2 = n^{-1} \sum_{j=1}^{n} (x_{ij} - \mu)^2\,, \quad i=1, \cdots, k\,, \qquad V_{(k)}' = \max_{i=1}^{k} V_i'^2\,, \tag{2.8a}$$

and $d_{4,1}$ satisfies

$$P^* = \int_0^{\infty} [F_n(u/d_{4,1})]^{k-1} dF_n(u)\,. \tag{2.9}$$

The function $F_n(u)$ is the cumulative distribution function of a chi-square variable with n degrees of freedom. We tabulate some values of $d_{4,1}$ in Table 2.3.

Case 4.2. $\mu < a$. For this case, it is clear that we are looking for the population with the least of the σ_i. Accordingly, the parameter-free level P^* procedure turns out to be the following

Procedure 4.2. Retain π_i in the subset if

$$V_i'^2 \leqq d_{4,2} V_{(1)}'^2 \tag{2.10}$$

where the $V_i'^2$'s are defined in (2.8a), $V_{(1)}'^2 = \min_{i=1}^{k} V_i'^2$, and $d_{4,2}$ satisfies

$$P^* = \int_0^{\infty} [1 - F_n(v/d_{4,2})]^{k-1} dF_n(v) \tag{2.11}$$

with F_n, as before, denoting the chi-squared cumulative distribution

Table 2.3. Values of the constant $d_{4,1}$ needed to make the procedure (2.8) parameter-free at level P^*

	$k=2$				$k=3$				$k=4$			
n \ P^*	.75	.90	.95	.99	.75	.90	.95	.99	.75	.90	.95	.99
2	.3333	.1111	.0526	.0101	.2078	.0723	.0347	.0067	.1663	.0587	.0283	.0055
4	.4845	.2435	.1565	.0626	.3505	.1830	.1194	.0485	.2997	.1586	.1041	.0426
6	.5611	.3274	.2334	.1181	.4313	.2601	.1881	.0969	.3792	.2315	.1683	.0873
8	.6099	.3862	.2909	.1659	.4856	.3166	.2416	.1403	.4340	.2862	.2195	.1284
10	.6446	.4305	.3358	.2062	.5256	.3605	.2846	.1778	.4750	.3293	.2612	.1644
12	.6711	.4657	.3722	.2407	.5568	.3960	.3202	.2105	.5074	.3644	.2961	.1960
14	.6921	.4945	.4026	.2704	.5821	.4255	.3503	.2391	.5340	.3939	.3257	.2238
16	.7094	.5186	.4285	.2966	.6032	.4506	.3762	.2644	.5562	.4191	.3514	.2486
18	.7239	.5394	.4510	.3197	.6211	.4724	.3989	.2870	.5753	.4411	.3741	.2708
20	.7364	.5575	.4708	.3404	.6367	.4915	.4190	.3073	.5919	.4605	.3942	.2909
22	.7472	.5734	.4883	.3591	.6504	.5084	.4370	.3258	.6065	.4778	.4122	.3092
24	.7568	.5876	.5041	.3761	.6625	.5236	.4532	.3427	.6196	.4933	.4285	.3259
