ON THE USE OF NON-GAUSSIAN PROCESS IN THE IDENTIFICATION OF A LINEAR DYNAMIC SYSTEM

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1. Introduction and summary

When we apply the cross-spectral technique to estimation of the frequency response function of a system operating under a stationary input, we encounter two types of significant difficulties. One arises in the case where a non-negligible error is introduced into the measurement of input and the other arises in the case where some sort of feedback loop exists which connects the output to the input. If we apply the ordinary method of estimation of frequency response functions (confer, for example, [1], [3]) to these cases, there may appear a significant bias which entirely invalidates the results of our analysis. There is no way out of these difficulties when we limit ourselves to second order statistics, such as auto-covariances and cross-covariances, and this means that so long as the processes under observation are all Gaussian we can not completely get rid of these difficulties. Thus we can infer that the ways out of these difficulties will lie only in the use of non Gaussian properties of the related quantities or in the use of higher order moment functions and spectra.

In this paper we shall show that under certain conditions, which are not very much unrealistic, of the related quantities we can really get a way out of these difficulties by using the "mixed spectra", of which a general use was recently proposed by the present author [2]. The results of the discussions suggest a wide range of applicability of higher order moments (or semi-invariants) and spectra (or spectral semi-invariants) in the analysis of linear systems, and this shows that the use of higher order moments is not necessarily limited to the analysis of non-linearities. It may also be of interest to note that the problems treated in this paper are in close relationships with identification problems treated extensively in relation to econometric models.

2. Measurement error of input

In this section we treat the case with measurement error at the input. Consider the system illustrated in Fig. 1.

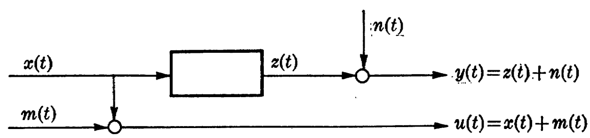

Fig. 1

Concerning the quantities of Fig. 1 we assume:

$x(t)$: a stationary non Gaussian process with power spectral density function $p_{xx}(f)$ and with a bounded third order moment function,

$m(t)$: a stationary Gaussian process with power spectral density function $p_{mm}(f)$, independent of $x(t)$ and $n(t)$,

$n(t)$: a stationary process with power spectral density function $p_{nn}(f)$, independent of $x(t)$ and $m(t)$.

Here the stationarity is required only of the moments up to a necessary order. We adopt the above assumptions for the sake of simplicity of the explanation and, as will be seen from the following discussions, various modifications of these assumptions may be possible. It should be noted that all the processes under consideration are assumed to have power spectral density functions and thus have zero means. We shall hereafter adopt the notation $p_{\eta\xi}(f)$ to represent the cross- (or auto- when $\xi=\eta$) spectral density function between the stationarily correlated stationary processes $\xi(t)$ and $\eta(t)$. We further assume that the operation of the system represented by a square in Fig. 1 is given by

$$z(t)=\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

where $h(\tau)$ is the impulsive response function of the system which is supposed to belong to L_1 and L_2 and its Fourier transform $A(f)=\int_{-\infty}^{\infty} \exp(-i2\pi f\tau)h(\tau)\,d\tau$ gives the frequency response function $A(f)$ of the system. We consider the case where observations are available only of $u(t)$ and $y(t)$ which are the input and output of the system contaminated by the noises $m(t)$ and $n(t)$ respectively. Taking into account the relations

$$p_{yu}(f)=A(f)p_{xx}(f)\ ,$$

$$p_{uu}(f)=p_{xx}(f)+p_{mm}(f)\ ,$$

we can see that $p_{yu}(f)/p_{uu}(f)$ is not necessarily equal to $A(f)$ and the ordinary method of estimation of the frequency response function by using the ratio of the sample cross-spectrum of the input and output to the sample power spectrum of the input will not necessarily give a consistent estimate of $A(f)$ in this case.

