ON SOME MODEL OF QUEUEING SYSTEM WITH STATE-DEPENDENT SERVICE TIME DISTRIBUTIONS

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1. Introduction

In some single server queueing situations the following queueing model will be natural. The service-time distribution of a customer is $H_a(t)$ or $H_b(t)$ according as the length of the waiting line behind him at the moment of entering into service is less than or not less than a given integer $N(>0)$, where $H_a(t)$ and $H_b(t)$ are distribution functions in $t\geqq 0$, which are arbitrary except that $H_a(0+)<1$ and $H_b(0+)<1$, and they may be different. For example, we can conceive a service system in which the speed of service is accelerated when the congestion of the system exceeded some level.

In this paper we consider the model which is the standard $M/G/1$ queueing system except the above-mentioned assumption as to the service time. A more general formulation is given in Harris [2], in which steady-state properties are investigated. In what follows we shall study some transient behaviors for our model in the case of $N=1$. We may treat the case $N\geqq 2$ in the same way, but the reason we restrict ourselves to the case $N=1$ is to avoid excessive complexity in expressing the results.

We shall use the following notations after Takács [1]:

λ=the intensity of the input Poisson process;

$\alpha_b=\int_0^\infty x dH_b(x)$;

$\xi(t)$=the number of customers in the system at the instant t $(0\leqq t<\infty)$;

τ_n=the instant of the nth arrival of customer ($n=1, 2, \cdots$, and $\tau_0=0$);

$\theta_n=\tau_{n+1}-\tau_n$ $(n=0, 1, 2, \cdots)$;

η_n=the waiting time of the customer of the nth arrival $(n=1, 2, \cdots)$;

χ_n=the nth service time $(n=1, 2, \cdots)$;

τ'_n=the instant of the nth departure of customer $(n=1, 2, \cdots)$.

In addition we put $\tau'_0=0$ if $\xi(0)>0$ and the initial service starts at time zero, or $\xi(0)=0$;

$\zeta_n=\xi(\tau_n'+0)=$the number of customers in the system immediately after the nth departure $(n=1, 2, \cdots)$.

Moreover, we put $\zeta_0=\xi(0)$ if $\xi(0)>0$ and the initial service starts at time zero, or $\xi(0)=0$;

$$\psi_a(s)=\int_0^\infty e^{-sx}dH_a(x) \quad \text{for } Rs\geqq 0,$$

$$\psi_b(s)=\int_0^\infty e^{-sx}dH_b(x) \quad \text{for } Rs\geqq 0.$$

2. Busy period

Let X_0 be the length of the initial busy period and let $X_1, X_2, \cdots$ be the lengths of the successive busy periods. It is obvious that X_0, $X_1, X_2, \cdots$ form a sequence of mutually independent random variables and every X_n $(n\geqq 1)$ has the same distribution function. Let $G(x)$ be the distribution function of X_n $(n\geqq 1)$, and let $\hat{G}(x|\xi(0)=i)$ be the distribution function of X_0 under the condition that $\xi(0)=i$ for $i\geqq 1$. For the case where the server is idle at time zero, that is $\xi(0)=0$, we define $\hat{G}(x|\xi(0)=0)=1$ for $x\geqq 0$ and $\hat{G}(x|\xi(0)=0)=0$ for $x<0$. We put

$$\Gamma(s)=\int_0^\infty e^{-sx}dG(x)\,, \tag{1}$$

$$\hat{\Gamma}(s|\xi(0)=i)=\int_0^\infty e^{-sx}d\hat{G}(x|\xi(0)=i)\,, \tag{2}$$

for $Rs\geqq 0$. In this section we determine these Laplace-Stieltjes transforms. First we cite the following lemma given in Takács [1].

LEMMA 1. *If $Rs\geqq 0$ and $|w|\leqq 1$ then $z=\gamma_b(s, w)$, the root of the equation*

$$z=w\psi_b(s+\lambda(1-z)) \tag{3}$$

which has the smallest absolute value, is

$$\gamma_b(s, w)=\sum_{j=1}^\infty \frac{\lambda^{j-1}w^j}{j!}\int_0^\infty e^{-(\lambda+s)x}x^{j-1}dH_b^{j*}(x)\,, \tag{4}$$

where $H_b^{j*}(x)$ denotes the jth iterated convolution of $H_b(x)$ with itself.

This is a continuous function of s and w if $Rs\geqq 0$ and $|w|\leqq 1$ and, further, $z=\gamma_b(s, w)$ is the only root of (3) in the unit circle $|z|<1$ if $Rs\geqq 0$ and $|w|<1$ or $Rs>0$ and $|w|\leqq 1$ or $Rs\geqq 0$, $|w|\leqq 1$ and $\lambda\alpha_b>1$. Specifically, $\omega_b=\gamma_b(0, 1)$ is the smallest positive real root of the equation

$$z=\psi_b(\lambda(1-z))\,. \tag{5}$$

If $\lambda\alpha_b>1$ then $\omega_b<1$ and if $\lambda\alpha_b\leqq 1$ then $\omega_b=1$.

Now consider an arbitrary busy period and let X be the length of this period, let χ be the length of the first service time in this period, and let ν be the number of customers who arrived during the time χ. Then

$$(6) \qquad G(x)=\int_0^\infty P\{X\leqq x|\chi=u,\ \nu=0\}P\{\nu=0|\chi=u\}dH_a(u) \\ +\int_0^\infty P\{X\leqq x|\chi=u,\ \nu=1\}P\{\nu=1|\chi=u\}dH_a(u) \\ +\sum_{j=2}^{\infty}\int_0^\infty P\{X\leqq x|\chi=u,\ \nu=j\}P\{\nu=j|\chi=u\}dH_a(u)\,.$$

Each term of the right-hand side of (6) is calculated as follows. First,

$$(7) \qquad P\{X\leqq x|\chi=u,\ \nu=0\}=\begin{cases}1 & u\leqq x\,,\\ 0 & u>x\,.\end{cases}$$

Under the condition $\chi=u$, $\nu=1$, it is obvious that X is decomposed so that

$$(8) \qquad X=u+X^{(1)}\,,$$

where $X^{(1)}$ is a random variable independent of χ and ν, which has distribution function $G(x)$. Therefore,

$$(9) \qquad P\{X\leqq x|\chi=u,\ \nu=1\}=\begin{cases}G(x-u) & u\leqq x\,,\\ 0 & u>x\,.\end{cases}$$