26	.7653	.6004	.5183	.3916	.6733	.5374	.4679	.3582	.6313	.5074	.4434	.3413
28	.7729	.6119	.5313	.4059	.6832	.5499	.4814	.3725	.6420	.5202	.4570	.3556
30	.7798	.6225	.5432	.4191	.6921	.5613	.4938	.3857	.6516	.5320	.4696	.3688
32	.7861	.6322	.5542	.4314	.7003	.5719	.5053	.3981	.6605	.5429	.4812	.3812
34	.7919	.6411	.5643	.4428	.7078	.5817	.5159	.4096	.6687	.5530	.4921	.3927
36	.7972	.6493	.5737	.4535	.7147	.5907	.5258	.4205	.6763	.5624	.5022	.4036
38	.8021	.6570	.5825	.4635	.7212	.5992	.5351	.4306	.6834	.5711	.5116	.4138
40	.8067	.6642	.5907	.4730	.7272	.6071	.5438	.4402	.6900	.5794	.5205	.4235
42	.8109	.6709	.5985	.4819	.7328	.6145	.5520	.4493	.6961	.5871	.5289	.4326
44	.8149	.6772	.6057	.4903	.7380	.6215	.5597	.4579	.7019	.5944	.5368	.4412
46	.8186	.6831	.6126	.4983	.7430	.6281	.5670	.4660	.7074	.6012	.5443	.4495
48	.8221	.6887	.6191	.5059	.7477	.6343	.5739	.4738	.7125	.6077	.5514	.4573
50	.8254	.6940	.6252	.5131	.7521	.6402	.5805	.4812	.7174	.6139	.5581	.4648
52	.8285	.6990	.6310	.5200	.7562	.6459	.5867	.4883	.7220	.6198	.5645	.4719
54	.8315	.7038	.6366	.5265	.7602	.6512	.5927	.4950	.7264	.6253	.5707	.4787
56	.8343	.7083	.6419	.5328	.7640	.6563	.5984	.5015	.7305	.6307	.5765	.4852
58	.8369	.7126	.6469	.5388	.7675	.6611	.6038	.5077	.7345	.6358	.5821	.4915
60	.8395	.7167	.6518	.5446	.7710	.6658	.6090	.5136	.7383	.6406	.5875	.4975
65	.8453	.7263	.6630	.5580	.7788	.6766	.6211	.5275	.7471	.6519	.5999	.5116
70	.8505	.7349	.6731	.5702	.7860	.6863	.6320	.5401	.7550	.6622	.6113	.5244
75	.8552	.7427	.6823	.5814	.7924	.6951	.6420	.5517	.7622	.7615	.6216	.5362
80	.8595	.7498	.6907	.5917	.7982	.7032	.6512	.5624	.7687	.6801	.6311	.5471
85	.8635	.7563	.6985	.6012	.8036	.7107	.6596	.5723	.7748	.6879	.6399	.5572
90	.8671	.7624	.7057	.6100	.8086	.7176	.6675	.5815	.7803	.6952	.6480	.5666
95	.8704	.7679	.7123	.6182	.8132	.7240	.6747	.5901	.7855	.7020	.6556	.5753
100	.8735	.7731	.7185	.6259	.8174	.7300	.6815	.5981	.7902	.7083	.6627	.5835

