ON BOUMAN-VELDEN-YAMAMOTO'S ASYMPTOTIC EVALUATION FORMULA FOR THE PROBABILITY OF VISUAL RESPONSE IN A CERTAIN EXPERIMENTAL RESEARCH IN QUANTUM-BIOPHYSICS OF VISION

SADAO IKEDA

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1. Introduction and summary

To determine the threshold number of light-quanta in human vision is an important problem in quantum-biophysics of vision, and it has been studied by some biophysicians basing upon a certain biological experimentation [1], [2], [3].

In the theory of quantum-biophysics of vision, it is assumed that when a light-stimulus is given to human eye, the effective absorptions of light-quanta occur independently from one another with the same probability for each, and if at least k quanta cumulate their energies at some time-point during the stimulation, then they elicit a visual sensation of human eye. Value of k, the minimum number of quanta necessary to cause a visual sensation, is called the threshold number of light-quanta in human vision.

In an experimental situation, with which we shall be concerned in this article, the light-stimulus is of constant intensity with average number μ of light-quanta which are absorbed effectively in the unit time. In such a situation, it is adequate to assume that absorptions of light-quanta occur according to a Poisson process with parameter μ [3].

It can also be assumed that the life-time τ of light-quanta has a certain life-distribution of the discrete or of the continuous type, and for each quantum which is absorbed effectively in the stimulus, its life-time is a random realization of τ.

Under such an experimental situation, let $W_k(\mu, t)$ be the probability of visual response, i.e., the probability that the subjective recognizes the flash, in the duration t of the light-stimulus. This quantity is fundamental to the analysis of data for estimating the value of k in the experimental research under consideration. Theoretical derivation of the exact form for evaluating $W_k(\mu, t)$ is difficult, and an asymptotic evaluation formula under a certain limiting is considered: Bouman and Van der Velden [1] anticipated the proportionality

$$W_k(\mu, t) \propto \alpha^{k-1}\mu^k t \,, \text{ as } t \gg 1 \text{ and } \mu \ll 1 \,, \tag{1.1}$$

where α designates the mean value of τ. Recently, Yamamoto [3] proposed without any rigorous proof, an improved evaluation formula such as

$$W_k(\mu, t) \sim 1 - \exp\left[-\frac{\alpha^{k-1}\mu^k t}{(k-1)!}\right], \text{ as } (t\to\infty)_k \,, \tag{1.2}$$

where $(t\to\infty)_k$ designates a limiting process of parameters μ and t such that $t\to\infty$ and $\mu\to 0$ under the restriction $\mu^k t\to\lambda$ for some positive constant λ. We shall call the formula (1.2) the Bouman-Velden-Yamamoto asymptotic evaluation formula as in the title of this paper.

Later, Yamamoto [4] proved the validity of (1.2) under the assumption that τ has an exponential distribution with mean α, and the present author [5], [6] showed the validity of (1.2) in the case when τ has the unit distribution with the whole mass at $\tau=\alpha$. The question, however, whether the asymptotic evaluation formula (1.2) is true or not for other distributions of τ, has been left open.

Isii [7] treated the problem for general distribution of τ, and proved the validity of (1.2) assuming that τ has moment of a certain order greater than unity.

This article treats the same problem in an elementary way and gives a complete answer to the above question: *If only a finite mean value α of τ exists, then the asymptotic evaluation formula (1.2) holds true.*

In the following section, notations and preliminary results are stated. In section 3 it is shown that the number of visual responses in the duration t of stimulus has a limiting Poisson distribution under some mild restrictions imposed on the distribution of τ, and that the above stated result can be derived as a corollary, in the case when k is greater than unity. In the case when $k=1$, the validity of (1.2) is easily confirmed, and it will be omitted in the present discussion.

2. Notations and preliminaries

Since the absorptions of light-quanta occur according to a Poisson process with parameter μ, it is assumed, in the discussion of the present paper, that (i) the number S_t of quanta which are absorbed in any time-interval with length t is distributed according to a Poisson distribution with mean μt, and (ii) numbers of quanta which are absorbed in mutually exclusive intervals are mutually independently distributed.

Let (a, b) be time-interval with length t, then, by assumption (i) given above, the number S_t of light-quanta which are absorbed in this interval has probabilities

$$(2.1) \qquad P(S_t=s)=e^{-\mu t}\frac{(\mu t)^s}{s\,!}, \quad s=0, 1, 2, \cdots.$$

Let $T_1^s, T_2^s, \cdots, T_s^s$ be time-points at which s quanta are absorbed successively under the condition that $S_t=s$, and put

$$(2.2) \qquad U_0^s=T_1^s-a, \; U_1^s=T_2^s-T_1^s, \; \cdots, \; U_s^s=b-T_s^s,$$

which are time-intervals between successive absorptions of quanta. Clearly these $(s+1)$ conditional variables given $S_t=s$ are subject to the restriction such that $\sum_{i=0}^s U_i^s=t$, and the conditional joint probability density function of these variables is given by

$$(2.3) \qquad p_t(u_1, u_2, \cdots, u_s\,|\,s)=\frac{s\,!}{t^s}, \quad (0<u_i;\; \sum_{i=1}^s u_i<t),$$

for any given s, $s=1, 2, \cdots$, and, in particular, for $s=0$, U_0^0 is distributed according to the unit distribution with mass-point t.