For $j\geqq 2$, under the condition $\chi=u$, $\nu=j$, we can see, using the same argument as in Takács ([1], p. 61), that we may calculate the distribution function of X by introducing appropriate random variables and decomposition of X into them. In fact when we introduce mutually independent random variables $X^{(1)}, X_1^{(2)}, \cdots, X_{j-1}^{(2)}$, independent of χ and ν, where the distribution function of $X^{(1)}$ is $G(x)$ and the distribution function of $X_k^{(2)}$ is $K(x)$, which is the distribution function of the busy period in the standard $M/G/1$ queue with the service-time distribution function $H_b(x)$, we can calculate the distribution function of X by considering that X is decomposed as follows:

$$(10) \qquad X=u+X_1^{(2)}+\cdots+X_{j-1}^{(2)}+X^{(1)}\,.$$

From this we have

$$(11) \qquad P\{X\leqq x|\chi=u,\ \nu=j\}=\begin{cases}G*K^{(j-1)*}(x-u) & u\leqq x\,,\\ 0 & u>x\,.\end{cases}$$

As, under the condition $\chi=u$, ν distributes according to the Poisson distribution with parameter λu, we have from (6), (7), (9) and (11)

$$G(x)=\int_0^x e^{-\lambda u}dH_a(u)+\int_0^x G(x-u)e^{-\lambda u}\lambda u dH_a(u)+\sum_{j=2}^{\infty}\int_0^x G*K^{(j-1)*}(x-u)e^{-\lambda u}\frac{(\lambda u)^j}{j!}dH_a(u)\,. \tag{12}$$

Hence we obtain

$$\Gamma(s)=\frac{\Lambda(s)\phi_a(s+\lambda)}{\Lambda(s)+\phi_a(s+\lambda)-\phi_a[s+\lambda(1-\Lambda(s))]}\,, \tag{13}$$

where

$$\Lambda(s)=\int_0^\infty e^{-sx}dK(x)\,. \tag{14}$$

But it is known (Takács [1]) that

$$\Lambda(s)=\gamma_b(s)\,, \tag{15}$$

where $\gamma_b(s)$ is the only root of the equation (3) with $w=1$ in the unit circle. From (13) and (15) we have

$$\Gamma(s)=\frac{\gamma_b(s)\phi_a(s+\lambda)}{\gamma_b(s)+\phi_a(s+\lambda)-\phi_a[s+\lambda(1-\gamma_b(s))]}\,.$$

The distribution function $G(x)$, whose Laplace-Stieltjes transform is $\Gamma(s)$, may be an improper distribution function. We next consider a condition that $G(x)$ be a proper distribution function. As $G(x)$ is bounded ($\leqq 1$) and non-decreasing function, $G(\infty)=\lim_{x\to\infty} G(x)$ exists. Hence, by Abel's theorem and noting Lemma 1, we get

$$G(\infty)=\lim_{s\to 0+}\Gamma(s)=\frac{\omega_b\phi_a(\lambda)}{\omega_b+\phi_a(\lambda)-\phi_a(\lambda(1-\omega_b))}\,.$$

Thus we have

$$0<\frac{\omega_b\phi_a(\lambda)}{\omega_b+\phi_a(\lambda)-\phi_a(\lambda(1-\omega_b))}\leqq 1\,.$$

In this inequality we have equality if and only if $\omega_b=1$. For, put

$$f_1(s)=s[1-\phi_a(\lambda)]\,,$$

$$f_2(s)=\phi_a(\lambda(1-s))-\phi_a(\lambda)\,,$$

for $0<s\leqq 1$. Then $f_2''(s)=\lambda^2\int_0^\infty e^{-\lambda(1-s)x}x^2dH_a(x)>0$, $f_1(0)=f_2(0)=0$ and $f_1(1)=$

$f_2(1)=1-\psi_a(\lambda)$. So $f_2(s)\leqq f_1(s)$ for $0<s\leqq 1$, and the equality holds if and only if $s=1$.

Now, by Lemma 1, $\omega_b=1$ if and only if $\lambda\alpha_b\leqq 1$. Hence $G(\infty)=1$ if and only if $\lambda\alpha_b\leqq 1$.

Summarizing the above results, we get the following.

THEOREM 1. *$\Gamma(s)$ is given by*

$$\Gamma(s)=\frac{\gamma_b(s)\psi_a(s+\lambda)}{\gamma_b(s)+\psi_a(s+\lambda)-\psi_a[s+\lambda(1-\gamma_b(s))]}\,, \tag{16}$$

where $\gamma_b(s)$ is the only root in the unit circle $|z|<1$ of the equation

$$z=\psi_b(s+\lambda(1-z))\,. \tag{17}$$

$G(x)$ is a proper distribution function if and only if $\lambda\alpha_b\leqq 1$. In any case

$$G(\infty)=\frac{\omega_b\psi_a(\lambda)}{\omega_b+\psi_a(\lambda)-\psi_a(\lambda(1-\omega_b))}\,, \tag{18}$$

and this is <1 if and only if $\lambda\alpha_b>1$.

Next we consider $\hat{\Gamma}(s|\xi(0)=i)$. Here and in what follows, when condition $\xi(0)=i$ is imposed for $i\geqq 1$, we tacitly assume that the initial service has started at the instant of time zero.

It is obvious that

$$\hat{\Gamma}(s|\xi(0)=0)\equiv 1 \tag{19}$$

and

$$\hat{\Gamma}(s|\xi(0)=1)=\Gamma(s)\,. \tag{20}$$

For $i\geqq 2$ we have

$$\begin{aligned}\hat{G}(x|\xi(0)=i)&\\ =\int_0^\infty & P\{X_0\leqq x|\chi_i^{(0)}=u,\,\nu_i^{(0)}=0\}P\{\nu_i^{(0)}=0|\chi_i^{(0)}=u\}dH_b^{(i-1)*}(u)\\ &+\sum_{j=1}^{\infty}\int_0^\infty P\{X_0\leqq x|\chi_i^{(0)}=u,\,\nu_i^{(0)}=j\}P\{\nu_i^{(0)}=j|\chi_i^{(0)}=u\}dH_b^{(i-1)*}(u)\,,\end{aligned} \tag{21}$$

where $\chi_i^{(0)}$ is the sum of the lengths of the first $(i-1)$ service times (accordingly the distribution function of $\chi_i^{(0)}$ is $H_b^{(i-1)*}$), and $\nu_i^{(0)}$ is the number of customers who arrived during the time $\chi_i^{(0)}$.