Table 2.4. Values of the constant $d_{4,2}$ needed to make the procedure (2.10) parameter-free at level P^*

	k=2				k=3				k=4			
n \ P^*	.75	.90	.95	.99	.75	.90	.95	.99	.75	.90	.95	.99
2	3.0000	9.0000	19.0000	99.0000	6.0000	18.0000	38.0000	198.0000	9.0000	27.0000	57.0000	297.0000
4	2.0642	4.1073	6.3884	15.9821	3.1571	6.1368	9.4165	23.0689	4.0103	7.7091	11.7539	28.5158
6	1.7821	3.0546	4.2839	8.4667	2.4750	4.1375	5.7179	11.0315	2.9684	4.9038	6.7296	12.8338
8	1.6396	2.5894	3.4381	6.0297	2.1608	3.3315	4.3613	7.4687	2.5137	3.8320	4.9826	8.4346
10	1.5513	2.3226	2.9782	4.8493	1.9764	2.8927	3.6599	5.8252	2.2550	3.2654	4.1052	6.4619
12	1.4902	2.1474	2.6866	4.1555	1.8534	2.6141	3.2295	4.8885	2.0860	2.9128	3.5769	5.3575
14	1.4449	2.0224	2.4837	3.6979	1.7647	2.4202	2.9370	4.2840	1.9659	2.6707	3.2226	4.6538
16	1.4097	1.9281	2.3335	3.3721	1.6971	2.2766	2.7243	3.8608	1.8754	2.4933	2.9675	4.1657
18	1.3814	1.8539	2.2172	3.1281	1.6437	2.1654	2.5619	3.5478	1.8045	2.3570	2.7744	3.8073
20	1.3580	1.7938	2.1242	2.9379	1.6001	2.0764	2.4335	3.3063	1.7472	2.2488	2.6226	3.5325
22	1.3383	1.7440	2.0478	2.7849	1.5638	2.0034	2.3291	3.1138	1.6997	2.1604	2.4999	3.3145
24	1.3214	1.7019	1.9838	2.6591	1.5330	1.9421	2.2423	2.9567	1.6596	2.0867	2.3984	3.1372
26	1.3067	1.6657	1.9292	2.5536	1.5064	1.8899	2.1689	2.8257	1.6252	2.0241	2.3128	2.9900
28	1.2938	1.6342	1.8821	2.4637	1.4833	1.8448	2.1059	2.7147	1.5953	1.9703	2.2396	2.8656
30	1.2823	1.6065	1.8409	2.3860	1.4628	1.8054	2.0510	2.6192	1.5690	1.9233	2.1762	2.7590
32	1.2720	1.5819	1.8045	2.3181	1.4446	1.7706	2.0029	2.5361	1.5457	1.8820	2.1205	2.6664
34	1.2628	1.5599	1.7721	2.2583	1.4283	1.7395	1.9601	2.4632	1.5249	1.8453	2.0713	2.5853
36	1.2544	1.5400	1.7430	2.2051	1.4136	1.7117	1.9219	2.3985	1.5061	1.8124	2.0275	2.5135
38	1.2467	1.5220	1.7167	2.1574	1.4002	1.6865	1.8876	2.3407	1.4891	1.7827	1.9880	2.4495
40	1.2397	1.5056	1.6928	2.1142	1.3879	1.6637	1.8564	2.2887	1.4736	1.7558	1.9524	2.3919
42	1.2332	1.4906	1.6710	2.0752	1.3767	1.6428	1.8281	2.2417	1.4594	1.7313	1.9200	2.3400
44	1.2272	1.4767	1.6509	2.0395	1.3663	1.6236	1.8021	2.1988	1.4463	1.7088	1.8903	2.2927
46	1.2216	1.4639	1.6325	2.0069	1.3567	1.6059	1.7782	2.1597	1.4342	1.6882	1.8631	2.2496
48	1.2164	1.4520	1.6154	1.9768	1.3478	1.5895	1.7562	2.1237	1.4230	1.6690	1.8381	2.2100
50	1.2115	1.4409	1.5995	1.9490	1.3395	1.5743	1.7358	2.0905	1.4125	1.6513	1.8149	2.1736
52	1.2070	1.4306	1.5847	1.9232	1.3317	1.5602	1.7168	2.0598	1.4028	1.6348	1.7933	2.1398
54	1.2027	1.4209	1.5709	1.8992	1.3244	1.5469	1.6991	2.0313	1.3937	1.6194	1.7733	2.1085
56	1.1986	1.4118	1.5579	1.8769	1.3176	1.5345	1.6825	2.0047	1.3851	1.6050	1.7545	2.0794
58	1.1948	1.4033	1.5458	1.8559	1.3111	1.5229	1.6670	1.9799	1.3770	1.5915	1.7369	2.0522
60	1.1912	1.3952	1.5343	1.8363	1.3051	1.5119	1.6524	1.9566	1.3694	1.5787	1.7204	2.0268
65	1.1830	1.3769	1.5084	1.7921	1.2912	1.4871	1.6194	1.9044	1.3522	1.5500	1.6833	1.9697
70	1.1757	1.3608	1.4857	1.7537	1.2790	1.4654	1.5907	1.8592	1.3370	1.5249	1.6509	1.9204
75	1.1693	1.3465	1.4656	1.7200	1.2682	1.4461	1.5654	1.8196	1.3236	1.5027	1.6224	1.8773
80	1.1634	1.3337	1.4477	1.6901	1.2585	1.4290	1.5428	1.7846	1.3116	1.4830	1.5971	1.8392
85	1.1581	1.3222	1.4316	1.6634	1.2497	1.4136	1.5227	1.7533	1.3008	1.4653	1.5745	1.8052
90	1.1533	1.3117	1.4171	1.6393	1.2418	1.3997	1.5044	1.7253	1.2910	1.4493	1.5541	1.7748
95	1.1489	1.3022	1.4038	1.6175	1.2345	1.3870	1.4879	1.6999	1.2820	1.4347	1.5355	1.7473
100	1.1449	1.2934	1.3917	1.5977	1.2278	1.3754	1.4727	1.6768	1.2738	1.4214	1.5187	1.7222

function, n degrees of freedom. We tabulate some values of $d_{4,2}$ in Table 2.4.