Now, let us assume that $m_{xxx}(\tau, \sigma)=\mathrm{E}x(t+\tau)x(t+\sigma)x(t)$ $(-\infty<\tau<\infty)$ is not identically vanishing for some σ $(-\infty<\sigma<\infty)$. Throughout the present paper we shall adopt the notation $m_{\xi\eta\zeta}(\tau, \sigma)$ in place of $\mathrm{E}\xi(t+\tau)\eta(t+\sigma)\zeta(t)$ when the latter is stationary or independent of t. Then, taking into account the relation

$$m_{zxx}(\tau, \sigma)=\int_{-\infty}^{\infty} h(\rho)m_{xxx}(\tau-\rho, \sigma)\, d\rho,$$

we get

$$A(f)=\frac{B_{zxx}(f;\sigma)}{B_{xxx}(f;\sigma)}$$

where

$$B_{zxx}(f;\sigma)=\int_{-\infty}^{\infty} \exp(-i2\pi f\tau)m_{zxx}(\tau, \sigma)\, d\tau$$

and

$$B_{xxx}(f;\sigma)=\int_{-\infty}^{\infty} \exp(-i2\pi f\tau)m_{xxx}(\tau, \sigma)\, d\tau$$

are "mixed spectra" introduced in the previous paper [2] and it is assumed that $B_{xxx}(f;\sigma)\neq 0$. Under the present assumptions, it can be shown that

$$\begin{aligned} m_{uuu}(\tau, \sigma)&=\mathrm{E}u(t+\tau)u(t+\sigma)u(t)\\ &=m_{xxx}(\tau, \sigma)\\ m_{yuu}(\tau, \sigma)&=\mathrm{E}y(t+\tau)u(t+\sigma)u(t)\\ &=m_{zxx}(\tau, \sigma)\end{aligned}$$

and we get

$$A(f)=\frac{B_{yuu}(f;\sigma)}{B_{uuu}(f;\sigma)}\ .$$

We can see that so long as a pair of consistent estimates of $B_{uuu}(f;\sigma)$ and $B_{yuu}(f;\sigma)$ are available we can get a consistent estimate of $A(f)$.

3. Existence of feedback

Here we shall treat the case where some feedback exists from the output to the input. We consider the problem in the most general situation shown in Fig. 2 and we try to estimate the response characteristics of the component A. Obviously, this situation contains that of the preceding section as a special case where the gain of B is zero.

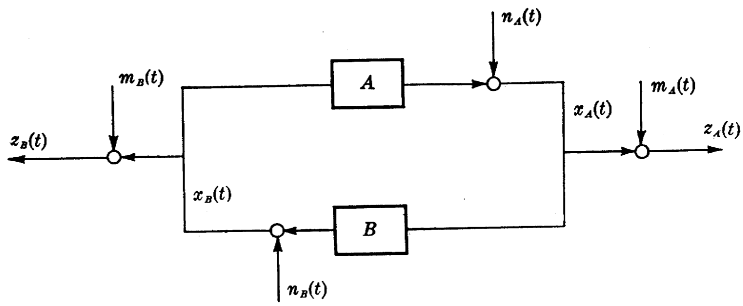

Fig. 2

It is intended that $n_A(t)$ and $n_B(t)$ are to designate the internal noises of the components A and B, respectively, and $m_A(t)$ and $m_B(t)$ are to represent the measurement errors of the input and the output. It is assumed that $n_A(t)$, $n_B(t)$, $m_A(t)$ and $m_B(t)$ are mutually independent. For the sake of simplicity we shall assume that all the quantities under consideration have power spectral densities and they form a multiple stationary process, i.e., stationarily correlated. We adopt the convention of preceding section for the representation of power and cross spectral density functions. We further assume that the components A and B are linear and time-invariant with impulsive response functions $h_A(\tau)$ and $h_B(\tau)$, which belong to L_1 and at the same time to L_2, and have the corresponding frequency response functions $A(f)$ and $B(f)$. The following relations hold between the related quantities:

$$\text{(A)} \qquad x_A(t)=\int_{-\infty}^{\infty} x_B(t-\tau)h_A(\tau)\,d\tau+n_A(t)$$

$$\text{(B)} \qquad x_B(t)=\int_{-\infty}^{\infty} x_A(t-\tau)h_B(\tau)\,d\tau+n_B(t)$$

$$z_A(t)=x_A(t)+m_A(t)$$

$$z_B(t)=x_B(t)+m_B(t)\ .$$

It is assumed that only $z_B(t)$ and $z_A(t)$ are observable and our aim is to get an estimate of $A(f)$. From the above relations we get