In the same way as above we have

$$P\{X_0\leqq x|\chi_i^{(0)}=u,\,\nu_i^{(0)}=0\}=\begin{cases}G(x-u) & u\leqq x\,,\\ 0 & u>x\end{cases} \tag{22}$$

and

(23) $$P\{X_0\leqq x\mid \chi_i^{(0)}=u,\ \nu_i^{(0)}=j\}=\begin{cases} G*K^{j*}(x-u) & u\leqq x\,,\\ 0 & u>x\end{cases}$$

for $j\geqq 1$. As, under the condition $\chi_i^{(0)}=u$, $\nu_i^{(0)}$ distributes according to the Poisson distribution with parameter λu, we have from (21), (22) and (23)

(24) $$\hat{G}(x\mid\xi(0)=i)=\int_0^x G(x-u)e^{-\lambda u}dH_b^{(i-1)*}(u) +\sum_{j=1}^{\infty}\int_0^x G*K^{j*}(x-u)e^{-\lambda u}\frac{(\lambda u)^j}{j!}dH_b^{(i-1)*}(u)\,.$$

From (24) and $\int_0^\infty e^{-sx}dK(x)=\Lambda(s)=\gamma_b(s)$, we have for $i\geqq 2$

$$\hat{\Gamma}(s\mid\xi(0)=i)=\Gamma(s)[\psi_b\{s+\lambda(1-\gamma_b(s))\}]^{i-1}=\Gamma(s)[\gamma_b(s)]^{i-1}\,.$$

Thus we have

THEOREM 2. *For $i\geqq 1$, if the initial service has started at time zero, $\hat{\Gamma}(s\mid\xi(0)=i)$ is given by*

(25) $$\hat{\Gamma}(s\mid\xi(0)=i)=\Gamma(s)[\gamma_b(s)]^{i-1}\,,$$

where $\Gamma(s)$ is given by (16). *As to the case $i=0$, $\hat{\Gamma}(s\mid\xi(0)=0)\equiv 1$.*

3. The transition probabilities of $\{\zeta_n\}$

As is easily seen, $\{\zeta_0, \zeta_1, \zeta_2, \cdots\}$ forms a homogeneous Markov chain. Let (p_{ik}) be the matrix of transition probabilities of this Markov chain and put $(p_{ik}^{(n)})=(p_{ik})^n$, $n=1, 2, \cdots$. In this section we shall study probabilities $p_{ik}^{(n)}$:

(26) $$p_{ik}^{(n)}=P\{\zeta_n=k\mid\zeta_0=i\}\,.$$

Let us determine the generating function

(27) $$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}p_{ik}^{(n)}w^n z^k=\sum_{n=0}^{\infty}U_{in}(z)w^n$$

for $|w|<1$ and $|z|\leqq 1$, where $p_{ik}^{(0)}=1$ for $i=k$ and $p_{ik}^{(0)}=0$ for $i\neq k$, and

(28) $$U_{in}(z)=E\{z^{\zeta_n}\mid\zeta_0=i\}\,.$$

Define ν_n is the number of customers who arrived during the nth service time ($n\geqq 1$). Then we obviously have

$$P\{\nu_{n+1}=j\,|\,\zeta_n\leqq 1,\ \zeta_0=i\}=q_j\ ,$$

$$P\{\nu_{n+1}=j\,|\,\zeta_n\geqq 2,\ \zeta_0=i\}=p_j\ ,$$

for all $n\geqq 0$, where

$$q_j=\int_0^\infty e^{-\lambda x}\frac{(\lambda x)^j}{j!}dH_a(x)\ ,$$

$$p_j=\int_0^\infty e^{-\lambda x}\frac{(\lambda x)^j}{j!}dH_b(x)\ .$$

From these it is easily seen that

$$E[z^{\nu_{n+1}}|\zeta_n\leqq 1,\ \zeta_0=i]=\psi_a(\lambda(1-z))\ , \tag{29}$$

$$E[z^{\nu_{n+1}}|\zeta_n\geqq 2,\ \zeta_0=i]=\psi_b(\lambda(1-z))\ . \tag{30}$$

Now we can write that

$$\zeta_{n+1}=[\zeta_n-1]^+ +\nu_{n+1}\ , \tag{31}$$

where $[a]^+=\max\{a,0\}$. Therefore, for $n\geqq 1$ we have

$$\begin{aligned}U_{i,n+1}(z)&=E\{z^{[\zeta_n-1]^+ +\nu_{n+1}}|\zeta_0=i\}\\&=P\{\zeta_n\leqq 1|\zeta_0=i\}E\{z^{\nu_{n+1}}|\zeta_n\leqq 1,\ \zeta_0=i\}\\&\quad+P\{\zeta_n\geqq 2|\zeta_0=i\}\frac{1}{z}E\{z^{\nu_{n+1}}|\zeta_n\geqq 2,\ \zeta_0=i\}E\{z^{\zeta_n}|\zeta_n\geqq 2,\ \zeta_0=i\}\ ,\end{aligned}$$

because, under the condition $\zeta_n\geqq 2$, ν_{n+1} and ζ_n are mutually independent. From this, using (29) and (30), we have

$$U_{i,n+1}(z)=\frac{U_{in}(z)-(p_{i0}^{(n)}+zp_{i1}^{(n)})}{z}\psi_b(\lambda(1-z))+(p_{i0}^{(n)}+p_{i1}^{(n)})\psi_a(\lambda(1-z))\ . \tag{32}$$

Noting $U_{i0}(z)=z^i$, we have the following relation by forming the generating functions for both sides of (32):

$$\begin{aligned}&[z-w\psi_b(\lambda(1-z))]\sum_{n=0}^\infty U_{in}(z)w^n\\&\qquad=z^{i+1}+wz\{\psi_a(\lambda(1-z))-\psi_b(\lambda(1-z))\}\sum_{n=0}^\infty p_{i1}^{(n)}w^n\\&\qquad\quad+w\{z\psi_a(\lambda(1-z))-\psi_b(\lambda(1-z))\}\sum_{n=0}^\infty p_{i0}^{(n)}w^n\ .\end{aligned} \tag{33}$$

Differentiating both sides of (33) with respect to z, and then letting $z\to 0$ we have

$$[1+\lambda w\psi_b'(\lambda)]\sum_{n=0}^\infty p_{i0}^{(n)}w^n-w\psi_b(\lambda)\sum_{n=0}^\infty p_{i1}^{(n)}w^n$$

$$
=\begin{cases} 1+w\sum_{n=0}^{\infty} p_{01}^{(n)}w^n[\psi_a(\lambda)-\psi_b(\lambda)]+w\sum_{n=0}^{\infty} p_{00}^{(n)}w^n[\psi_a(\lambda)+\lambda\psi_b'(\lambda)] & \text{for } i=0\,, \\ w\sum_{n=0}^{\infty} p_{i1}^{(n)}w^n[\psi_a(\lambda)-\psi_b(\lambda)]+w\sum_{n=0}^{\infty} p_{i0}^{(n)}w^n[\psi_a(\lambda)+\lambda\psi_b'(\lambda)] & \text{for } i\geqq 1\,. \end{cases}
$$