From (2.3) it is easily observed that any subset of size $n(\leqq s)$ of these variables $\{U_i^s\}(i=0, 1, 2, \cdots, s)$ has the conditional joint probability density function given $S_t=s$ such as

$$(2.4) \qquad p_t(u_1, \cdots, u_n\,|\,s)=\frac{s(s-1)\cdots(s-n+1)}{t^n}\left(1-\frac{1}{t}\sum_{i=1}^n u_i\right)^{s-n}, \quad (0<u_i;\; \sum_{i=1}^n u_i<t),$$

and therefore, $(1/t)\sum_{m=1}^n U_{i_m}^s$ $(=V_n^s$, say) is distributed as a beta-distribution $B(n, s-n+1)$, whose probability density function is

$$(2.5) \qquad p_t(v\,|\,s)=\frac{\Gamma(s+1)}{\Gamma(n)\Gamma(s-n+1)}v^{n-1}(1-v)^{s-n}, \quad (0<v<1).$$

Next, let (a, b) and (c, d) be two mutually exclusive time-intervals with length t and h respectively, and let S_t and S_h be numbers of quanta which are absorbed in these intervals. Then, by assumption (ii), S_t and S_h are mutually independent in the stochastic sense, and, since the time-intervals between successive absorptions are distributed depending only on the number of quanta which are absorbed in each interval, the conditional variables $\{U_i^s\}(i=0, 1, \cdots, s)$ given $S_t=s$ and $\{U_j'^{s'}\}(j=0, 1, \cdots, s')$ given $S_h=s'$ are mutually independently distributed under the condition that $S_t=s$ and $S_h=s'$.

Now, let $F(\tau)$ be the cumulative distribution function of the life-distribution of τ, then, it is obvious that

$$(2.6)\qquad \int_0^\infty (1-F(u))du=\int_0^\infty u\, dF(u)(=\alpha, \text{ say}) ,$$

provided that either of these integrals exists. It is also easily seen that, if the mean value α exists, then it holds that

$$(2.7)\qquad \int_0^\infty\int_0^\infty\cdots\int_0^\infty \prod_{i=1}^n (1-F(u_1+u_2+\cdots+u_i))du_1du_2\cdots du_n=\frac{\alpha^n}{n\,!} ,$$

for every positive integer n.

3. Limiting distribution of the number of visual responses in the case when $k\geqq 2$

Let $[0, t)$ be a time-interval during the stimulation, where 0 is the starting point of the stimulus. As was stated in the first section, a visual response occurs if at least k quanta cumulate their energies at some time-point during the stimulation, and hence, it occurs just after the absorption of some light-quantum. Successively to the occurrence of a visual response, there may be a time-interval during which the visual sensation continues, i.e., at least k quanta survive simultaneously in that whole interval, and, any light-quantum which is absorbed in such an interval can not contribute to the occurrence of a new visual response. The number of times of visual responses thus counted in the time-interval $[0, t)$ will be called simply " the number of visual responses in the interval $[0, t)$ ".

Now, let us define the following events:

(3.1) The symbol $\{X_j=1\}$ designates the event that a visual response occurs just after the jth absorption of light-quantum, while $\{X_j=0\}$ designates the complementary event of $\{X_j=1\}$,

and we put symbolically

$$(3.2)\qquad N_i=\sum_{j=0}^{i} X_j\ ,\ i=0, 1, 2, \cdots .$$

That is to say, $\{N_i=n\}$ means exactly n visual responses occur before the $(i+1)$st absorption of quantum. In these definitions, it is evident that both of the events $\{X_0=1\}$ and $\{N_0=1\}$ are empty, or, more precisely, $\{X_j=1\}(0\leqq j\leqq k-1)$ and $\{N_i\geqq 1\}(0\leqq i\leqq k-1)$ are all empty.

The probability that the number of visual responses in $[0, t)$ is equal to n, n being any non-negative integer, is given by

$$(3.3) \qquad P_n(\mu,\, t)=\sum_{s\geqq 0} P(S_t=s)P(N_s=n \mid S_t=s)\,,$$

for which it is clear that

$$(3.4) \qquad P_n(\mu,\, 0)=\delta_{0,n}, \qquad \delta_{i,j} \text{ being the Kronecker delta.}$$