$$\mathrm{E}x_A(t+\tau)x_B(t)=\int_{-\infty}^{\infty}\mathrm{E}x_B(t+\tau-\sigma)x_B(t)h_B(\sigma)\,d\sigma+\mathrm{E}n_A(t+\tau)x_B(t)$$

or equivalently

$$R(\tau\,;\,x_A,\,x_B)=\int_{-\infty}^{\infty}h_B(\sigma)R(\tau-\sigma\,;\,x_B,\,x_B)\,d\sigma+R(\tau\,;\,n_A,\,x_B)$$

where

$$R(\tau\,;\,\xi,\,\eta)=\mathrm{E}\xi(t+\tau)\eta(t)\ .$$

Unfortunately, in the present situation $R(\tau\,;\,n_A,\,x_B)$ is not identically equal to zero and $p_{x_A x_B}(f)/p_{x_B x_B}(f)$ is not necessarily equal to $A(f)$. Thus even when the observations of $x_B(t)$ and $x_A(t)$ are available we cannot generally expect to get an unbiased estimate of $A(f)$ by applying the ordinary cross-spectral method to this situation. Thus, so long as the related quantities are all Gaussian, we cannot generally get a good estimate of $A(f)$. In contrast to this, in some practical situations, it is often possible to expect that the artificially constructed controller B generates an output which is not Gaussian, while other noises are nearly Gaussian, and we shall here consider this case, i.e., the case where $m_A(t)$, $m_B(t)$ and $n_A(t)$ are Gaussian and $n_B(t)$ is not Gaussian. To be more specific, we assume that $n_B(t)$ has a bounded and stationary third order moment function which is not identically vanishing, i.e., we assume

$$\mathrm{E}n_B(t+\tau)n_B(t+\sigma)n_B(t)=m_{n_B n_B n_B}(\tau,\,\sigma)\not\equiv 0\ .$$

From the relations (A) and (B) it holds that

$$x_A(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_A(t-\tau-\sigma)h_B(\sigma)\,d\sigma\,h_A(\tau)\,d\tau+\int_{-\infty}^{\infty}n_B(t-\tau)h_A(\tau)\,d\tau+n_A(t)$$

$$x_B(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_B(t-\tau-\sigma)h_A(\sigma)\,d\sigma\,h_B(\tau)\,d\tau+\int_{-\infty}^{\infty}n_A(t-\tau)h_B(\tau)\,d\tau+n_B(t)$$

and we get the relations (with probability one)

(C) $$d_A(t)=n_A(t)+\int_{-\infty}^{\infty}n_B(t-\tau)h_A(\tau)\,d\tau$$

(D) $$d_B(t)=\int_{-\infty}^{\infty}n_A(t-\tau)h_B(\tau)\,d\tau+n_B(t)$$

where $d_A(t)$ and $d_B(t)$ are by definition

$$d_A(t)=x_A(t)-\int_{-\infty}^{\infty}x_A(t-\rho)h_A*h_B(\rho)\,d\rho$$

$$d_B(t)=x_B(t)-\int_{-\infty}^{\infty}x_B(t-\rho)h_A*h_B(\rho)\,d\rho$$

and

$$h_A * h_B(\rho)=\int_{-\infty}^{\infty} h_A(\rho-\sigma)h_B(\sigma)\,d\sigma \ .$$

From the present equation (C) we get, assuming the existence of necessary moment functions,

$$\begin{aligned} m_{d_A x_B x_B}(\tau,\sigma) &= \mathrm{E}\, d_A(t+\tau)x_B(t+\sigma)x_B(t) \\ &= m_{n_A x_B x_B}(\tau,\sigma)+\int_{-\infty}^{\infty} m_{n_B x_B x_B}(\tau-\rho,\sigma)h_A(\rho)\,d\rho \ . \end{aligned}$$

Taking into account that $x_B(t)$ is composed of linear transformations of $n_B(t)$ and $n_A(t)$, the latter being Gaussian, we get

$$m_{n_A x_B x_B}(\tau,\sigma)=0 \ ,$$

and accordingly

$$\text{(E)} \qquad m_{d_A x_B x_B}(\tau,\sigma)=\int_{-\infty}^{\infty} m_{n_B x_B x_B}(\tau-\rho,\sigma)h_A(\rho)\,d\rho \ .$$