From this we have

$$
(34)\qquad \sum_{n=0}^{\infty} p_{01}^{(n)}w^n=\frac{1}{w\psi_a(\lambda)}\left[\{1-w\psi_a(\lambda)\}\sum_{n=0}^{\infty} p_{00}^{(n)}w^n-1\right],
$$

and

$$
(35)\qquad \sum_{n=0}^{\infty} p_{i1}^{(n)}w^n=\frac{[1-w\psi_a(\lambda)]}{w\psi_a(\lambda)}\sum_{n=0}^{\infty} p_{i0}^{(n)}w^n\,, \qquad \text{for } i\geqq 1\,.
$$

On the other hand from (33) we have

$$
(36)\qquad \sum_{n=0}^{\infty} U_{in}(z)w^n
$$

$$
=\Big[z^{i+1}+wz\{\psi_a(\lambda(1-z))-\psi_b(\lambda(1-z))\}\sum_{n=0}^{\infty} p_{i1}^{(n)}w^n
$$

$$
+w\{z\psi_a(\lambda(1-z))-\psi_b(\lambda(1-z))\}\sum_{n=0}^{\infty} p_{i0}^{(n)}w^n\Big]\Big/[z-w\psi_b(\lambda(1-z))]\,.
$$

The left-hand side of (36) is a regular function of z if $|z|<1$ and $|w|<1$, and the denominator of the right-hand side of (36) has exactly one zero in the unit circle $|z|<1$ by Lemma 1. Then this zero must be a zero of the numerator of the right-hand side of (36). Thus if we denote the unique root of $z=w\psi_b(\lambda(1-z))$ in the unit circle by $z=\gamma=g(w)$, we get for $i\geqq 0$

$$
(37)\qquad \gamma^i+[w\psi_a(\lambda(1-\gamma))-\gamma]\sum_{n=0}^{\infty} p_{i1}^{(n)}w^n+[w\psi_a(\lambda(1-\gamma))-1]\sum_{n=0}^{\infty} p_{i0}^{(n)}w^n=0\,.
$$

From (34), (35) and (37) we have, for $|w|<1$,

$$
(38)\qquad \left\{\sum_{n=0}^{\infty} p_{00}^{(n)}w^n\right\}[\gamma\{1-w\psi_a(\lambda)\}-w\{\psi_a(\lambda(1-\gamma))-\psi_a(\lambda)\}]
$$

$$
=\gamma-w[\psi_a(\lambda(1-\gamma))-\psi_a(\lambda)]\,,
$$

and

$$
(39)\qquad \left\{\sum_{n=0}^{\infty} p_{i0}^{(n)}w^n\right\}[\gamma\{1-w\psi_a(\lambda)\}-w\{\psi_a(\lambda(1-\gamma))-\psi_a(\lambda)\}]
$$

$$
=w\psi_a(\lambda)\gamma^i \qquad\qquad \text{for } i\geqq 1\,.
$$

If in (38) or (39)

$$\gamma\{1-w\phi_a(\lambda)\}-w\{\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)\}=0$$

for $|w|<1$, $w\neq 0$, then $w\phi_a(\lambda)\gamma=0$ in any case. But if $w\neq 0$, $\gamma=g(w)\neq 0$ because of Lemma 1, hence $w\phi_a(\lambda)\gamma\neq 0$, which is a contradiction. Therefore,

$$\gamma\{1-w\phi_a(\lambda)\}-w\{\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)\}\neq 0$$

for $|w|<1$, $w\neq 0$. Then, from (38) and (39) we have, for $|w|<1$, $w\neq 0$,

$$\sum_{n=0}^{\infty} p_{00}^{(n)}w^n=\frac{\gamma-w[\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)]}{\gamma[1-w\phi_a(\lambda)]-w[\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)]}, \tag{40}$$

and

$$\sum_{n=0}^{\infty} p_{i0}^{(n)}w^n=\frac{w\phi_a(\lambda)\gamma^i}{\gamma[1-w\phi_a(\lambda)]-w[\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)]} \quad \text{for } i\geqq 1. \tag{41}$$

Using (40), (34) and (41), (35) we have for $|w|<1$, $w\neq 0$,

$$\sum_{n=0}^{\infty} p_{01}^{(n)}w^n=\frac{w[\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)]}{\gamma[1-w\phi_a(\lambda)]-w[\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)]} \tag{42}$$

and

$$\sum_{n=0}^{\infty} p_{i1}^{(n)}w^n=\frac{[1-w\phi_a(\lambda)]\gamma^i}{\gamma[1-w\phi_a(\lambda)]-w[\phi_a(\lambda(1-\gamma))-\phi_a(\lambda)]} \quad \text{for } i\geqq 1. \tag{43}$$

Putting (40), (42) or (41), (43) into (36), we finally have the following.

THEOREM 3. *The higher transition probabilities* $\{p_{ik}^{(n)}\}$ *are given by the following generating functions:*

$$\begin{aligned}(44)\quad &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{0k}^{(n)}w^n z^k\\ &=\Big\{z[w\phi_a(\lambda)\{1-g(w)\}-\{g(w)-w\phi_a(\lambda(1-g(w)))\}]\\ &\qquad +wg(w)[z\phi_a(\lambda(1-z))-\phi_b(\lambda(1-z))]\\ &\qquad +w^2(1-z)\phi_b(\lambda(1-z))[\phi_a(\lambda(1-g(w)))-\phi_a(\lambda)]\Big\}\Big/\\ &\quad [z-w\phi_b(\lambda(1-z))][g(w)\{1-w\phi_a(\lambda)\}-w\{\phi_a(\lambda(1-g(w)))-\phi_a(\lambda)\}],\\ &\qquad\qquad |w|<1,\ w\neq 0,\ |z|\leqq 1,\end{aligned}$$

and for $i\geqq 1$,

$$(45)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{ik}^{(n)}w^n z^k$$

$$=\Big\{z^{i+1}[w\phi_a(\lambda)\{1-g(w)\}+\{g(w)-w\phi_a(\lambda(1-g(w)))\}]$$
$$+[g(w)]^i[wz\{\phi_a(\lambda(1-z))-\phi_b(\lambda(1-z))\}+w^2(z-1)\phi_a(\lambda)\phi_b(\lambda(1-z))]\Big\}\Big/$$
$$[z-w\phi_b(\lambda(1-z))][g(w)\{1-w\phi_a(\lambda)\}-w\{\phi_a(\lambda(1-g(w)))-\phi_a(\lambda)\}]\,,$$
$$|w|<1,\ w\neq 0,\ |z|\leqq 1\,,$$

where $g(w)$ *is the only one root in the unit circle* $|z|<1$ *of the equation*

$$z=w\phi_b(\lambda(1-z))\,. \tag{46}$$

4. The distribution of $\xi(t)$

In this section we study the distribution of $\xi(t)$ for arbitrary t. Let $P_{ik}(t)=P\{\xi(t)=k|\xi(0)=i\}$. By our assumption we have

$$P_{ik}(t)=P\{\xi(t)=k|\zeta_0=i\}\,.$$

We shall determine the above $\{P_{ik}(t)\}$, and to do that, we begin with the following.