Now, let S_{t+h} be the number of light-quanta which are absorbed in the interval $[0,\, t+h)$. Then, from (3.3) we have

$$(3.5) \qquad P_n(\mu,\, t+h)=\sum_{s\geqq 0} P(S_{t+h}=s)P(N_s=n \mid S_{t+h}=s)\,,$$

for every non-negative integer n. Dividing the interval $[0,\, t+h)$ into two sub-intervals $[0,\, t)$ and $[t,\, t+h)$, and letting S_t and S_h be numbers of light-quanta which are absorbed in respective sub-intervals, one can readily obtain

$$(3.6) \quad P_n(\mu,\, t+h)=\sum_{s\geqq 0}\sum_{s'\geqq 0} P(S_t=s)P(S_h=s')P(N_{s+s'}=n \mid S_t=s,\, S_h=s')\,.$$

Since, for small h, $P(S_h=0)=1-\mu h+o(h)$, $P(S_h=1)=\mu h+o(h)$ and $P(S_h\geqq 2)=o(h)$, it follows from (3.6) that

$$(3.7) \qquad P_n(\mu,\, t+h)=(1-\mu h)\sum_{s\geqq 0} P(S_t=s)P(N_s=n \mid S_t=s,\, S_h=0)$$
$$+\mu h\sum_{s\geqq 0} P(S_t=s)P(N_{s+1}=n \mid S_t=s,\, S_h=1)+o(h)\,.$$

Here, we note that (a) under the condition $S_t=s$ and $S_h=0$, the event $\{N_s=n\}$ is dependent only on time-intervals $\{U_i^s\}(i=1,\, 2,\, \cdots,\, s-1)$ and life-times $\{\tau_i\}(i=1,\, 2,\, \cdots,\, s-1)$, τ_i being life-time of the ith quantum absorbed, and hence, this event is independent of the condition $S_h=0$, and (b) the same is seen for the condition $S_h=1$. Using (a), it readily follows that

$$\sum_{s\geqq 0} P(S_t=s)P(N_s=n \mid S_t=s,\, S_h=0)=P_n(\mu,\, t)\,.$$

Since the event $\{N_{s+1}=n\}$ is, under the condition $S_t=s$ and $S_h=1$, the union of two events, $\{N_s=n-1$ and $X_{s+1}=1\}$ and $\{N_s=n$ and $X_{s+1}=0\}$, which are mutually exclusive, and the latter is the complementary event of $\{N_s=n$ and $X_{s+1}=1\}$ with respect to $\{N_s=n\}$, it easily follows from (b) that

$$\sum_{s\geqq 0} P(S_t=s)P(N_{s+1}=n \mid S_t=s,\, S_h=1)$$
$$=\sum_{s\geqq 0} P(S_t=s)P(N_s=n-1,\, X_{s+1}=1 \mid S_t=s,\, S_h=1)+P_n(\mu,\, t)$$
$$-\sum_{s\geqq 0} P(N_s=n)P(N_s=n,\, X_{s+1}=1 \mid S_t=s,\, S_h=1)\,,$$

for every non-negative integer n, where, for $n=0$, $\{N_s=-1\}$ defines the empty event for every s.

Hence, (3.7) turns out to be

$$
\begin{aligned}
(3.8)\quad & P_n(\mu, t+h) \\
&= P_n(\mu, t)+\mu h\sum_{s\geqq 0} P(S_t=s)P(N_s=n-1,\ X_{s+1}=1 \mid S_t=s,\ S_h=1) \\
&\quad -\mu h\sum_{s\geqq 0} P(S_t=s)P(N_s=n,\ X_{s+1}=1 \mid S_t=s,\ S_h=1)+o(h)\,,
\end{aligned}
$$

for every non-negative integer n.

In order to delete the condition $S_h=1$ in the second and third members on the right-hand side of (3.8), let us investigate the event $\{X_{s+1}=1\}$. Clearly, under the condition $S_t=s$ and $S_h=1$, this event depends upon the random variables, $\tau_1, \cdots, \tau_s$, $U_1^s, \cdots, U_s^s$ and U'_{s+1}, where, as before, τ_i designates the life-time of the ith quantum absorbed, U_i^s's are time-intervals between successive absorptions in the interval $[0, t)$, while U'_{s+1} stands for the time-interval between t and time-point of the $(s+1)$st absorptions. Let us define events such as

$$E(i, j)=\{\tau_i> U_i^s+ \cdots +U_j^s\}\,,\quad (i\leqq j\,;\ i, j=1, \cdots, s-1)\,,$$

and

$$E'(l, s)=\{\tau_l> U_l^s+ \cdots +U_s^s+U'_{s+1}\}\,,\quad (l=1, \cdots, s)\,.$$

Then, it is clear that the event $\{X_{s+1}=1\}$ can be expressed as a function of these events with the operations, union, intersection and complementation, which is written as

$$(3.9)\quad R'_s(t+h)=\varphi(E(i, j),\ E'(l, s);\ i\leqq j=1, \cdots, s-1;\ l=1, \cdots, s)\,,$$

where $R'_s(t+h)$ stands for the event $\{X_{s+1}=1\}$.

