ON THE DISTRIBUTION OF THE LATENT ROOTS OF A POSITIVE DEFINITE RANDOM SYMMETRIC MATRIX I*

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1. Introduction

Many problems in multivariate analysis involve the distribution problems of the latent roots of positive definite random symmetric matrices. In particular, the distributions of the latent roots of a Wishart matrix and those of a multivariate quadratic form are very fundamental in the normal multivariate case. In this paper, we shall give the density functions of the following statistics composed of the latent roots of a non-central Wishart matrix and of a non-central multivariate quadratic form.

(i) The latent roots of the determinantal equations $\det(\Sigma-\lambda XX')=0$ and $\det(\Sigma-\lambda XAX')=0$ (sections 5.1 and 7.1),

(ii) the maximum latent root (sections 5.2 and 7.2),

(iii) the traces (sections 5.3 and 7.3).

To treat the distribution problems of the latent roots of a non-central Wishart matrix, we shall introduce a generalized Hermite polynomial with a matrix argument, discuss some properties of it and give its generating function (section 3). We shall also introduce a new function which is appropriate to discuss the distribution of a non-central multivariate quadratic form (section 6).

2. Notations and preliminary results

Let S be an $m\times m$ positive definite symmetric matrix. There exists a zonal polynomial $C_\kappa(S)$ which is given by James [6] corresponding to each partition $\kappa=(k_1, \cdots, k_m)$, $k_1\geqq\cdots\geqq k_m\geqq 0$ of integer k into not more than m parts. The following integrals are used in the sequel, which are fundamental properties of the zonal polynomials:

$$(1)\qquad \int_{O(m)} C_\kappa(AHBH')\,d(H)=\frac{C_\kappa(A)\,C_\kappa(B)}{C_\kappa(I_m)}\ ,$$

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$$(2)\qquad \int_{O(n)} (\operatorname{tr} XH)^{2k}\, d(H) = \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{n}{2}\right)_\kappa} C_\kappa\left(\frac{1}{4}XX'\right),$$

where the invariant orthogonal measure $d(H)$ is normalized so as to make the volume of the orthogonal groups $O(m)$ and $O(n)$ equal to unity, A and B are $m\times m$ symmetric matrices, X is an $n\times n$ rectangular matrix and

$$(3)\qquad (a)_\kappa = \prod_{i=1}^{m}\left(a-\frac{1}{2}(i-1)\right)_{k_i}, \qquad (a)_k = a(a+1)\cdots(a+k-1).$$

Constantine [1] gave an important Γ-type integral (Laplace transform) of a zonal polynomial, i.e.,

$$(4)\qquad \int_{R>0} \operatorname{etr}(-RZ)(\det R)^{\alpha-p} C_\kappa(R)\, dR = \Gamma_m(\alpha;\kappa)(\det Z)^{-\alpha} C_\kappa(Z^{-1}),$$

where

$$\Gamma_m(\alpha;\kappa) = \pi^{m(m+1)/4}\prod_{i=1}^{m}\Gamma\left(\alpha+k_i-\frac{1}{2}(i-1)\right)$$

and

$$\operatorname{Re}(\alpha)+k_m > p-1, \qquad p=\frac{1}{2}(m+1).$$

If α is such that the gamma functions are defined, then the coefficient of binomial type is

$$(5)\qquad (\alpha)_\kappa = \Gamma_m(\alpha;\kappa)/\Gamma_m(\alpha),$$

where

$$\Gamma_m(\alpha) = \pi^{m(m+1)/4}\prod_{i=1}^{m}\Gamma\left(\alpha-\frac{1}{2}(i-1)\right).$$

Let $A_\gamma(R)$ be a Bessel function of a matrix argument, that is,

$$(6)\qquad A_\gamma(R) = \frac{1}{\Gamma_m(\gamma+p)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(-R)}{(\gamma+p)_\kappa k!} = \frac{1}{\Gamma_m(\gamma+p)}\, {}_0F_1(\gamma+p;\,-R).$$

Then the generalized Laguerre polynomial of an $m\times m$ matrix S corresponding to a partition κ of k is defined as

$$(7)\qquad \operatorname{etr}(-R)L_\kappa^\gamma(S) = \int_{R>0} A_\gamma(RS)(\det R)^\gamma \operatorname{etr}(-R) C_\kappa(R)\, dR.$$

It should be noted that (7) is the same as the γ-Hankel transform of a function $\operatorname{etr}(-R)C_k(R)$, (Herz [5]). Constantine [2] has discussed some properties of the generalized Laguerre polynomial, that is, he showed:

(i) The Laplace transform of $L_\kappa^\gamma(S)$ is

$$\int_{S>0} \operatorname{etr}(-RS)(\det S)^\gamma L_\kappa^\gamma(S)\, dS = \Gamma_m(\gamma+p;\, \kappa)(\det R)^{-\gamma-p} C_\kappa(I-R^{-1})$$

(ii) $L_\kappa^\gamma(0)=(\gamma+p)_\kappa C_\kappa(I_m)$.

(iii) $|L_\kappa^\gamma(S)| \leqq (\gamma+p)_\kappa C_\kappa(I_m)\operatorname{etr}(S)$.

(iv) $L_\kappa^\gamma(S)$'s are orthogonal polynomials with respect to the weight function $\operatorname{etr}(-S)(\det S)^\gamma$, that is,

$$\int_{S>0} \operatorname{etr}(-S)(\det S)^\gamma L_\kappa^\gamma(S) L_\tau^\gamma(S)\, dS = \delta_{kl}\delta_{\kappa\tau}\, k!\, \Gamma_m(\gamma+p;\, \kappa) C_\kappa(I_m) ,$$

where k and l are degrees of L_κ^γ and L_τ^γ, respectively.

(v) The generating function is

$$\det(I-Z)^{-\gamma-p} \int_{O(m)} \operatorname{etr}(-SHZ(I-Z)^{-1}H')\, d(H) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{L_\kappa^\gamma(S) C_\kappa(Z)}{k!\, C_\kappa(I_m)} , \quad \|Z\|<1 ,$$

where $\|Z\|$ means the maximum of the absolute values of the latent roots of Z.