LEMMA 2. *The* $P_{i0}(t)$ $(i\geqq 0)$ *satisfies the following integral equation*

$$P_{i0}(t)=\hat{G}(t|\xi(0)=i)-\lambda\int_0^t[1-G(t-u)]P_{i0}(u)du\,. \tag{47}$$

Moreover,

$$\lambda\int_0^t P_{i0}(u)du=\sum_{n=1}^{\infty}P\{\tau_n\leqq t,\ \eta_n=0|\zeta_0=i\}\,. \tag{48}$$

The proof given in Takács ([1], p. 66) holds valid in the case of our model, too, and hence (48) holds.

LEMMA 3. *Let us define*

$$U_{in}(s,z)=E\{\exp(-s\tau_n')z^{\zeta_n}|\zeta_0=i\}\,, \tag{49}$$
$$n=0,1,2,\cdots,\qquad \textit{for } Rs>0 \textit{ and } |z|\leqq 1.$$

Then we have

$$\sum_{n=0}^{\infty}U_{in}(s,z) \tag{50}$$
$$=\Big[z^{i+1}+\Big\{\sum_{n=0}^{\infty}U_{in}(s,0)\Big\}\Big\{\frac{\lambda z}{\lambda+s}\phi_a(s+\lambda(1-z))-\phi_b(s+\lambda(1-z))\Big\}$$
$$+z\Big\{\sum_{n=0}^{\infty}\Big(\frac{\partial U_{in}(s,z)}{\partial z}\Big)_{z=0}\Big\}\{\phi_a(s+\lambda(1-z))-\phi_b(s+\lambda(1-z))\}\Big]\Big/$$
$$[z-\phi_b(s+\lambda(1-z))]\,,$$

where

(51) $$\sum_{n=0}^{\infty} U_{in}(s, 0) = \frac{\psi_a(s+\lambda)[\gamma(s)]^i - [\psi_a(s+\lambda(1-\gamma(s))) - \gamma(s)]\delta_i^{(0)}}{\gamma(s)\left[1-\frac{\lambda}{\lambda+s}\psi_a(s+\lambda)\right] - [\psi_a(s+\lambda(1-\gamma(s))) - \psi_a(s+\lambda)]},$$

and

(52) $$\sum_{n=0}^{\infty} \left(\frac{\partial U_n(s, z)}{\partial z}\right)_{z=0} = \frac{\left[1-\frac{\lambda}{\lambda+s}\psi_a(s+\lambda)\right][\gamma(s)]^i - \left[1-\frac{\lambda}{\lambda+s}\psi_a(s+\lambda(1-\gamma(s)))\right]\delta_i^{(0)}}{\gamma(s)\left[1-\frac{\lambda}{\lambda+s}\psi_a(s+\lambda)\right] - [\psi_a(s+\lambda(1-\gamma(s))) - \psi_a(s+\lambda)]},$$

$\delta_i^{(0)}$ *being* 1 *or* 0 *according as* $i=0$ *or* $i \geqq 1$, *and* $\gamma(s)$ *being the only one root in the unit circle* $|z|<1$ *of the equation*

(53) $$z = \psi_b(s+\lambda(1-z)).$$

PROOF. Let ν_n be the number of arrivals during the nth service and let θ_n^* be the time interval between the nth departure and the immediately following arrival of the customer. We write $\chi_{n+1}^{(a)}$ for χ_{n+1} if $\zeta_n \leqq 1$, and write $\chi_{n+1}^{(b)}$ for χ_{n+1} if $\zeta_n \geqq 2$. Then we have

(54) $$\zeta_{n+1} = [\zeta_n - 1]^+ + \nu_{n+1}$$

and

(55) $$\tau'_{n+1} = \begin{cases} \tau'_n + \theta_n^* + \chi_{n+1}^{(a)} & \text{if } \zeta_n = 0, \\ \tau'_n + \chi_{n+1}^{(a)} & \text{if } \zeta_n = 1, \\ \tau'_n + \chi_{n+1}^{(b)} & \text{if } \zeta_n \geqq 2. \end{cases}$$

Here we know that $\chi_n^{(a)}$ or $\chi_n^{(b)}$ is distributed according to $H_a(t)$ or $H_b(t)$, respectively, that θ_n^* has an exponential distribution with parameter λ, and that all τ'_n, θ_n^* and $\chi_{n+1}^{(a)}$ or $\chi_{n+1}^{(b)}$ are mutually independent. Therefore, from (54) and (55), we have

$$\begin{aligned} U_{i,n+1}(s, z) &= P\{\zeta_n=0|\zeta_0=i\}E\{\exp(-s(\tau'_n+\theta_n^*+\chi_{n+1}^{(a)}))z^{\nu_{n+1}}|\zeta_n=0, \zeta_0=i\} \\ &\quad + P\{\zeta_n=1|\zeta_0=i\}E\{\exp(-s(\tau'_n+\chi_{n+1}^{(a)}))z^{\nu_{n+1}}|\zeta_n=1, \zeta_0=i\} \\ &\quad + \sum_{k=2}^{\infty} P\{\zeta_n=k|\zeta_0=i\}E\{\exp(-s(\tau'_n+\chi_{n+1}^{(b)}))z^{\zeta_n-1+\nu_{n+1}}|\zeta_n=k, \zeta_0=i\} \\ &= P\{\zeta_n=0|\zeta_0=i\}E\{\exp(-s\theta_n^*)\}E\{\exp(-s\tau'_n)|\zeta_n=0\} \\ &\quad \cdot E\{\exp(-s\chi_{n+1}^{(a)})z^{\nu_{n+1}}|\zeta_n=0\} \end{aligned}$$

$$+P\{\zeta_n=1|\zeta_0=i\}E\{\exp(-s\tau_n')|\zeta_n=1\}E\{\exp(-s\chi_{n+1}^{(a)})z^{\nu_{n+1}}|\zeta_n=1\}$$
$$+\sum_{k=2}^{\infty}P\{\zeta_n=k|\zeta_0=i\}E\{\exp(-s\tau_n')z^{k-1}|\zeta_n=k\}$$
$$\cdot E\{\exp(-s\chi_{n+1}^{(b)})z^{\nu_{n+1}}|\zeta_n=k\}.$$

Now it is easily seen that

$$E\{\exp(-s\theta_n^*)\}=\frac{\lambda}{\lambda+s},$$
$$E\{\exp(-s\chi_{n+1}^{(a)})z^{\nu_{n+1}}|\zeta_n\leqq 1\}=\psi_a(s+\lambda(1-z)),$$
$$E\{\exp(-s\chi_{n+1}^{(b)})z^{\nu_{n+1}}|\zeta_n=k\}=\psi_b(s+\lambda(1-z)), \qquad k\geqq 2.$$