Case 5. μ's known, variable; σ's unknown and variable.

Since we wish to find that population with least value of $(\mu_i - a)/\sigma_i$, we see immediately that this case splits into the following three cases:

Case 5.1. *All μ_i known and greater than a.* For this case, we recognize that we wish to select the population having greatest value of $\sigma_i/(\mu_i - a)$, $i=1, \cdots, k$. We use the following parameter-free procedure at level P^*.

Procedure 5.1. Retain population π_i in the subset if

$$Q_i^2 \geqq d_{5,1} Q_{(k)}^2 \tag{2.12}$$

where $Q_i^2 = U_i^2/(\mu_i - a)$, $i=1, \cdots, k$ with $U_i^2 = n^{-1}\sum_{j=1}^{n}(X_{ij} - \mu_i)^2$, *and where* $d_{5,1} = d_{4,1}$. $\left(Q_{(k)}^2 = \max_{i=1}^{k} Q_i^2\right)$.

Case 5.2. *All μ_i known and less than a.* For this case, we wish to select the population having least value of $\sigma_i/(a - \mu_i)$. We use then, the following parameter-free procedure at level P^*.

Procedure 5.2. Retain population π_i in the subset if

$$Q_i'^2 \leqq d_{5,2} Q_{(1)}'^2 \tag{2.13}$$

where $Q_i'^2 = U_i^2/(a - \mu_i)$ *and* $d_{5,2} = d_{4,2}$. $\left(Q_{(1)}'^2 = \min_{i=1}^{k} Q_i'^2\right)$.

Case 5.3. *All μ_i known, with* $\mu_{[1]} \leqq \mu_{[2]} \leqq \cdots \leqq \mu_{[k_1-1]} < \mu_{[k_1]} < a < \mu_{[k_1+1]} \leqq \cdots \leqq \mu_{[k]}$, *where* $1 < k_1 < k$. Here, $\mu_{[i]}$ denote the ordered values of the μ_i, $i=1, \cdots, k$. The properties of the normal distribution come into play and we note that, since as μ decreases the coverage of the interval $(-\infty, a)$ increases, we may eliminate from consideration the $k - k_1$ populations which have means greater than "a", and then apply procedure (2.13) for $k = k_1$.

Case 6. μ's unknown, $\mu_i \equiv \mu$, $i=1, \cdots, k$; σ^2's unknown and variable.

In this situation we are faced with the unpleasant fact that not only do we not know the common value μ of the μ_i, but also, we do not know whether μ is greater or smaller than the known number a. Since we wish to find the population with the least value of $(\mu - a)/\sigma_i$, $i=1, \cdots, k$ (see (2.1)), this means that we do not know whether the best population is the one with the largest σ_i (as is the case if $\mu > a$) or the one with the smallest σ_i (as is the case if $\mu<a$). We do however, assume that there exists an ordering of the σ_i such that

$$\sigma_{[1]}<\sigma_{[2]}\leqq\cdots\leqq\sigma_{[k-1]}<\sigma_{[k]}\,. \tag{2.14}$$

Further, we may gain information as to whether μ is less or greater than "a" by using the combined estimator $\bar{\bar{X}}$ of μ, where

$$\bar{\bar{X}}=\frac{n\bar{X}_1+\cdots+n\bar{X}_k}{kn}=\frac{1}{k}\sum_{i=1}^{n}\bar{X}_i \tag{2.15}$$

where $\bar{X}_i=n^{-1}\sum_{j=1}^{n}X_{ij}$. Denoting the usual unbiased estimator of σ_i^2 by V_i^2, i.e., $V_i^2=(n-1)^{-1}\sum_{j=1}^{n}(X_{ij}-\bar{X}_i)^2$, we now state the following procedure.

Procedure 6. Compute the estimator $\bar{\bar{X}}$ of μ given by (2.15).