On the other hand, from the definition of $d_A(t)$ we get

$$\text{(F)} \qquad m_{d_A x_B x_B}(\tau,\sigma)=m_{x_A x_B x_B}(\tau,\sigma)-\int_{-\infty}^{\infty} m_{x_A x_B x_B}(\tau-\rho,\sigma)h_A * h_B(\rho)\,d\rho \ .$$

From these relations we have

$$\begin{aligned} B_{d_A x_B x_B}(f;\sigma) &= \int_{-\infty}^{\infty} \exp(-i2\pi f\tau)m_{d_A x_B x_B}(\tau,\sigma)\,d\tau \\ &= A(f)B_{n_B x_B x_B}(f;\sigma) \\ &= (1-A(f)B(f))B_{x_A x_B x_B}(f;\sigma) \ , \end{aligned}$$

and

$$B_{x_A x_B x_B}(f;\sigma)=A(f)\frac{1}{1-A(f)B(f)}B_{n_B x_B x_B}(f;\sigma) \ .$$

By putting $d_B(t)$ in place of $d_A(t)$ in the above derivation we get another relation

$$B_{x_A x_B x_B}(f;\sigma)=\frac{1}{1-A(f)B(f)}B_{n_B x_B x_B}(f;\sigma) \ .$$

From these two equations it follows that

$$\frac{B_{x_A x_B x_B}(f;\sigma)}{B_{x_B x_B x_B}(f;\sigma)}=A(f) \ .$$

This last relation shows that by taking the ratio of the mixed spectra

$B_{x_A x_B x_B}(f;\sigma)$ to $B_{x_B x_B x_B}(f;\sigma)$ we can get the desired frequency response function $A(f)$ of the component A. Further, under the present assumption of normality of $m_A(t)$ and $m_B(t)$ we can get the relations

$$m_{z_B z_B z_B}(\tau,\sigma)=m_{x_B x_B x_B}(\tau,\sigma)$$

$$m_{z_A z_B z_B}(\tau,\sigma)=m_{x_A x_B x_B}(\tau,\sigma)$$

and we get finally

$$\frac{B_{z_A z_B z_B}(f;\sigma)}{B_{z_B z_B z_B}(f;\sigma)}=A(f)\ .$$

This is the desired result in our general situation, and we can see that if at least a pair of consistent estimates of the mixed spectra is available, we can get a consistent estimate of $A(f)$. In the present derivation of formulae, the crucial point was the assumption of non normality of $n_B(t)$ and normality of $n_A(t)$, $m_A(t)$ and $m_B(t)$. It will almost be obvious that the assumption of normality may be replaced by the assumption of the vanishing third order moment in this case.

It should be noticed that the statistical methods adopted for the estimation of frequency response functions such as the ordinary cross-spectral method and the method of this paper are essentially methods of projection of the input and the output into the space of signals where they become free of the disturbances which deteriorate the input-output relation of the system. $u(t+\sigma)u(t)$ of the preceding section and $z_B(t+\sigma)z_B(t)$ in this section are used for the purpose of construction of this type of spaces and take the role of the noise-free input used in the ordinary cross-spectral method of estimation of the frequency response function of an open loop system. It will be necessary to select as a signal such as $u(t+\sigma)u(t)$ or $z_B(t+\sigma)z_B(t)$ the one which will show high coherencies with the observed input and the output to get a reliable estimate, and this will be fairly difficult in many practical applications.

We can see that the notion of "mixed spectra" is conveniently used for the analysis of a linear system operating under a non Gaussian input. It may be inferred that we can use it for the analysis of non-linear systems by extending the dimension of the frequency space as an occasion demands. In that case, spectra should be understood in the sense of spectral semi-invariants.

4. Relation to econometric identification problems

There are a great many papers concerning the identification problems in econometrics (confer, for example, the references of [4]). We can see that by taking the Fourier transforms (properly defined) of related quantities we can transform the problem treated in this paper just into the form treated in econometric papers. From the historical review of the paper by Reiersøl ([4], p. 378), we can see that our method adopted in this paper corresponds to the method of "instrumental variables" introduced by Reiersøl and Geary. Further, we can see ([4], p. 378) that semi-invariants were used for the identification problems by Geary. This corresponds to the use of spectral semi-invariants in our problem. From the point of view of this correspondence, the problem of a measurement error treated in the former section of this paper falls into a category of the problem treated by these authors, but the case with feedback does not seem to have its equivalence.

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