Corresponding to the definition of $E'(l, s)$ given above, let us consider the events

$$E(l, s)=\{\tau_l> U_l^s+ \cdots +U_s^s\}\,,\quad (l=1, \cdots, s)\,,$$

and, by exchanging $E'(l, s)$ on the right-hand side of (3.9) for $E(l, s)$, put

$$(3.9)'\quad R_s(t)=\varphi(E(i, j),\ E(l, s);\ i\leqq j=1, \cdots, s-1;\ l=1, \cdots, s)\,.$$

Then, it is observed that

$$(3.10)\qquad \Delta(R'_s(t+h),\ R_s(t))\subset \bigcup_{l=1}^{s} (E(l, s)-E'(l, s))\,,$$

where $\Delta(A, B)$ designates the difference between events A and B. Since $E(l, s)-E'(l, s)$ is included by the event $\{\tau_l-h\leqq U_l^s+ \cdots +U_s^s<\tau_l\}$ for each s, $l=1, 2, \cdots, s$, it follows from (3.10), by using (2.5), that

$$P(\Delta(R'_s(t+h),\ R_s(t))\,|\,S_t=s)\leqq\sum_{l=1}^{s}P(\tau_l-h\leqq U_l^s+\cdots+U_s^s<\tau_l\,|\,S_t=s)$$

$$=\sum_{l=1}^{s}\int_0^{t+h}dF(\tau)\int_{(\tau-h)/t}^{\tau/t}\frac{\Gamma(s+1)}{\Gamma(s-l+1)\Gamma(l)}v^{s-l}(1-v)^{l-1}dv$$

$$=s\int_0^{t+h}dF(\tau)\int_{(\tau-h)/t}^{\tau/t}dv\leqq\frac{h}{t}s\,,$$

from which we get

$$\sum_{s\geqq 0}P(S_t=s)P(\Delta(R'_s(t+h),\ R_s(t))\,|\,S_t=s)\leqq\mu h,\ \text{for small } h. \tag{3.11}$$

Hence, replacing the event $\{X_{s+1}=1\}$ on the right-hand side of (3.8) by $R_s(t)$, one can easily obtain

$$\begin{aligned} P_n(\mu,\ t+h) &= P_n(\mu,\ t)+\mu h\sum_{s\geqq 0}P(S_t=s)P(N_s=n-1,\ R_s(t)\,|\,S_t=s)\\ &\quad -\mu h\sum_{s\geqq 0}P(S_t=s)P(N_s=n,\ R_s(t)\,|\,S_t=s)+o(h)\,, \end{aligned} \tag{3.12}$$

for small $h(>0)$ and for every non-negative integer n, where we have dropped the condition $S_h=1$ because the events $\{N_s=n\}$ and $R_s(t)$ are independent of that condition. Thus, one can state the following

LEMMA 3.1. *The probability $P_n(\mu, t)$ satisfies the differential equation*

$$\begin{aligned} P'_n(\mu,\ t) &= \mu\sum_{s\geqq 0}P(S_t=s)P(N_s=n-1,\ R_s(t)\,|\,S_t=s)\\ &\quad -\mu\sum_{s\geqq 0}P(S_t=s)P(N_s=n,\ R_s(t)\,|\,S_t=s)\,, \end{aligned} \tag{3.13}$$

for every non-negative integer n, where the left-hand member designates the derivative of $P_n(\mu, t)$ with respect to t.

Now, let us define as

$$A_s(t)=\bigcap_{l=s-k+2}^{s}E(l,\ s)\ \text{and}\ B_s(t)=\bigcup_{\substack{(i_1,\ \cdots,\ i_{k-1})\\ \neq(s-k+2,\ \cdots,\ s)}}\ \bigcap_{j=1}^{k-1}E(i_j,\ s)\,,$$

where $E(l, s)$'s are the same as before, and the union in the definition of $B_s(t)$ is taken over all choices of $k-1$ integers, $\{i_1<\cdots<i_{k-1}\}$, out of s integers, $\{1, 2, \cdots, s\}$, excluding one, $\{s-k+2, \cdots, s\}$. Here, $A_s(t)$ and $B_s(t)$ are empty if $s\leqq k-2$ and $s\leqq k-1$ respectively. Then, it can easily be seen that

$$(3.14)\qquad A_s(t)\cap\overline{B_s(t)}\subset R_s(t)\subset A_s(t)\cup B_s(t)\,,$$

where $\overline{B_s(t)}$ stands for the complementary event of $B_s(t)$.