(i) If $\bar{\bar{X}}>a$, then retain π_i in the subset if

$$V_i^2\geqq d_{6,1}V_{(k)}^2 \tag{2.16}$$

where $V_{(k)}^2=\max_{i=1}^{k} V_i^2$, and where $d_{6,1}$ is given by

$$d_{6,1}(n)=d_{4,1}(n-1) \qquad \text{for } n\geqq 2\,. \tag{2.17}$$

(ii) If $\bar{X}\leqq a$, then retain π_i in the subset if

$$V_i^2\leqq d_{6,2}V_{(1)}^2 \tag{2.18}$$

where $V_{(1)}^2=\min_{i=1}^{k} V_i^2$, and where $d_{6,2}$ is given by

$$d_{6,2}(n)=d_{4,2}(n-1) \qquad \text{for } n\geqq 2\,. \tag{2.19}$$

We have the following theorem.

THEOREM. *The procedure* 6 *given above* (*see* (2.16)–(2.19)) *is parameter-free at level* P^*.

PROOF. A correct selection may be made if either one of the following events occurs:

E_1: $\bar{\bar{X}}>a$, $V^2\geqq d_{6,1}V_{(k)}^2$, when $\mu>a$, and V^2 is from that population with largest σ_i^2, or

E_2: $\bar{\bar{X}}>a$, $V_\cdot^2\geqq d_{6,1}V_{(k)}^2$, when $\mu>a$, and $V_\cdot^2$ is from that population with smallest σ_i^2, or

E_3: $\bar{\bar{X}}\leqq a$, $V_{\cdot\cdot}^2\leqq d_{6,2}V_{(1)}^2$, when $\mu<a$, and $V_{\cdot\cdot}^2$ is from that population with the smallest σ_i^2, or

E_4: $\bar{\bar{X}} \leqq a$, $V^2_{\cdot\cdot\cdot} \leqq d_{6,2} V^2_{(1)}$, when $\mu < a$, and $V^2_{\cdot\cdot}$ is from that population with the largest σ_i^2.

We note that $E_i \cap E_j = \phi$, the null set, for $i=1, 2$ and $j=3, 4$. Hence we have that

$$
\begin{aligned}
(2.20) \qquad P(CS) &= P(E_1 \cap E_2) + P(E_3 \cap E_4) \\
&= P(E_1) + P(E_2)[1 - P(E_1 \mid E_2)] \\
&\quad + P(E_3) + P(E_4)[1 - P(E_3 \mid E_4)] \\
&\geqq P(E_1) + P(E_3),
\end{aligned}
$$

and using the property of independence of sample means and variances when sampling from the normal, we may write the above as

$$(2.21) \qquad P(CS) \geqq P(\bar{\bar{X}} > a \mid \mu > a) \cdot P_1 + P(\bar{\bar{X}} \leqq a \mid \mu < a) \cdot P_2$$

where

$$
(2.22) \qquad
\begin{aligned}
P_1 &= P(V^2_{(k)} \leqq V^2 / d_{6,1} \mid V^2 \text{ from population with largest } \sigma_i^2), \\
P_2 &= P(V^2_{(1)} \geqq V^2_{\cdot\cdot} / d_{6,2} \mid V^2_{\cdot\cdot} \text{ from population with smallest } \sigma_i^2).
\end{aligned}
$$

We further note that from (2.21), we have

$$(2.23) \qquad P(CS) \geqq P(\bar{\bar{X}} > a \mid \mu > a)[\min P_1] + P(\bar{\bar{X}} \leqq a \mid \mu < a)[\min P_2]$$

and it is left to show that since $d_{6,1}$ and $d_{6,2}$ satisfy (2.17) and (2.19), respectively, that

$$(2.24) \qquad \min P_1 = \min P_2 = P^*.$$

This enables us to write

$$(2.25) \qquad P(CS) \geqq P^* \{P(\bar{\bar{X}} > a \mid \mu > a) + P(\bar{\bar{X}} \leqq a \mid \mu < a)\}.$$

Now $\bar{\bar{X}}$ is clearly a normal variable and in particular has mean μ. But a normal variable is such that (i) if its mean exceeds "a", the probability of $\bar{\bar{X}}$ exceeding "a" is greater than 1/2 and (ii) if its mean does not exceed a, the probability of $\bar{\bar{X}}$ not exceeding a is greater than 1/2. This means that the expression in the braces { } of the right-hand side of (2.25) is at least one, so that we have

$$(2.26) \qquad P(CS) \geqq P^*,$$

that is to say, the procedure would be parameter-free at level P^*.