3. The generalized Hermite polynomials

The distributions of the latent roots of a non-central Wishart matrix and of related statistics will be expressed as series of generalized Hermite polynomials $H_\kappa(T)$ in the elements of an $m\times n$ matrix T corresponding to the partition κ of k. $H_\kappa(T)$ is a particular form of Herz's general definition ([5] p. 503) and is given from the relation

$$(8) \qquad \operatorname{etr}(-TT')H_\kappa(T) = \frac{(-1)^k}{\pi^{mn/2}} \int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU')C_\kappa(UU')\, dU ,$$

where U and T are $m\times n$ $(m \leqq n)$ matrices and $C_\kappa(UU')$ is a zonal polynomial of degree k with a partition κ. It should be noted that (8) can be regarded as the Fourier transform of $\operatorname{etr}(-UU')C_\kappa(-UU')$. Hence we have the inverse Fourier transform

$$\frac{1}{\pi^{mn/2}} \int_T \operatorname{etr}(2iTU')\operatorname{etr}(-TT')H_\kappa(T)\, dT = \operatorname{etr}(-UU')C_\kappa(-UU') .$$

The following lemma which gives a relation between the Fourier transform and the Hankel transform is very important.

LEMMA. *Let $f(U)=f(UU')$ be a real-valued function defined for a* *positive semi-definite matrix UU' where U is an $m\times n$ $(m\leqq n)$ matrix. If $f(UU')$ is square integrable over U, that is,*

$$\int_{R>0}|f(R)|^2(\det R)^\gamma\,dR<\infty\ ,\qquad \gamma=\frac{n}{2}-p\ ,$$

then we have for the Fourier transform

$$\pi^{mn/2}g(TT')=\int_U \operatorname{etr}(-2iTU')f(UU')\,dU,$$

the γ-Hankel transform of f, i.e.

$$g(TT')=\int_{R>0}A_\gamma(TT'R)(\det R)^\gamma f(R)\,dR\ .$$

PROOF. This lemma is a special case of Herz ([5], Theorem 3.4). We here show that the Fourier transform of $f(UU')$ can be reformulated by the $\left(\frac{n}{2}-p\right)$-Hankel transform of $f(UU')$.

$$\begin{aligned} g(TT')&=\frac{1}{\pi^{mn/2}}\int_U \operatorname{etr}(-2iTU')f(UU')\,dU\\ &=\frac{1}{\pi^{mn/2}}\int_U dU\int_{O(n)}\operatorname{etr}(-2iTH'U')f(UU')\,d(H)\ . \end{aligned}$$

From (2) and (6), we have

$$\begin{aligned} g(TT')&=\frac{\Gamma_m\left(\frac{n}{2}\right)}{\pi^{mn/2}}\int_U A_{(n/2)-p}(TT'UU')f(UU')\,dU\\ &=\int_{R>0}A_{(n/2)-p}(TT'R)(\det R)^{(n/2)-p}f(R)\,dR\ . \end{aligned}$$

The second equality is shown by Hsu's lemma. Hence the R. H. S. is the $\left(\frac{n}{2}-p\right)$-Hankel transform of $f(R)$, which completes the proof.

THEOREM 1.

$$H_\kappa(T)=(-1)^k L_\kappa^{(n/2)-p}(TT')\ . \tag{10}$$

PROOF. Let $f(UU')=\operatorname{etr}(-UU')C_\kappa(-UU')$ in the lemma. Then from (7) and (8), we obtain (10) immediately.

COROLLARY 1.

$$H_\kappa(T)=H_\kappa(H_1T)=H_\kappa(TH_2)\ , \tag{11}$$

where $H_1\in O(m)$ and $H_2\in O(n)$, respectively.

PROOF. The invariance with respect to H_1 is seen from (8) by a simple calculation. The invariance with respect to H_2 is also seen from theorem 1.

COROLLARY 2.

$$H_\kappa(0)=(-1)^k\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m) . \tag{12}$$

PROOF. (12) is obvious from theorem 1 and (ii) with $\gamma=\frac{n}{2}-p$. (12) is also given by a direct calculation of (8) for $T=0$.

Remark. From corollary 2 we can consider that the g.H.p. $H_\kappa(T)$ corresponds to a generalization of a univariate Hermite polynomial of even degree.

COROLLARY 3. *The g.H.p.'s are orthogonal functions with respect to a weight function* $\operatorname{etr}(-TT')$.

$$\int_T \operatorname{etr}(-TT')H_\kappa(T)H_\tau(T)\,dT=\delta_{kl}\delta_{\kappa\tau}\pi^{mn/2}\,k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m) ,$$

where $H_\kappa(T)$ *and* $H_\tau(T)$ *are g.H.p.'s corresponding to* k *degrees and* l *degrees zonal polynomials* $C_\kappa(UU')$ *and* $C_\tau(UU')$, *respectively.*

PROOF. From theorem 1 and (iv), we get

$$\begin{aligned}
&\int_T \operatorname{etr}(-TT')H_\kappa(T)H_\tau(T)\,dT \\
&\quad=(-1)^{k+l}\int_T \operatorname{etr}(-TT')L_\kappa^{(n/2)-p}(TT')L_\tau^{(n/2)-p}(TT')\,dT \\
&\quad=\frac{\pi^{mn/2}}{\Gamma_m\left(\frac{n}{2}\right)}(-1)^{k+l}\int_{Z>0}\operatorname{etr}(-Z)(\det Z)^{(n/2)-p}L_\kappa^{(n/2)-p}(Z)L_\tau^{(n/2)-p}(Z)\,dZ \\
&\quad=\delta_{kl}\,\delta_{\kappa\tau}\,\pi^{mn/2}k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m) ,
\end{aligned}$$

where the second equality is shown to hold by Hsu's lemma.

COROLLARY 4.

$$|H_\kappa(T)|\leq\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)\operatorname{etr}(TT') . \tag{13}$$

Here we consider the generating function of the g.H.p.'s. In the univariate case the generating function is given by

$$\exp(-s^2+2ts)=\sum_{k=0}^{\infty}\frac{H_k(t)}{k!}s^k\ .$$

Herz [5] has given the generating function of the g.H.p.'s $H_\eta(T)$ by using an extension of a Hilbert-Schmidt kernel of a generalized Weber function $\operatorname{etr}\left(-\frac{1}{2}TT'\right)H_\eta(T)$. We give it in the following way.