Hence we have

(56)
$$\begin{aligned}&U_{i,n+1}(s, z)\\&=P\{\zeta_n=0|\zeta_0=i\}E\{\exp(-s\tau_n')|\zeta_n=0\}\\&\quad\cdot\left\{\frac{\lambda}{\lambda+s}\psi_a(s+\lambda(1-z))-\frac{1}{z}\psi_b(s+\lambda(1-z))\right\}\\&\quad+P\{\zeta_n=1|\zeta_0=i\}E\{\exp(-s\tau_n')|\zeta_n=1\}\\&\quad\cdot\{\psi_a(s+\lambda(1-z))-\psi_b(s+\lambda(1-z))\}\\&\quad+\frac{1}{z}U_{in}(s, z)\psi_b(s+\lambda(1-z)).\end{aligned}$$

Noting

(57)
$$U_{in}(s, 0)=P\{\zeta_n=0|\zeta_0=i\}E\{\exp(-s\tau_n')|\zeta_n=0\},$$

and

(58)
$$\left(\frac{\partial U_{in}(s, z)}{\partial z}\right)_{z=0}=P\{\zeta_n=1|\zeta_0=i\}E\{\exp(-s\tau_n')|\zeta_n=1\},$$

and summing both sides of (56) over n, we have

(59)
$$\begin{aligned}&\left\{\sum_{n=0}^{\infty}U_{in}(s, z)\right\}\{z-\psi_b(s+\lambda(1-z))\}\\&=z\{U_{i0}(s, z)-\lim_{n\to\infty}U_{in}(s, z)\}\\&\quad+\left\{\sum_{n=0}^{\infty}U_{in}(s, 0)\right\}\left\{\frac{\lambda z}{\lambda+s}\psi_a(s+\lambda(1-z))-\psi_b(s+\lambda(1-z))\right\}\\&\quad+z\left\{\sum_{n=0}^{\infty}\left(\frac{\partial U_{in}(s, z)}{\partial z}\right)_{z=0}\right\}\{\psi_a(s+\lambda(1-z))-\psi_b(s+\lambda(1-z))\}.\end{aligned}$$

In the right-hand side of (59), $\sum_{n=0}^{\infty}U_{in}(s, 0)$ and $\sum_{n=0}^{\infty}\left(\frac{\partial U_{in}(s, z)}{\partial z}\right)_{z=0}$ are absolutely convergent for $Rs>0$, and $\lim_{n\to\infty}U_{in}(s, z)=0$ for $Rs>0$, $|z|\leqq 1$. For, from (57) we have

$$\begin{aligned}|U_{in}(s,0)| &\leqq P\{\zeta_n=0|\zeta_0=i\}E\{\exp(-(Rs)\tau_n')|\zeta_n=0\}\\ &\leqq \sum_{j=0}^{\infty} P\{\zeta_n=j|\zeta_0=i\}E\{\exp(-(Rs)\tau_n')|\zeta_n=j\}\\ &=E\{\exp(-(Rs)\tau_n')|\zeta_0=i\}.\end{aligned}$$

In the same way, from (58) we have

$$\left|\left(\frac{\partial U_{in}(s,z)}{\partial z}\right)_{z=0}\right| \leqq E\{\exp(-(Rs)\tau_n')|\zeta_0=i\}.$$

But, for $n \geqq 1$,

$$\chi_1+\chi_2+\cdots+\chi_n \leqq \tau_n',$$

and χ_j is distributed according to $H_a(x)$ or $H_b(x)$, and hence for s fixed

$$E\{\exp(-(Rs)\tau_n')|\zeta_0=i\} \leqq [\max\{\psi_a(Rs), \psi_b(Rs)\}]^n, \qquad n\geqq 1.$$

Therefore, for $Rs>0$

$$(60) \qquad \sum_{n=1}^{\infty} E\{\exp(-(Rs)\tau_n')|\zeta_0=i\} \leqq \max\left[\frac{\psi_a(Rs)}{1-\psi_a(Rs)}, \frac{\psi_b(Rs)}{1-\psi_b(Rs)}\right] < \infty.$$

Hence $\sum_{n=0}^{\infty} U_{in}(s,z)$ and $\sum_{n=0}^{\infty}\left(\frac{\partial U_{in}(s,z)}{\partial z}\right)_{z=0}$ are absolutely convergent for $Rs>0$. Moreover, $|U_{in}(s,z)| \leqq E\{\exp(-(Rs)\tau_n')|\zeta_0=i\}$ for $|z|\leqq 1$, so from (60) we have $\lim_{n\to\infty} U_{in}(s,z)=0$ for $Rs>0$, $|z|\leqq 1$. Accordingly in (59) $\sum_{n=0}^{\infty} U_{in}(s,z)$ is convergent for $Rs>0$, $|z|\leqq 1$. Differentiating both sides of (59) with respect to z, and then letting $z\to 0$, we have

$$\begin{aligned}\{1+\lambda\psi_b'(s+\lambda)\}&\left\{\sum_{n=0}^{\infty} U_{in}(s,0)\right\}-\psi_b(s+\lambda)\left\{\sum_{n=0}^{\infty}\left(\frac{\partial U_{in}(s,z)}{\partial z}\right)_{z=0}\right\}\\ =U_{i0}(s,0)&+\left\{\sum_{n=0}^{\infty} U_{in}(s,0)\right\}\left\{\frac{\lambda}{\lambda+s}\psi_a(s+\lambda)+\lambda\psi_b'(s+\lambda)\right\}\\ &+\left\{\sum_{n=0}^{\infty}\left(\frac{\partial U_{in}(s,z)}{\partial z}\right)_{z=0}\right\}\{\psi_a(s+\lambda)-\psi_b(s+\lambda)\}.\end{aligned}$$

From this we have

$$(61) \qquad \sum_{n=0}^{\infty}\left(\frac{\partial U_{in}(s,z)}{\partial z}\right)_{z=0} = \frac{\left\{1-\frac{\lambda}{\lambda+s}\psi_a(s+\lambda)\right\}\left\{\sum_{n=0}^{\infty} U_{in}(s,0)\right\}-U_{i0}(s,0)}{\psi_a(s+\lambda)}.$$

On the other hand, from (59) we get

$$(62)\quad \sum_{n=0}^{\infty} U_{in}(s, z)$$
$$=\Big[zU_{i0}(s, z)+\Big\{\sum_{n=0}^{\infty} U_{in}(s, 0)\Big\}\Big\{\frac{\lambda z}{\lambda+s}\phi_a(s+\lambda(1-z))-\phi_b(s+\lambda(1-z))\Big\}$$
$$+z\Big\{\sum_{n=0}^{\infty}\Big(\frac{\partial U_{in}(s, z)}{\partial z}\Big)_{z=0}\Big\}\{\phi_a(s+\lambda(1-z))-\phi_b(s+\lambda(1-z))\}\Big]\Big/$$
$$[z-\phi_b(s+\lambda(1-z))] .$$