Replacing $R_s(t)$ on the right-hand side of (3.13) by $A_s(t)$, we get the equation

$$(3.15)\qquad \begin{aligned} P'_n(\mu,\, t)=\mu\sum_{s\geqq 0}P(S_t=s)P(N_s=n-1,\, A_s(t)\,|\,S_t=s)\\ -\mu\sum_{s\geqq 0}P(S_t=s)P(N_s=n,\, A_s(t)\,|\,S_t=s)+\gamma_{1n}(\mu,\, t)\,,\end{aligned}$$

where

$$(3.16)\qquad |\gamma_{1n}(\mu,\, t)\,|\leqq 2\mu\sum_{s\geqq 0}P(S_t=s)P(B_s(t)\,|\,S_t=s)\,.$$

We shall evaluate the right-hand side of this inequality: Since

$$P(B_s(t)\,|\,S_t=s)\leqq \sum_{\substack{(i_1,\,\cdots,\, i_{k-1})\\ \neq(s-k+2,\,\cdots,\, s)}} P(\bigcap_{j=1}^{k-1}E(i_j,\, s)\,|\,S_t=s)$$

$$=\sum_{\substack{(i_1,\,\cdots,\, i_{k-1})\\ \neq(s-k+2,\,\cdots,\, s)}}\iint\cdots\int\prod_{j=1}^{k-1}(1-F(u_{i_j}+\,\cdots\,+u_s))p_t(u_{i_1},\,\cdots,\, u_s\,|\,s)du_{i_1}\cdots du_s$$

$$=\sum_{i=k}^{s}\sum_{\substack{j_1+\cdots+j_{k-1}=i\\ j_m\geqq 1}}\iint_{0<\sum_{l=1}^{i}u_l<t}\cdots\int\prod_{m=1}^{k-1}(1-F(u_1+\,\cdots\,+u_{j_1+\cdots+j_m}))$$

$$\cdot\frac{s(s-1)\cdots(s-i+1)}{t^i}\left(1-\frac{1}{t}\sum_{l=1}^{i}u_l\right)^{s-i}du_1du_2\cdots du_i$$

$$\leqq\sum_{i=k}^{s}\iint_{0<\sum_{m=1}^{k-1}v_m<t}\cdots\int\prod_{m=1}^{k-1}(1-F(v_m))\sum_{\substack{j_1+\cdots+j_{k-1}=i\\ j_m\geqq 1}}\prod_{m=1}^{k-1}\frac{v_m^{j_m-1}}{(j_m-1)\,!}\cdot\frac{s(s-1)\cdots(s-i+1)}{t^i}$$

$$\cdot\left(1-\frac{1}{t}\sum_{m=1}^{k-1}v_m\right)^{s-i}dv_1dv_2\cdots dv_{k-1}$$

$$=\frac{s(s-1)\cdots(s-k+2)}{t^{k-1}}\iint_{0<v<t}\cdots\int\prod_{m=1}^{k-1}(1-F(v_m))\sum_{i=1}^{s-k+1}\binom{s-k+1}{i}\left(\frac{v}{t}\right)^i$$

$$\cdot\left(1-\frac{v}{t}\right)^{s-k+1-i}dv_1\cdots dv_{k-1}\quad\left(v=\sum_{m=1}^{k-1}v_m\right)$$

$$=\frac{s(s-1)\cdots(s-k+2)}{t^{k-1}}\iint_{0<v<t}\cdots\int\prod_{m=1}^{k-1}(1-F(v_m))\left[1-\left(1-\frac{v}{t}\right)^{s-k+1}\right]$$

$$\cdot\, dv_1\cdots dv_{k-1}\,,$$

we have

$$\sum_{s\geq 0} P(S_t=s)P(B_s(t)\,|\,S_t=s)$$

$$\leqq \mu^{k-1}\iint_{0<v<\infty}\cdots\int \prod_{m=1}^{k-1}(1-F(v_m))[1-e^{-\mu v}]dv_1\cdots dv_{k-1}\,.$$

Thus, by (3.16), we get inequality

$$|\gamma_{1n}(\mu,\,t)|\leqq\zeta_1(\mu),\qquad \text{uniformly in } t(>0) \text{ and } n(\geqq 0)\,, \tag{3.17}$$

for every fixed $\mu(>0)$, where

$$\zeta_1(\mu)=2\mu^k\iint_{\substack{0<v_m<\infty\\ m=1,\,\cdots,\,k-1}}\cdots\int\prod_{m=1}^{k-1}(1-F(v_m))\cdot[1-e^{-\mu\sum_{m=1}^{k-1}v_m}]dv_1\cdots dv_{k-1}\,. \tag{3.18}$$

Next, we shall evaluate the first two members of the right-hand side of (3.15).

Let δ be a fixed positive constant less than unity, and put $b=\mu^{-(1+\delta)}$ and $a=t-b$ when $t>b$.