THEOREM 2. *Let S and T be $m\times n$ $(m\leqq n)$ matrices. Then the generating function of g.H.p.'s is given by*

$$(14)\qquad \int_{O(m)}\int_{O(n)}\operatorname{etr}(-SS'+2H_1TH_2S')d(H_1)d(H_2)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(T)C_\kappa(SS')}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\ ,$$

where $H_1\in O(m)$ and $H_2\in O(n)$, respectively.

PROOF. We prove this theorem by the uniqueness of a Fourier transform. Multiply $\operatorname{etr}(+2iTM')\operatorname{etr}(-TT')$ by both sides of (14) and integrate term by term with respect to T. Using (1) and (2), the left-hand side of (14) becomes

$$\begin{aligned}\text{L.H.S.}&=\int_T\operatorname{etr}(2iTM')\operatorname{etr}(-TT')\\&\quad\times\int_{O(m)}\int_{O(n)}\operatorname{etr}(-SS'+2H_1TH_2S')\,d(H_1)\,d(H_2)\,dT\\&=\int_{O(m)}\int_{O(n)}\operatorname{etr}(-SS')\\&\quad\times\int_T\operatorname{etr}(-TT'+2T(H_1'SH_2'+iM)')\,dT\,d(H_1)\,d(H_2)\\&=\pi^{mn/2}\operatorname{etr}(-MM')\int_{O(m)}\int_{O(n)}\operatorname{etr}(2iH_1'SH_2'M')\,d(H_1)\,d(H_2)\\&=\pi^{mn/2}\operatorname{etr}(-MM')\int_{O(m)}{}_0F_1\left(\frac{n}{2};\ -H_1'SS'H_1MM'\right)d(H_1)\\&=\pi^{mn/2}\operatorname{etr}(-MM')\,{}_0F_1^{(m)}\left(\frac{n}{2};\ -SS',MM'\right)\ .\end{aligned}$$

On the other hand, the right-hand side becomes, due to (9),

$$\begin{aligned}\text{R.H.S.}&=\int_T\operatorname{etr}(2iTM')\operatorname{etr}(-TT')\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(T)C_\kappa(SS')}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\,dT\\&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(SS')}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\int_T\operatorname{etr}(2iTM')\operatorname{etr}(-TT')H_\kappa(T)\,dT\end{aligned}$$

$$=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(SS')}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\pi^{mn/2}\operatorname{etr}(-MM')C_\kappa(-MM')$$

$$=\pi^{mn/2}\operatorname{etr}(-MM')\,{}_0F_1^{(m)}\left(\frac{n}{2};\ -SS',\ MM'\right),$$

which equals the previous expression. Q. E. D.

Remark. Corollary 1 is obvious from the orthogonal invariance of the orthogonal measure and theorem 2.

COROLLARY 5.

$$\text{(15)}\quad \sum_{\kappa} H_\kappa(T)=(-1)^k\operatorname{etr}(TT')\frac{\Gamma\left(\frac{mn}{2}+k\right)}{\Gamma\left(\frac{mn}{2}\right)}\,{}_1F_1\left(\frac{mn}{2}+k;\ \frac{mn}{2};\ -\operatorname{tr}TT'\right).$$

PROOF. To prove (15), we need the following equality which was given by Khatri [10],

$$\text{(16)}\quad \int_{R>0}\operatorname{etr}(-R)(\det R)^{\alpha-p}(\operatorname{tr}R)^kC_\tau(SR)\,dR$$

$$=\Gamma_m(\alpha;\ \kappa)\frac{\Gamma(m\alpha+k+l)}{\Gamma(m\alpha+l)}C_\tau(S)$$

where $C_\tau(S)$ is a zonal polynomial of l degrees with partition τ. Now, from the definition of g.H.p.'s,

$$\sum_{\kappa} H_\kappa(T)=\frac{(-1)^k}{\pi^{mn/2}}\operatorname{etr}(TT')\int_U\operatorname{etr}(-2iTU')\operatorname{etr}(-UU')\sum_{\kappa}C_\kappa(UU')\,dU$$

$$=\frac{(-1)^k}{\pi^{mn/2}}\operatorname{etr}(TT')$$

$$\times\int_U\int_{O(n)}\operatorname{etr}(-2iTH'U')\operatorname{etr}(-UU')(\operatorname{tr}UU')^k\,d(H)\,dU$$

$$=\frac{(-1)^k}{\pi^{mn/2}}\operatorname{etr}(TT')\int_U{}_0F_1\left(\frac{n}{2};\ -TT'UU'\right)\operatorname{etr}(-UU')(\operatorname{tr}UU')^k\,dU,$$

which Hsu's lemma shows to be equal to

$$=\frac{(-1)^k}{\Gamma_m\left(\frac{n}{2}\right)}\operatorname{etr}(TT')$$

$$\times\int_{R>0}{}_0F_1\left(\frac{n}{2};\ -TT'R\right)(\det R)^{(n/2)-p}\operatorname{etr}(-R)(\operatorname{tr}R)^k\,dR$$

$$=\frac{(-1)^k}{\Gamma_m\left(\frac{n}{2}\right)}\operatorname{etr}(TT')\sum_{l=0}^{\infty}\sum_{\tau}\frac{(-1)^l}{l!\left(\frac{n}{2}\right)_\tau}$$

$$\times\int_{R>0}\operatorname{etr}(-R)(\det R)^{(n/2)-p}(\operatorname{tr} R)^k C_\tau(TT'R)\,dR\,,$$

which is, in turn, seen from (16) to be equal to

$$=(-1)^k\operatorname{etr}(TT')\sum_{l=0}^{\infty}\sum_{\tau}\frac{(-1)^l}{l!}\frac{\Gamma\left(\frac{mn}{2}+k+l\right)}{\Gamma\left(\frac{mn}{2}+l\right)}C_\tau(TT')$$

$$=(-1)^k\operatorname{etr}(TT')\frac{\Gamma\left(\frac{mn}{2}+k\right)}{\Gamma\left(\frac{mn}{2}\right)}{}_1F_1\left(\frac{mn}{2}+k;\ \frac{mn}{2};\ -\operatorname{tr} TT'\right)$$

where ${}_1F_1$ is a univariate confluent hypergeometric function. If we set $T=0$ in (15), from (12) and ${}_1F_1=1$, we obtain

$$\sum_{\kappa}\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)=\left(\frac{mn}{2}\right)_k\,. \tag{17}$$

COROLLARY 6.