The left-hand side of (62) is a regular function of z if $|z|<1$, and the denominator of the right-hand side of (62) has exactly one zero $z=\gamma(s)$ in the unit circle $|z|<1$ by Lemma 1. Then this zero must be a zero of the numerator of the right-hand side of (62). Therefore, we get

$$(63)\quad \gamma(s)U_{i0}(s, \gamma(s))+\Big\{\frac{\lambda\gamma(s)}{\lambda+s}\phi_a(s+\lambda(1-\gamma(s)))-\gamma(s)\Big\}\Big\{\sum_{n=0}^{\infty} U_{in}(s, 0)\Big\}$$
$$+\gamma(s)\{\phi_a(s+\lambda(1-\gamma(s)))-\phi_b(s+\lambda(1-\gamma(s)))\}\Big\{\sum_{n=0}^{\infty}\Big(\frac{\partial U_{in}(s, z)}{\partial z}\Big)_{z=0}\Big\}$$
$$=0 .$$

From (61) and (63) we have

$$(64)\quad \Big\{\sum_{n=0}^{\infty} U_{in}(s, 0)\Big\}\Big[\phi_a(s+\lambda(1-\gamma(s)))+\Big(\frac{\lambda\gamma(s)}{\lambda+s}-1\Big)\phi_a(s+\lambda)-\gamma(s)\Big]$$
$$=U_{i0}(s, 0)[\phi_a(s+\lambda(1-\gamma(s)))-\gamma(s)]-\phi_a(s+\lambda)U_{i0}(s, \gamma(s)) .$$

However,

$$(65)\quad \phi_a(s+\lambda(1-\gamma(s)))+\Big(\frac{\lambda\gamma(s)}{\lambda+s}-1\Big)\phi_a(s+\lambda)-\gamma(s)\neq 0 ,$$

for, if the left-hand side of (65) is zero, the right-hand side of (64) is also zero. Then, in the case of $i=0$, we have $\frac{\lambda\gamma(s)}{\lambda+s}\phi_a(s+\lambda)=0$ because of $U_{00}(s, 0)=U_{00}(s, \gamma(s))=1$, and in the case of $i\geqq 1$, we have $\phi_a(s+\lambda)[\gamma(s)]^i=0$ because of $U_{i0}(s, 0)=0$ and $U_{i0}(s, \gamma(s))=[\gamma(s)]^i$. In any case we have $\gamma(s)=0$, but this is impossible as $z=\gamma(s)$ is a zero of $z-\phi_b(s+\lambda(1-z))$. Accordingly we obtain (51) from (64). Putting (51) into (61) we have (52).

Now let $\Pi_{ik}(s)$ be the Laplace transform of $P_{ik}(t)$, that is,

$$(66)\quad \Pi_{ik}(s)=\int_0^{\infty} e^{-st}P_{ik}(t)dt ,$$

for $Rs>0$ and $i,\ k=0, 1, 2, \cdots$. We shall determine them.

THEOREM 4. *The* $\Pi_{i0}(s)$ *is given by*

$$\Pi_{i0}(s)=\begin{cases}\dfrac{1}{s+\lambda[1-\Gamma(s)]} & \textit{for } i=0\,,\\[2ex] \dfrac{\Gamma(s)[\gamma(s)]^{i-1}}{s+\lambda[1-\Gamma(s)]} & \textit{for } i\geqq 1\,,\end{cases}\tag{67}$$

where

$$\Gamma(s)=\frac{\gamma(s)\psi_a(s+\lambda)}{\gamma(s)+\psi_a(s+\lambda)-\psi_a(s+\lambda(1-\gamma(s)))}\,.\tag{68}$$

PROOF. From Lemma 2 we have

$$\Pi_{i0}(s)=\frac{\hat{\Gamma}(s|\xi(0)=i)}{s+\lambda-\lambda\Gamma(s)}\,,$$

where $\Gamma(s)$ and $\hat{\Gamma}(s\,|\,\xi(0)=i)$ are the Laplace-Stieltjes transforms of $G(x)$ and $\hat{G}(x|\xi(0)=i)$, respectively. Then, from Theorems 1 and 2 we get the wanted result.

Define $\delta_i^{(0)}=1$ if $i=0$, $\delta_i^{(0)}=0$ if $i\neq 0$, $\delta_i^{(1)}=1$ if $i=1$ and $\delta_i^{(1)}=0$ if $i\neq 1$. In the following we prove

THEOREM 5. *For $Rs>0$ and $|z|<1$ $\sum_{k=0}^{\infty}\Pi_{ik}(s)z^k$ is given by*

$$\sum_{k=0}^{\infty}\Pi_{ik}(s)z^k=\frac{\Phi_i(s,z)}{s+\lambda(1-z)}\,,\tag{69}$$

where

$$\begin{aligned}\Phi_i(s,z)=&\,\Pi_{i0}(s)[s+\lambda\{1-z\psi_a(s+\lambda(1-z))\}]\\ &+\delta_i^{(1)}\{1+(z-1)\psi_a(s+\lambda(1-z))-z\psi_b(s+\lambda(1-z))\}\\ &+z\left(\frac{\partial\left\{\sum_{n=0}^{\infty}U_{in}(s,z)\right\}}{\partial z}\right)_{z=0}\{\psi_b(s+\lambda(1-z))-\psi_a(s+\lambda(1-z))\}\\ &+\left[\left\{\sum_{n=0}^{\infty}U_{in}(s,z)\right\}-(\delta_i^{(0)}+\delta_i^{(1)})z^i\right]\{1-\psi_b(s+\lambda(1-z))\}\,.\end{aligned}\tag{70}$$

Here $\sum_{n=0}^{\infty}U_{in}(s,z)$ is given in Lemma 3 and $\Pi_{i0}(s)$ is given in Theorem 4.