In the first place, let us consider the case when $t\leqq b$. In this case, it is seen that

$$\sum_{n=1}^{\infty}\sum_{s\geq 0}P(S_t=s)P(N_s=n,\,A_s(t)\,|\,S_t=s)$$

$$\leqq\sum_{s\geq k}P(S_t=s)P(\bigcup_{i=1}^{s-k+1}\{\tau_i>U_i^s\}\,,\;A_s(t)\,|\,S_t=s)$$

$$\leqq\sum_{s\geq k}P(S_t=s)(s-k+1)\iint_{0<\sum_{i=1}^{k}u_i<t}\cdots\int(1-F(u_k))\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))$$

$$\cdot\,\frac{s(s-1)\cdots(s-k+1)}{t^k}\left(1-\frac{1}{t}\sum_{i=1}^{k}u_i\right)^{s-k}du_1\cdots du_k$$

$$\leqq\iint_{0<v<t}\cdots\int(1-F(u_k))\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))$$

$$\cdot\left[\mu^{k+1}t\left(1-\frac{v}{t}\right)+\mu^k\right]e^{-\mu v}\,du_1\cdots du_k\qquad\left(v=\sum_{i=1}^{k}u_i\right)$$

$$\leqq(\mu^{k-\delta}+\mu^k)\iint_{0<\sum_{i=1}^{k}u_i<b}\cdots\int(1-F(u_k))\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))e^{-\mu\sum_{i=1}^{k}u_i}$$

$$\cdot\,du_1\cdots du_k\,,$$

$$\sum_{s\geq 0} P(S_t=s)P(A_s(t)\mid S_t=s)$$

$$=\underset{0<\sum_{i=1}^{k-1}u_i<t}{\iint\cdots\int}\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))\mu^{k-1}e^{-\mu\sum_{i=1}^{k-1}u_i}du_1\cdots du_{k-1}$$

$$\leq\mu^{k-1}\underset{0<\sum_{i=1}^{k-1}u_i<b}{\iint\cdots\int}\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))e^{-\mu\sum_{i=1}^{k-1}u_i}du_1\cdots du_{k-1},$$

and

$$\sum_{n=1}^{\infty}P_n(\mu, t)=1-P_0(\mu, t)=\sum_{n=1}^{\infty}\sum_{s\geq 0}P(S_t=s)P(N_s=n\mid S_t=s)$$

$$\leq\sum_{s\geq 0}P(S_t=s)P(\bigcup_{i=1}^{s-1}\{\tau_i>U_i^s\}\mid S_t=s)$$

$$\leq\sum_{s\geq 1}P(S_t=s)(s-1)\int_0^t(1-F(u))\frac{s}{t}\left(1-\frac{u}{t}\right)^{s-1}du$$

$$=\mu^2 t\int_0^t(1-F(u))\left(1-\frac{u}{t}\right)e^{-\mu u}du$$

$$\leq\mu^{1-\delta}\int_0^b(1-F(u))e^{-\mu u}du\,.$$

Thus, if we put

$$(3.19)\quad \beta(\mu)=\mu^k\underset{0<\sum_{i=1}^{k-1}u_i<b}{\iint\cdots\int}\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))e^{-\mu\sum_{i=1}^{k-1}u_i}du_1du_2\cdots du_{k-1},$$

then, for every non-negative integer n and for small μ positive, it holds by (3.15) that

$$(3.20)\qquad P_n'(\mu, t)=\beta(\mu)[P_{n-1}(\mu, t)-P_n(\mu, t)]+\gamma_{1n}(\mu, t)+\gamma_{2n}^*(\mu, t)\,,$$

for every fixed small μ and for all t such that $0<t\leq b$, and

$$(3.21)\qquad |\gamma_{2n}^*(\mu, t)|\leq\zeta_2^*(\mu)\,,\ \text{uniformly in } t(\leq b) \text{ and } n(\geq 0)\,,$$

where

$$(3.22)\quad \zeta_2^*(\mu)=4\mu^{k+1-\delta}\int_0^b(1-F(u))e^{-\mu u}du\cdot\underset{0<v<b}{\iint\cdots\int}\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))$$

$$\cdot\ e^{-\mu v}du_1\cdots du_{k-1}\quad\left(v=\sum_{i=1}^{k-1}u_i\right).$$

In the second place, we investigate the case when $t>b$.

Let us divide the interval $[0, t)$ into two sub-intervals $[0, a)$ and $[a, t)$, and let us denote the numbers of quanta which are absorbed in respective intervals by S_a and S_b. Further, let $\{U^s_{ia}\}(i=0, 1, \cdots, s)$ and $\{U^{s'}_{jb}\}(j=0, 1, \cdots, s')$ be time-intervals between successive absorptions in respective sub-intervals $[0, a)$ and $[a, t)$ under the respective conditions $S_a=s$ and $S_b=s'$. Then, as before, S_a and S_b, and hence, U^s_{ia}'s and $U^{s'}_{jb}$'s are mutually independent in the stochastic sense.