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(T)}{k!}=\frac{1}{2^{mn/2}}\operatorname{etr}\left(\frac{1}{2}TT'\right) \tag{18}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(T)}{k!\left(\frac{n}{2}\right)_\kappa}=\operatorname{etr}(-I_m)\,{}_0F_1\left(\frac{n}{2};\ TT'\right)\,. \tag{19}$$

PROOF. (18) can be shown from the definition of a generalized Hermite polynomials. (19) can be shown when we set $SS'=I_m$ in (14).

We give a relation between the generating function of g.H.p.'s and one of the generalized Laguerre polynomials. To give this relation, we use the following form which is given when we exchange S to TT', Z to $-Z$ and $\gamma=\frac{n}{2}-p$ in (v).

$$\begin{aligned}\det(I+Z)^{-n/2}&\int_{O(m)}\operatorname{etr}(TT'H_1Z(I+Z)^{-1}H_1)\,d(H_1)\\&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^kL_\kappa^{(n/2)-p}(TT')C_\kappa(Z)}{k!\,C_\kappa(I_m)}\,,\qquad \|Z\|<1\,.\end{aligned} \tag{20}$$

COROLLARY 7. *The relation between the generating function of the*

Laguerre polynomials and one of the g.H.p.'s is given as follows. If we multiply $\pi^{-mn/2}(\det Z)^{-n/2}\operatorname{etr}(-SS'Z^{-1})$ *by both sides of* (14), *and integrate it with respect to S, then we obtain* (20).

PROOF. Multiply $\pi^{-mn/2}(\det Z)^{-n/2}\operatorname{etr}(-SS'Z^{-1})$ by both sides of (14) and integrate it with respect to S,

$$\begin{aligned}\text{L.H.S.}&=\pi^{-mn/2}(\det Z)^{-n/2}\int_S \operatorname{etr}(-SS'Z^{-1})\\&\quad\times\int_{O(m)}\int_{O(n)}\operatorname{etr}(-SS'+2H_1SH_2T')\,dS\,d(H_1)\,d(H_2)\\&=(\det Z)^{-n/2}\det(I+Z^{-1})^{-n/2}\\&\quad\times\int_{O(m)}\int_{O(n)}\operatorname{etr}(TT'H_1(I+Z^{-1})^{-1}H_1')\,d(H_1)\,d(H_2)\\&=\det(I+Z)^{-n/2}\int_{O(m)}\operatorname{etr}(TT'H_1Z(I+Z)^{-1}H_1')\,d(H_1)\,.\end{aligned}$$

On the other hand, from (4) and Hsu's lemma, we get

$$\begin{aligned}\text{R.H.S.}&=\pi^{-mn/2}(\det Z)^{-n/2}\int_S\operatorname{etr}(-SS'Z^{-1})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(T)C_\kappa(SS')}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\,dS\\&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(T)C_\kappa(Z)}{k!\,C_\kappa(I_m)}\\&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^kL_\kappa^{(n/2)-p}(TT')C_\kappa(Z)}{k!\,C_\kappa(I_m)}\,,\end{aligned}$$

which completes our assertion.

4. The Jacobian of the maximum latent root and integrals

In this section we give the useful transform and related Beta type integrals.

LEMMA. *Let S be an* $m\times m$ *positive definite random symmetric matrix. If we decompose S, so that*

$$S=H\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}H'\,,$$

where H is an $m\times m$ *orthogonal matrix which has only* $(m-1)$ *independent variables* $h_{21}, h_{31},\cdots,h_{m1}$ *and V is an* $(m-1)\times(m-1)$ *positive definite random symmetric matrix in the range* $\lambda_1 I_{m-1}>V>0$. *Then the Jacobian of this transform is given by*

$$J(S\rightarrow\lambda_1, H, V)=\det(\lambda_1 I-V)\frac{1}{\sqrt{1-\sum_{i=2}^{m}h_{i1}^2}}\ .$$

PROOF. See Hayakawa [4].

The following corollary is very important to give the density function of the maximum latent root.

COROLLARY 8.

$$\text{(21)}\qquad \int_{I_{m-1}>W>0}(\det W)^{\alpha-p}\det(I-W)\,C_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix}dW$$

$$=\frac{\Gamma_m(\alpha)\Gamma_m(p)}{\Gamma_m(\alpha+p)}\frac{\Gamma\left(\frac{m}{2}\right)}{\pi^{m/2}}(\alpha m+k)\frac{(\alpha)_\kappa}{(\alpha+p)_\kappa}C_\kappa(I_m)\ ,$$

$$\text{(22)}\qquad \int_{1>w_2>\cdots>w_m>0}\left(\prod_{i=2}^{m}w_i\right)^{\alpha-p}\prod_{i=2}^{m}(1-w_i)\prod_{i<j}(w_i-w_j)\,C_\kappa\begin{pmatrix}1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_m\end{pmatrix}\prod_{i=2}^{m}dw_i$$

$$=\frac{\Gamma_m(\alpha)\Gamma_m(p)}{\Gamma_m(\alpha+p)}\frac{\Gamma_m\left(\frac{m}{2}\right)}{\pi^{m^2/2}}(\alpha m+k)\frac{(\alpha)_\kappa}{(\alpha+p)_\kappa}C_\kappa(I_m)\ .$$

PROOF. It is well-known that the following integral holds,

$$\text{(23)}\qquad \int_{I>S>0}(\det S)^{\alpha-p}C_\kappa(S)\,dS=\frac{\Gamma_m(\alpha)\Gamma_m(p)}{\Gamma_m(\alpha+p)}\frac{(\alpha)_\kappa}{(\alpha+p)_\kappa}C_\kappa(I_m)\ .$$

To prove (21), we decompose S so that

$$S=H\begin{bmatrix}\lambda_1 & \\ & \lambda_1 W\end{bmatrix}H'$$

where W is an $(m-1)\times(m-1)$ positive definite symmetric matrix in the range $I_{m-1}>W>0$. Then the Jacobian is given by

$$J(S\rightarrow\lambda_1, H, W)=\lambda_1^{(m-1)(m+2)/2}\det(I_{m-1}-W)\frac{1}{\sqrt{1-\sum_{i=2}^{m}h_{i1}^2}}\ .$$

Hence the left-hand side of (23) becomes

$$\text{L.H.S.}=\int_0^1 \lambda_1^{\alpha m+k-1}d\lambda_1 \int_{I_{m-1}>W>0} (\det W)^{\alpha-p}\det(I-W)\,C_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix}dW$$

$$\times \int_{\sum_{i=2}^m h_{i1}^2\leq 1} \frac{dh_{21}\cdots dh_{m1}}{\sqrt{1-\sum_{i=2}^m h_{i1}^2}}$$

$$=\frac{1}{\alpha m+k}\frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}\int_{I_{m-1}>W>0}(\det W)^{\alpha-p}\det(I-W)\,C_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix}dW,$$

since

$$\int_0^1 \lambda_1^{\alpha m+k-1}d\lambda_1=\frac{1}{\alpha m+k} \quad \text{and} \quad \int_{\sum_{i=2}^m h_{i1}^2\leq 1}\frac{\prod_{i=2}^m dh_{i1}}{\sqrt{1-\sum_{i=2}^m h_{i1}^2}}=\frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)},$$

therefore, we have (21).