PROOF. When $\zeta_0=i$, the event $\{\xi(t)=k\}$ for $k\geqq 1$, $i\geqq 0$ can be decomposed in the following way into the mutually exclusive events:

$$\begin{aligned}&\{\xi(t)=k\}\\ &\quad=\{\zeta_0>0,\ t<\tau_1',\ \text{and } k-i \text{ customers arrive during } (0,t]\}\end{aligned}$$

$$\cup\left[\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{k}\{\tau'_n\leqq t<\tau'_{n+1},\ \zeta_n=j,\ \text{and } k-j \text{ customers arrive during } (\tau'_n, t]\}\right]$$
$$\cup\left[\bigcup_{n=1}^{\infty}\{\eta_n=0,\ \tau_n\leqq t<\tau_n+\chi_n,\ \text{and } k-1 \text{ customers arrive during } (\tau_n, t]\}\right].$$

But we have

$$P\{\zeta_0>0,\ t<\tau'_1,\ \text{and } k-i \text{ customers arrive during } (0,t]\}$$
$$=P\{\zeta_0>0,\ t<\chi_1,\ \text{and } k-i \text{ customers arrive during } (0,t]\}$$
$$=\begin{cases}[1-H_a(t)]e^{-\lambda t}\dfrac{(\lambda t)^{k-1}}{(k-1)!} & \text{for } i=1,\\[2mm] [1-H_b(t)]e^{-\lambda t}\dfrac{(\lambda t)^{k-1}}{(k-i)!} & \text{for } 2\leqq i\leqq k,\\[2mm] 0 & \text{otherwise,}\end{cases}$$

$$P\{\tau'_n\leqq t<\tau'_{n+1},\ \zeta_n=j,\ \text{and } k-j \text{ customers arrive during } (\tau'_n, t]\}$$
$$=\int_0^t P\{\tau'_n\leqq t<\tau'_n+\chi_{n+1},\ \zeta_n=j,\ \text{and } k-j \text{ customers arrive during } (\tau'_n, t]\,|\,\tau'_n=u,\ \zeta_n=j\}dP\{\tau'_n\leqq u,\ \zeta_n=j\}$$
$$=\begin{cases}\displaystyle\int_0^t[1-H_a(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{k-1}}{(k-1)!}dP\{\tau'_n\leqq u,\zeta_n=1\} & \text{for } j=1,\\[2mm] \displaystyle\int_0^t[1-H_b(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{k-j}}{(k-j)!}dP\{\tau'_n\leqq u,\zeta_n=j\} & \text{for } 2\leqq j\leqq k\end{cases}$$

and

$$P\{\eta_n=0,\ \tau_n\leqq t<\tau_n+\chi_n,\ \text{and } k-1 \text{ customers arrive during } (\tau_n, t]\}$$
$$=\int_0^t P\{\eta_n=0,\ \tau_n\leqq t<\tau_n+\chi_n,\ \text{and } k-1 \text{ customers arrive during } (\tau_n, t]\,|\,\tau_n=u,\ \eta_n=0\}dP\{\tau_n\leqq u,\eta_n=0\}$$
$$=\int_0^t[1-H_a(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{k-1}}{(k-1)!}dP\{\tau_n\leqq u,\eta_n=0\}.$$

Hence, we obtain, for $k\geqq 1$, $i\geqq 0$,

$$P_{ik}(t)=\delta_i^{(1)}[1-H_a(t)]e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}+\delta_{ik}^{*}[1-H_b(t)]e^{-\lambda t}\frac{(\lambda t)^{k-i}}{(k-i)!}$$
$$+\int_0^t[1-H_a(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{k-1}}{(k-1)!}dN_{i1}(u) \tag{71}$$

$$+\sum_{j=2}^{k}\int_0^t [1-H_b(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{k-j}}{(k-j)!}dN_{ij}(u)$$
$$+\lambda\int_0^t [1-H_a(t-u)]e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^{k-1}}{(k-1)!}P_{i0}(u)du\ ,$$

where $\delta_{ik}^*=1$ for $2\leqq i\leqq k$, and $\delta_{ik}^*=0$ otherwise, and

$$N_{ij}(u)=\sum_{n=1}^{\infty}P\{\tau_n'\leqq u,\ \zeta_n=j\,|\,\zeta_0=i\}\ ,\qquad j\geqq 1\ , \tag{72}$$

and (48) was used. From (71) we have

$$\begin{aligned}\sum_{k=1}^{\infty}P_{ik}(t)z^k=&\ \delta_i^{(1)}[1-H_a(t)]ze^{-\lambda(1-z)t}\\ &+[1-\delta_i^{(0)}][1-\delta_i^{(1)}][1-H_b(t)]z^ie^{-\lambda(1-z)t}\\ &+z\int_0^t[1-H_a(t-u)]e^{-\lambda(1-z)(t-u)}dN_{i1}(u)\\ &+\sum_{j=2}^{\infty}z^j\int_0^t[1-H_b(t-u)]e^{-\lambda(1-z)(t-u)}dN_{ij}(u)\\ &+\lambda z\int_0^t[1-H_a(t-u)]e^{-\lambda(1-z)(t-u)}P_{i0}(u)du\ .\end{aligned} \tag{73}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty}\Pi_{ik}(s)z^k=&\int_0^{\infty}e^{-st}\left[\sum_{k=1}^{\infty}P_{ik}(t)z^k\right]dt+\Pi_{i0}(s)\\ =&\ \delta_i^{(1)}\frac{[1-\phi_a(s+\lambda(1-z))]}{s+\lambda(1-z)}\\ &+[1-\delta_i^{(0)}][1-\delta_i^{(1)}]z^i\frac{[1-\phi_b(s+\lambda(1-z))]}{s+\lambda(1-z)}\\ &+\frac{[1-\phi_a(s+\lambda(1-z))]}{s+\lambda(1-z)}z\left[\int_0^{\infty}e^{-st}dN_{i1}(t)\right]\\ &+\frac{[1-\phi_b(s+\lambda(1-z))]}{s+\lambda(1-z)}\sum_{j=2}^{\infty}z^j\left[\int_0^{\infty}e^{-st}dN_{ij}(t)\right]\\ &+\frac{[s+\lambda\{1-z\phi_a(s+\lambda(1-z))\}]}{s+\lambda(1-z)}\Pi_{i0}(s)\ .\end{aligned} \tag{74}$$

On the other hand,

$$\sum_{j=1}^{\infty}z^j\left[\int_0^{\infty}e^{-st}dN_{ij}(t)\right]=\sum_{n=1}^{\infty}U_{in}(s,z)\ . \tag{75}$$

For, the left-hand side of (75) is:

$$=\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\infty}e^{-st}z^jd_tP\{\tau_n'\leqq t,\ \zeta_n=j\,|\,\zeta_0=i\}$$

$$=\sum_{n=1}^{\infty} E\{\exp(-s\tau_n')z^{\zeta_n}|\zeta_0=i\}=\sum_{n=1}^{\infty} U_{in}(s, z)\,.$$

Moreover, from (75) we have

$$\int_0^\infty e^{-st}dN_{i1}(t)=\left(\frac{\partial\left\{\sum_{n=1}^{\infty} U_n(s, z)\right\}}{\partial z}\right)_{z=0}. \tag{76}$$

From (74), (75), (76), (67) and Lemma 3 we obtain the above theorem.

References

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