Now, for every non-negative integer n, it is evident that

$$(3.23)\qquad \sum_{s\geqq 0} P(S_t=s)P(N_s=n,\ A_s(t)\,|\,S_t=s)$$
$$=\sum_{s\geqq 0}\sum_{s'\geqq 0} P(S_a=s)P(S_b=s')P(N_{s+s'}=n,\ A_{s+s'}(t)\,|\,S_a=s,\ S_b=s').$$

In order to evaluate this probability, we introduce and evaluate the following

$$(3.24)\qquad \sum_{s\geqq 0}\sum_{s'\geqq 0} P(S_a=s)P(S_b=s')P(N_s=n,\ A_{s+s'}(t)\,|\,S_a=s,\ S_b=s')$$
$$=\sum_{s\geqq 0}\sum_{s'\geqq k-1} P(S_a=s)P(S_b=s')P(N_s=n\,|\,S_a=s)P(A_{s+s'}(t)\,|\,S_b=s')$$
$$+\sum_{s\geqq 0}\sum_{s'\leqq k-2} P(S_a=s)P(S_b=s')P(N_s=n,\ A_{s+s'}(t)\,|\,S_a=s,\ S_b=s'),$$

where we used the fact that, if $s'\geqq k-1$, then the events $\{N_s=n\}$ and $A_{s+s'}(t)$ are independent of the conditions $S_b=s'$ and $S_a=s$ respectively. The first member on the right-hand side of this equality is exactly equal to

$$(3.25)\qquad \mu^{-1}\beta(\mu)P_n(\mu, a),\ \beta(\mu) \text{ being the same as (3.19).}$$

The second member can be evaluated as follows: This is not greater than

$$(3.26)\qquad \sum_{s'\leqq k-2} P(S_b=s')=e^{-\mu b}\left[1+\frac{(\mu b)}{1!}+\cdots+\frac{(\mu b)^{k-2}}{(k-2)!}\right]$$
$$=e^{-\mu^{-\delta}}\sum_{l=0}^{k-2}\frac{\mu^{-l\delta}}{l!}\leqq \mu^k, \text{ for small } \mu.$$

Next, we shall consider the difference between two quantities given by (3.23) and (3.24). Under the condition $S_a=s$ and $S_b=s'$, it holds that

$$\{N_{s+s'}=n\}=\sum_{j=0}^{\min(s',\,n)}\{N_s=n-j,\ N^*_{s'}=j\},\ \left(N^*_{s'}=\sum_{i=s+1}^{s+s'}X_i\right),$$

and

$$\{N_s=n\}=\sum_{j=0}^{s'}\{N_s=n,\ N^*_{s'}=j\}\ .$$

Consequently, the difference between two events $\{N_{s+s'}=n\}$ and $\{N_s=n\}$ is included by the event $\{N^*_{s'}\geqq 1\}$, which is interpreted, under the condition $S_a=s$ and $S_b=s'$, as the event that at least one visual response occurs in the sub-interval $[a, t)$. Hence, the difference between those given by (3.23) and (3.24) is not greater than the first member of the following inequality:

$$(3.27)\quad \begin{aligned}&\sum_{s\geqq 0}\sum_{s'\geqq 0}P(S_a=s)P(S_b=s')P(N^*_{s'}\geqq 1,\ A_{s+s'}(t)\,|\,S_a=s,\ S_b=s')\\ &\leqq\sum_{s'\leqq k-2}P(S_b=s')\\ &\quad+\sum_{s\geqq 0}\sum_{s'\geqq k-1}P(S_a=s)P(S_b=s')P(N^*_{s'}\geqq 1,\ A_{s+s'}(t)\,|\,S_a=s,\ S_b=s')\ .\end{aligned}$$

Here, an evaluation of the second member has been given by (3.26). Evaluation of the third member can be given as follows: This is not greater than the first member of the following inequality

$$(3.28)\quad \begin{aligned}&\sum_{s'\geqq k-1}P(S_b=s')P(\bigcup_{j=0}^{s'-k+1}\{\tau_{s+j}>U^{s'}_{jb}\}\ ,\ A_{s+s'}(t)\,|\,S_b=s')\\ &\leqq(\mu^{k-\delta}+2\mu^k)\int\!\!\int_{0<\sum_{i=1}^k u_i<b}\!\!\cdots\int(1-F(u_k))\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))e^{-\mu\sum_{i=1}^k u_i}\\ &\qquad\cdot du_1\cdots du_k\ ,\end{aligned}$$

which is obtained by using a similar calculation to that used to derive the result (3.20).

Finally, we shall compare the probability $P_n(\mu, a)$ with $P_n(\mu, t)$. It is easily seen that

$$\begin{aligned}|P_n(\mu, a)-P_n(\mu, t)|&\leqq\sum_{s\geqq 0}\sum_{s'\geqq 0}P(S_a=s)P(S_b=s')P(N^*_{s'}\geqq 1\,|\,S_a=s,\ S_b=s')\\ &\leqq\sum_{s\geqq 0}\sum_{s'\geqq 0}P(S_a=s)P(S_b=s')P(\bigcup_{j=0}^{s'-1}\{\tau_{s+j}>U^{s'}_{jb}\}\cup B_s(a)\,|\,S_a=s,\ S_b=s')\end{aligned}$$