To prove (22), we further decompose W so that $W=H_1\Lambda_\omega H_1'$ where $H_1\in O(m-1)$ and $\Lambda_\omega=\operatorname{diag}(w_2,\cdots,w_m)$. Then the Jacobian is

$$J(W\to\Lambda_\omega, H_1)=\prod_{2\leq i<j\leq m}(w_i-w_j).$$

Hence, inserting these results into (21), we obtain (22), since

$$\int_{O(m-1)}d(H_1)=\frac{\pi^{(m-1)^2/2}}{\Gamma_{m-1}\left(\frac{m-1}{2}\right)}.$$

(22) is the same result as given in Sugiyama [9].

5. Non-central Wishart distribution

5.1 The probability density function of the latent roots of a non-central Wishart matrix with a known covariance

In this section we consider the p.d.f.'s of the latent roots, of the maximum latent root, and of the trace of a non-central Wishart matrix with known covariance matrix Σ. The joint p.d.f. of the latent roots of a non-central Wishart matrix was given by James [7] as series of zonal polynomials.

$$\text{(24)}\qquad \frac{\pi^{m^2/2}}{2^{mn/2}\Gamma_m\left(\frac{m}{2}\right)\Gamma_m\left(\frac{n}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\operatorname{etr}\left(-\frac{1}{2}\Lambda\right)(\det\Lambda)^{(n/2)-p}$$

$$\times \prod_{i<j}(\lambda_i-\lambda_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\frac{1}{2}\Lambda\right)C_\kappa\left(\frac{1}{2}\Sigma^{-1}MM'\right)}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\ .$$

However, (24) is not an appropriate form to discuss the p.d.f.'s of related statistics, particularly the p.d.f. of the maximum latent root λ_1 when we use (22). We shall give another form of the p.d.f. of Λ.

THEOREM 3. *Let X be distributed with p.d.f.,*

$$\frac{1}{\pi^{mn/2}(\det 2\Sigma)^{n/2}}\operatorname{etr}\left[-\frac{1}{2}\Sigma^{-1}(X-M)(X-M)'\right]\ . \tag{25}$$

Then the joint p.d.f. of the latent roots Λ of $\Sigma^{-1/2}XX'\Sigma^{-1/2}$ is given by

$$\frac{\pi^{m^2/2}}{2^{mn/2}\Gamma_m\left(\frac{n}{2}\right)\Gamma_m\left(\frac{m}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)(\det\Lambda)^{(n/2)-p}\prod_{i<j}(\lambda_i-\lambda_j) \tag{26}$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M\right)C_\kappa\left(\frac{1}{2}\Lambda\right)}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\ .$$

PROOF. We decompose $Y=\Sigma^{-1/2}X$ as follows:

$$Y=H_1\Lambda^{1/2}L' \tag{27}$$

where H_1 is an orthogonal matrix with positive elements in the first column and $\Lambda^{1/2}$ is a diagonal matrix of square roots of the latent roots $\lambda_1,\cdots,\lambda_m$ of $\Sigma^{-1/2}XX'\Sigma^{-1/2}=YY'$ and L is an $n\times m$ Stiefel matrix satisfying $L'L=I_m$. The Jacobian of this transform is

$$dY=\frac{2^m\pi^{m(m+n)/2}}{\Gamma_m\left(\frac{n}{2}\right)\Gamma_m\left(\frac{m}{2}\right)}(\det\Lambda)^{(n/2)-p}\prod_{i<j}(\lambda_i-\lambda_j)\,d\Lambda\, d(H_1)\, d(L)\ , \tag{28}$$

where $d(L)$ is a normalized Stiefel invariant measure with total volume unity. Thus, inserting (27) and (28) into (25), we obtain the joint p.d.f. of Λ, H_1 and L,

$$\frac{2^m\pi^{m^2/2}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)}{2^{mn/2}\Gamma_m\left(\frac{n}{2}\right)\Gamma_m\left(\frac{m}{2}\right)}(\det\Lambda)^{(n/2)-p}\prod_{i<j}(\lambda_i-\lambda_j) \tag{29}$$

$$\times\operatorname{etr}\left(-\frac{1}{2}\Lambda^{1/2}L'L\Lambda^{1/2}+H_1\Lambda^{1/2}L'M'\Sigma^{-1/2}\right)\ .$$

We integrate (29) with respect to H_1 and L. If we set $L \to H_2'L$, $H_2 \in O(n)$, then $L'H_2H_2'L = L'L = I_m$ and the Stiefel invariant measure $d(L)$ remains unchanged. Thus

$$\frac{1}{2^m}\int_{O(m)}\int_{L'L=I_m} \operatorname{etr}\left(-\frac{1}{2}\Lambda^{1/2}L'L\Lambda^{1/2} + H_1\Lambda^{1/2}L'M'\Sigma^{-1/2}\right) d(H_1)\,d(L)$$

$$= \frac{1}{2^m}\int_{L'L=I_m} d(L)\int_{O(m)}\int_{O(n)} \operatorname{etr}\left(-\frac{1}{2}\Lambda^{1/2}L'H_2H_2'L\Lambda^{1/2} + H_1\Lambda^{1/2}L'H_2M'\Sigma^{-1/2}\right) d(H_1)\,d(H_2)\,.$$

Hence we can see that the integral with respect to H_1 and H_2 is the same form as given in theorem 2 if we set $S = \frac{1}{\sqrt{2}}\Lambda^{1/2}L'$ and $T = \frac{1}{\sqrt{2}}\Sigma^{-1/2}M$ in (14), thus we have

$$\int_{L'L=I_m} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{H_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M\right) C_\kappa\left(\frac{1}{2}\Lambda\right)}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\, d(L)$$

$$= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{H_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M\right) C_\kappa\left(\frac{1}{2}\Lambda\right)}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}$$

which completes the proof.