It remains then to demonstrate that $\min P_1$ and $\min P_2$ have value P^* when the constants $d_{6,1}$ and $d_{6,2}$ given by (2.17) and (2.19) are used in procedure 6. We begin with P_1; we have that

(2.27) $$P_1=P(V_{(k)}^2 \leqq V^2/d_{6,1} \mid V^2 \text{ is from the population with largest } \sigma_i^2) = \int_0^\infty \left[\prod_{i=1}^{k-1} C(v^2/d_{6,1};\, \sigma_{[i]}^2)\right] dC(v^2;\, \sigma_{[k]}^2)$$

where

(2.27a) $$C(v^2;\, \sigma_{[i]}^2)=\int_0^{v^2} \frac{(n-1)^{(n-1)/2}(V_i^2)^{(n-1)/2-1}}{\Gamma((n-1)/2)\,(2\sigma_{[i]}^2)^{(n-1)/2}} \exp\{-(n-1)v_i^2/2\sigma_{[i]}^2\}\,dv_i^2\,.$$

It is easy to see that (2.27a) may, after appropriate transformation, be written as

(2.29) $$P_1=\int_0^\infty \int_0^{(W_k^2/d_{6,1})(\sigma_{[k]}^2/\sigma_{[k-1]}^2)} \cdots \int_0^{(W_k^2/d_{6,1})(\sigma_{[k]}^2/\sigma_{[1]}^2)} \cdot \left[\prod_{i=1}^{k} \frac{(n-1)^{(n-1)/2}(W_i^2)^{(n-1)/2-1}}{\Gamma((n-1)/2)2^{(n-1)/2}} \exp\{-(n-1)W_i^2/2\}\right] \cdot dW_1^2 \cdots dW_{k-1}^2 dW_k^2 = K_{d_{6,1}}\left(\frac{\sigma_{[k]}^2}{\sigma_{[k-1]}^2}, \cdots, \frac{\sigma_{[k]}^2}{\sigma_{[1]}^2}\right).$$

Now it is obvious that $K_{d_{6,1}}$ is a monotone increasing function in its arguments, where the $\sigma_{[i]}^2$'s obey (2.14). Hence the minimum value of P_1 is clearly

(2.30) $$K_{d_{6,1}}(1, \cdots, 1)\,.$$

But comparing $K_{d_{6,1}}(1, \cdots, 1)$ with (2.9) and because (2.17) holds, we have

(2.31) $$\min P_1=K_{d_{6,1}}(1, \cdots, 1)=P^*\,.$$

A similar proof shows that $\min P_2=P^*$, so that the argument leading to (2.26) holds, and the theorem is proved.

3. A short note on the calculations

Evaluation of expressions (2.7), (2.9) and (2.11) was done by numerical integration. Values of the Student-t and chi-square cumulative distribution functions required in the integration were provided by standard computer subprograms at the University of Wisconsin Computing Center. The subprograms evaluate these distribution functions through their relation to the incomplete beta and incomplete gamma distributions, where the incomplete beta distribution is evaluated by a continued fraction expansion and the incomplete gamma by expansions and approximations selected according to the magnitude of the parameters.

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REFERENCES

[1] W. A. Shewhart, Chapter 2 of "Statistical method from the viewpoint of quality control," Graduate School, U.S. Department of Agriculture, Washington, D.C., (1939), 50-79.

[2] R. E. Bechhoffer, "A single-sample multiple decision procedure for ranking means of normal populations with known variances," *Ann. Math. Statist.*, 25 (1954), 16-39.

[3] A. H. Bowker and G. J. Lieberman, *Engineering Statistics*, Prentice Hall, 1959.

[4] Irwin Guttman, "Best populations and tolerance regions," *Ann. Inst. Statist. Math.*, 13 (1961), 9-26.

[5] B. Ostle, *Statistics in Research*, 2nd ed., Iowa Press, 1963.