COROLLARY 9. *From* (12) *and* (26) *with* $M=0$, *we obtain the p.d.f. of the latent roots of a central Wishart matrix.*

5.2 The p.d.f. of the maximum latent root of the non-central Wishart matrix

Recently Hayakawa [4] and Sugiyama [9], [10] have considered the p.d.f.'s of the maximum latent roots of some positive definite random symmetric matrices. Hayakawa [4] has obtained the p.d.f. of the maximum latent root of a non-central Wishart matrix $\Sigma = I_m$ by using two expansions of zonal polynomials such that

$$C_\kappa(A+B) = \sum_{\sigma,\tau} a_{\sigma\tau}^{\kappa} C_\sigma(A) C_\tau(B)$$

and

$$C_\kappa(A) C_\tau(A) = \sum_{\sigma} b_{\kappa\tau}^{\sigma} C_\sigma(A)\,.$$

However, the explicit formulas for the $a^{\kappa}_{\sigma\tau}$ and $b^{\sigma}_{\kappa\tau}$ are not known. In this section we shall obtain the p.d.f. of the maximum latent root of a non-central Wishart matrix with known covariance without using $a^{\kappa}_{\sigma\tau}$'s and $b^{\sigma}_{\kappa\tau}$'s.

THEOREM 4. *The p.d.f. of the maximum latent root λ_1 of a non-central Wishart matrix of n degrees of freedom with known covariance is given by*

$$\frac{\Gamma_m(p)}{2^{mn/2}\Gamma\left(\frac{n}{2}+p\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\lambda_1^{mn/2-1} \tag{30}$$

$$\times\sum_{k=0}^{\infty}\frac{\left(\frac{mn}{2}+k\right)}{k!}\left(\frac{\lambda_1}{2}\right)^k\sum_{\kappa}\frac{H_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M\right)}{\left(\frac{n}{2}+p\right)_\kappa},$$

(30) *converges for $\lambda_1>0$.*

PROOF. Since the p.d.f. of the latent roots Λ is given by (26), if we set $\lambda_i=w_i\lambda_1$ $(i=2,\cdots,m)$ in (26) and integrate it with respect to $1>w_2>\cdots>w_m>0$, using (22), then we obtain (30). The convergence of the series is seen as follows. From (14), we get

$$\left|H_\kappa\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right)\right|\leqq\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m),$$

thus

$$\sum_{k=0}^{\infty}\frac{\left(\frac{mn}{2}+k\right)}{k!}\left(\frac{\lambda_1}{2}\right)^k\sum_{\kappa}\frac{H_\kappa\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right)}{\left(\frac{n}{2}+p\right)_\kappa}$$

$$\leqq\sum_{k=0}^{\infty}\frac{\left(\frac{mn}{2}+p\right)}{k!}\left(\frac{\lambda_1}{2}\right)^k\sum_{\kappa}\frac{\left(\frac{n}{2}\right)_\kappa}{\left(\frac{n}{2}+p\right)_\kappa}C_\kappa(I_m)\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)$$

and since $\left(\frac{n}{2}\right)_\kappa\Big/\left(\frac{n}{2}+p\right)_\kappa\leqq1$ for all κ, the right-hand side is dominated termwise by the series

$$\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{k=0}^{\infty}\frac{\left(\frac{mn}{2}+p\right)}{k!}\left(\frac{\lambda_1}{2}\right)^k\sum_{\kappa}C_\kappa(I_m)$$

$$=\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)\left[\frac{mn}{2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{m\lambda_1}{2}\right)^k+\frac{m\lambda_1}{2}\sum_{k=0}^{\infty}\frac{1}{(k-1)!}\left(\frac{m\lambda_1}{2}\right)^{k-1}\right]$$

$$=\frac{m}{2}e^{m\lambda_1/2}\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)(n+\lambda_1)\,.$$

Hence the series converges for all $\lambda_1>0$, which completes the proof.

COROLLARY 10. *The c.d.f. of the maximum latent root λ_1 of the non-central Wishart matrix with known covariance is given by*

$$P\{\lambda_1<x\}=\frac{\Gamma_m(p)}{2^{mn/2}\Gamma_m\left(\frac{n}{2}+p\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)x^{mn/2}\times\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{2}\right)^k\sum_{\kappa}\frac{H_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M\right)}{\left(\frac{n}{2}+p\right)_\kappa}\,. \tag{31}$$

COROLLARY 11. *From* (12) *and* (30) *with $M=0$, we obtain the p.d.f. of the maximum latent root λ_1 of a central Wishart matrix, which is the same form as given in Sugiyama* [9].

5.3 The p.d.f. of the trace of a non-central Wishart matrix with known covariance

THEOREM 5. *Let Λ be distributed with p.d.f.* (24). *Then the p.d.f. of $T=\operatorname{tr}\Lambda=\operatorname{tr}XX'$ is given by*

$$\frac{1}{2^{mn/2}\Gamma\left(\frac{mn}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)T^{mn/2-1}\times\sum_{k=0}^{\infty}\frac{1}{k!\left(\frac{mn}{2}\right)_\kappa}\left(\frac{T}{2}\right)^k\sum_{\kappa}H_\kappa\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right)\,. \tag{32}$$

(32) *converges for $T>0$.*

PROOF. We derive (32) from the inverse formula of the Laplace transform $g(t)=E_\Lambda(\operatorname{etr}(-t\Lambda))$.

$$g(t)=\frac{\pi^{m^2/2}}{2^{mn/2}\Gamma_m\left(\frac{n}{2}\right)\Gamma_m\left(\frac{m}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right)\left(\frac{1}{2}\right)^k}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}$$

$$\times\int_{\Lambda>0}\operatorname{etr}(-t\Lambda)(\det\Lambda)^{(n/2)-p}C_\kappa(\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)\,d\Lambda$$

$$=\frac{1}{2^{mn/2}}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}\right)^{k}t^{-mn/2-k}\sum_{\kappa}H\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right).$$

Applying (13) for $H_\kappa\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right)$, we see that the series is dominated termwise by the series

$$\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\left(\frac{n}{2}\right)_{\kappa}C_\kappa\left(\frac{t^{-1}}{2}I_m\right)$$
$$=\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)\det\left(I-\frac{t^{-1}}{2}I\right)^{-n/2}.$$

Hence, when $R(t)=C$ sufficiently large, the series can be integrated term-by-term with respect to t, the same being true for $\det\left(I_m-\frac{t^{-1}}{2}I_m\right)^{-n/2}$. Since

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}e^{tT}\,t^{(-mn/2)-k}\,dt=\frac{T^{(mn/2)+k-1}}{\Gamma\left(\frac{mn}{2}+k\right)},$$

as the p.d.f. of T we have

$$\frac{1}{2^{mn/2}\Gamma\left(\frac{mn}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)T^{mn/2-1}$$
$$\times\sum_{k=0}^{\infty}\frac{1}{k!\left(\frac{mn}{2}\right)_{\kappa}}\left(\frac{T}{2}\right)^{k}\sum_{\kappa}H_\kappa\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right).$$

The convergence of this series is seen as follows. Applying (13) for $H_\kappa\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right)$, we have a majorant series

$$\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)\sum_{k=0}^{\infty}\frac{1}{k!\left(\frac{mn}{2}\right)_{k}}\left(\frac{T}{2}\right)^{k}\sum_{\kappa}\left(\frac{n}{2}\right)_{\kappa}C_\kappa(I_m)$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{T}{2}\right)^{k}\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right)$$
$$=e^{T/2}\operatorname{etr}\left(\frac{1}{2}\Sigma^{-1}MM'\right),$$

where the first equality is given by (16). Hence the series (32) converges for $T>0$, which completes the proof.

COROLLARY 12. *The c.d.f. of the trace of a non-central Wishart matrix with known covariance is given by*

$$(33)\qquad P\{T<x\}=\frac{1}{2^{mn/2}\Gamma\left(\frac{mn}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)x^{mn/2}$$
$$\times\sum_{k=0}^{\infty}\frac{1}{k!\left(\frac{mn}{2}\right)_{k+1}}\left(\frac{x}{2}\right)^{k}\sum_{\kappa}H_{\kappa}\left(\frac{\Sigma^{-1/2}M}{\sqrt{2}}\right).$$

COROLLARY 13. *From* (17) *and* (32) *with* $M=0$, *we obtain the p.d.f. of* χ^2 *with mn degrees of freedom.*

6. $P_\kappa(T, A)$

In this section we define a new function $P_\kappa(T, A)$ which is convenient to discuss the p.d.f. of the latent roots of a non-central multivariate quadratic form. We define $P_\kappa(T, A)$ as follows:

$$(34)\qquad \operatorname{etr}(-TT')P_\kappa(T,A)$$
$$=\frac{(-1)^k}{\pi^{mn/2}}\int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU')C_\kappa(UAU')\,dU,$$

where U and T are $m\times n$ matrices and A is an $n\times n$ positive definite symmetric matrix. The polynomial $P_\kappa(T, A)$ has the following properties.

THEOREM 6.

$$(35)\qquad P_\kappa(T, I_n)=H_\kappa(T),$$

$$(36)\qquad P_\kappa(0, A)=(-1)^k C_\kappa(A)C_\kappa(I_m)/C_\kappa(I_n).$$

$$(37)\qquad |P_\kappa(T, A)|\leqq \operatorname{etr}(TT')\left(\frac{n}{2}\right)_\kappa C_\kappa(A)C_\kappa(I_m)/C_\kappa(I_n).$$

PROOF. From the definition (34), we can easily show these relations.

COROLLARY 14.

$$(38)\qquad \int_{O(n)}P_\kappa(TH, A)\,d(H)=\int_{O(n)}P_\kappa(T, HAH')\,d(H)=\frac{C_\kappa(A)}{C_\kappa(I_n)}H_\kappa(T).$$

PROOF. From (1) and (2), (38) is obtained by a simple calculation.

THEOREM 7. *The generating function of* $P_\kappa(T, A)$*'s is*

$$(39)\qquad \int_{O(m)}\int_{O(n)}\operatorname{etr}(-SH_2AH_2S'+2H_1SH_2A^{1/2}T')\,d(H_1)\,d(H_2)$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{P_\kappa(T,A)\,C_\kappa(SS')}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)},$$

where $H_1 \in O(m)$ *and* $H_2 \in O(n)$, *respectively. The series converges absolutely.*

PROOF. We prove (39) by a direct method. By inserting (34) into the right-hand side of (39), we have

$$\text{R.H.S.} = \frac{1}{\pi^{mn/2}} \operatorname{etr}(TT') \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k C_\kappa(SS')}{k! \left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}$$

$$\times \int_U \operatorname{etr}(-2iTU') \operatorname{etr}(-UU') C_\kappa(UAU') dU$$

$$= \frac{1}{\pi^{mn/2}} \operatorname{etr}(TT') \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k}{\left(\frac{n}{2}\right)_\kappa k!} \int_U \int_{O(m)} \operatorname{etr}(-2iTU') \operatorname{etr}(-UU')$$

$$\times C_\kappa(UAU'H_1SS'H_1') d(H_1) dU$$

$$= \frac{1}{\pi^{mn/2}} \operatorname{etr}(TT') \int_U \int_{O(m)} \int_{O(n)} \operatorname{etr}(-2iTU') \operatorname{etr}(-UU')$$

$$\times \operatorname{etr}(2iH_1SH_2A^{1/2}U') dU d(H_1) d(H_2)$$

$$= \int_{O(m)} \int_{O(n)} \operatorname{etr}(-SH_2AH_2'S' + 2H_1SH_2A^{1/2}T') d(H_1) d(H_2) .$$

The absolute convergence is proved by the same way as in the proof of theorem 5.

7. Non-central multivariate quadratic form

Recently, Hayakawa [3] and Khatri [8] have discussed the p.d.f. of a multivariate quadratic form in the central mean case by using a zonal polynomial expansion and have treated properties of some related statistics. In this section we consider the p.d.f. of the latent roots of a non-central multivariate quadratic form with known covariance, of the maximum latent root and of the trace of it, which may be used by the power function for the statistical criterion of the spread of the population.

7.1 The p.d.f. of the latent roots of the non-central multivariate quadratic form

THEOREM 8. *Let* X *be distributed with p.d.f.* (25). *Then the joint p.d.f. of the latent roots* Λ *of* XAX' *where* A *is an* $n \times n$ *positive definite symmetric matrix is given by*

$$\text{(40)}\quad \frac{1}{2^{mn/2}\Gamma_m\left(\frac{n}{2}\right)\Gamma_m\left(\frac{m}{2}\right)(\det A)^{m/2}}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)$$

$$\times(\det \Lambda)^{(n/2)-p}\prod_{i<j}(\lambda_i-\lambda_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{P_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M,\ A^{-1}\right)C_\kappa\left(\frac{1}{2}\Lambda\right)}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\ .$$

PROOF. If we set $XA^{1/2}=Y$ in (25), then $dX=(\det A)^{-m/2}dY$. Hence the p.d.f. of Y is given by

$$\text{(41)}\quad \frac{\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)}{(2\pi)^{mn/2}(\det\Sigma)^{n/2}(\det A)^{m/2}}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}YA^{-1}Y'+\Sigma^{-1}YA^{-1}M'\right)$$

and $XAX'=YY'$. Therefore, the latent roots of $\Sigma^{-1/2}XAX'\Sigma^{-1/2}$ are the same as those of $\Sigma^{-1/2}YY'\Sigma^{-1/2}$. If we use the decomposition (27), i.e., $W=H_1\Lambda^{1/2}L'=\Sigma^{-1/2}Y$, then the joint p.d.f. of Λ, H_1, and L becomes as follows,

$$\frac{2^m\pi^{m^2/2}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)}{2^{mn/2}\Gamma_m\left(\frac{n}{2}\right)\Gamma_m\left(\frac{m}{2}\right)(\det A)^{m/2}}(\det\Lambda)^{(n/2)-p}\prod_{i<j}(\lambda_i-\lambda_j)$$

$$\times\operatorname{etr}\left(-\frac{1}{2}\Lambda^{1/2}L'A^{-1}L\Lambda^{1/2}+H_1\Lambda^{1/2}L'A^{-1/2}M'\Sigma^{-1/2}\right).$$

If we set $L\to H_2'L$, $H_2\in O(n)$, then $L'H_2H_2'L=L'L=I_m$ and $d(L)$ remains invariant with respect to H_2. Thus the integral with respect to $O(m)$ and $O(n)$ is the same form as given in theorem 7 with $S=\frac{1}{\sqrt{2}}\Lambda^{1/2}L'$ and $T=\frac{1}{\sqrt{2}}\Sigma^{-1/2}M$. Hence

$$\frac{1}{2^m}\int_{L'L=I_m}\int_{O(m)}\int_{O(n)}\operatorname{etr}\left(-\frac{1}{2}\Lambda^{1/2}L'H_2A^{-1}H_2'L\Lambda^{1/2}+H_1\Lambda^{1/2}L'H_2A^{-1/2}M'\Sigma^{-1/2}\right)$$

$$\times d(H_1)\,d(H_2)\,d(L)$$

$$=\int_{L'L=I_m}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{P_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M,\ A^{-1}\right)C_\kappa\left(\frac{1}{2}\Lambda\right)}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}d(L)$$

$$=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{P_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M,\ A\right)C_\kappa\left(\frac{1}{2}\Lambda\right)}{k!\left(\frac{n}{2}\right)_\kappa C_\kappa(I_m)}\ ,$$

which completes the proof.

COROLLARY 15. *If we set* $A=I_n$, *then from* (35), *we obtain* (26).

COROLLARY 16. *If we set* $M=0$, *then we obtain the p.d.f. of the latent roots of the central multivariate quadratic form* XAX', *which is the same form as given in* Hayakawa ([3], p. 196, (25)).

7.2 The p.d.f. of the maximum latent root of XAX'

THEOREM 10. *Let* Λ *be distributed with p.d.f.* (40). *Then the p.d.f. of the maximum latent root* λ_1 *of* Λ *is given by*

$$(42)\qquad \frac{\Gamma_m(p)}{2^{mn/2}\Gamma_m\left(\frac{n}{2}+p\right)(\det A)^{m/2}}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)\lambda_1^{mn/2-1}$$
$$\times\sum_{k=0}^{\infty}\frac{\left(\frac{mn}{2}+k\right)}{k!}\left(\frac{\lambda_1}{2}\right)^k\sum_{\kappa}\frac{P_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M,A^{-1}\right)}{\left(\frac{n}{2}+p\right)_\kappa}.$$

PROOF. The proof is done in the same way as that of theorem 4.

COROLLARY 17. *The c.d.f. of* λ_1 *is given by*

$$(43)\qquad P\{\lambda_1<x\}=\frac{\Gamma_m\left(\frac{p}{2}\right)}{2^{mn/2}\Gamma_m\left(\frac{n}{2}+p\right)(\det A)^{m/2}}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)x^{mn/2}$$
$$\times\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{2}\right)^k\sum_{\kappa}\frac{P_\kappa\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M,A^{-1}\right)}{\left(\frac{n}{2}+p\right)_\kappa}.$$

7.3 The p.d.f. of the trace of XAX'

As it is well-known that the trace of a matrix is invariant for the orthogonal transformation, we have

$$T=\operatorname{tr}XAX'=\operatorname{tr}\Lambda .$$

THEOREM 10. *Let* Λ *be distributed with p.d.f.* (40). *Then the p.d.f. of* $T=\operatorname{tr}\Lambda$ *is givin by*

$$(44)\qquad \frac{\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)}{\Gamma\left(\frac{mn}{2}\right)(\det 2A)^{m/2}} T^{mn/2-1} \times \sum_{k=0}^{\infty} \frac{1}{k!\left(\frac{mn}{2}\right)_k}\left(\frac{T}{2}\right)^k \sum_{\kappa} P_{\kappa}\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M, A^{-1}\right).$$

PROOF. The proof is done in the same way as that of theorem 5.

COROLLARY 18. *The c.d.f. of T is given by*

$$(45)\qquad P\{T<x\} = \frac{\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}MM'\right)}{\Gamma\left(\frac{mn}{2}\right)(\det A)^{m/2}} T^{mn/2} \times \sum_{k=0}^{\infty} \frac{1}{k!\left(\frac{mn}{2}\right)_{k+1}}\left(\frac{T}{2}\right)^k \sum_{\kappa} P_{\kappa}\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M, A^{-1}\right).$$

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