$$(3.29)\quad \begin{aligned}&\leqq\sum_{s\geqq 0}\sum_{s'\geqq 0}P(S_a=s)P(S_b=s')P(\bigcup_{j=0}^{s'-1}\{\tau_{s+j}>U^{s'}_{jb}\}\,|\,S_a=s,\ S_b=s')\\ &\quad+\sum_{s\geqq 0}P(S_a=s)P(B_s(a)\,|\,S_a=s)\\ &\leqq\sum_{s'\geqq 0}P(S_b=s')s'\int_0^b(1-F(u))\frac{s'}{b}\left(1-\frac{u}{b}\right)^{s'-1}du\end{aligned}$$

$$+\sum_{s\geqq 0} P(S_a=s)\frac{s(s-1)-(s-k+2)}{a^{k-1}}\iint\cdots\int_{0<v<a}\prod_{m=1}^{k-1}(1-F(v_m))$$

$$\cdot\left[1-\left(1-\frac{v}{a}\right)^{s-k+1}\right]dv_1\cdots dv_{k-1}\quad\left(v=\sum_{m=1}^{k-1}v_m\right)$$

$$\leqq(\mu^{1-\delta}+\mu)\int_0^b(1-F(u))e^{-\mu u}du+\mu^{k-1}\iint\cdots\int_{0<v<\infty}\prod_{m=1}^{k-1}(1-F(v_m))(1-e^{-\mu v})$$

$$\cdot\, dv_1\cdots dv_{k-1}\,.$$

By using (3.23) through (3.29), it follows from (3.15) that, when $t>b$,

(3.30) $$P_n'(\mu, t)=\beta(\mu)[P_{n-1}(\mu, t)-P_n(\mu, t)]+\gamma_{1n}(\mu, t)+\gamma_{2n}^{**}(\mu, t)\,,$$

and

(3.31) $|\gamma_{2n}^{**}(\mu, t)|\leqq\zeta_2^{**}(\mu)$, uniformly in $t(>b)$ and $n(\geqq 0)$,

for every fixed small μ, where

(3.32) $$\zeta_2^{**}(\mu)=4\mu^{k+1}+2(\mu^{1-\delta}+\mu)\beta(\mu)\int_0^b(1-F(u))e^{-\mu u}du$$

$$+2\beta(\mu)\mu^{k-1}\iint\cdots\int_{0<v<\infty}\prod_{m=1}^{k-1}(1-F(v_m))\cdot(1-e^{-\mu v})dv_1\cdots dv_{k-1}$$

$$+2(\mu^{k+1-\delta}+2\mu^{k+1})\iint\cdots\int_{0<\sum_{i=1}^k u_i<b}(1-F(u_k))\prod_{i=1}^{k-1}(1-F(u_1+\cdots+u_i))$$

$$\cdot\, e^{-\mu\sum_{i=1}^k u_i}du_1\,du_2\cdots du_k\,.$$

Thus, putting

(3.33) $$\gamma_{2n}(\mu, t)=\begin{cases}\gamma_{2n}^{*}(\mu, t)\,, & \text{if}\quad t\leqq b\,,\\ \gamma_{2n}^{**}(\mu, t)\,, & \text{if}\quad t>b\,,\end{cases}$$

and

(3.34) $$\zeta_2(\mu)=\max\,(\zeta_2^{*}(\mu)\,,\ \zeta_2^{**}(\mu))\,,$$

we get by (3.20) and (3.30)

(3.35) $$P_n'(\mu, t)=\beta(\mu)[P_{n-1}(\mu, t)-P_n(\mu, t)]+\gamma_{1n}(\mu, t)+\gamma_{2n}(\mu, t)\,,$$

where

(3.36) $|\gamma_{2n}(\mu, t)|\leqq\zeta_2(\mu)$, uniformly in positive t and $n(\geqq 0)$,

for every fixed small μ.

Put

(3.37) $$Q_n(\mu, t)=\gamma_{1n}(\mu, t)+\gamma_{2n}(\mu, t) \text{ and } \zeta(\mu)=\zeta_1(\mu)+\zeta_2(\mu)\,.$$

Then, summarizing the results thus obtained, one can state the following

LEMMA 3.2. *$P_n(\mu, t)$'s satisfy the equations:*

(3.38) $$P'_n(\mu, t)=\beta(\mu)[P_{n-1}(\mu, t)-P_n(\mu, t)]+Q_n(\mu, t)\,,$$

where

(3.39) $$|Q_n(\mu, t)|\leqq\zeta(\mu)\,, \text{ uniformly in positive } t \text{ and } n(\geqq 0),$$

for every fixed small μ.

Now, as is easily verified, the solution of (3.36) under the initial condition (3.4) is given by

(3.40)
$$P_0(\mu, t)=e^{-\beta(\mu)t}[1+\int_0^t Q_0(\mu, x)e^{\beta(\mu)x}dx]\,,$$
$$P_n(\mu, t)=e^{-\beta(\mu)t}[\beta(\mu)\int_0^t P_{n-1}(\mu, x)e^{\beta(\mu)x}dx+\int_0^t Q_n(\mu, x)e^{\beta(\mu)x}dx] \quad (n\geqq 1)\,.$$

From this, we can show the following

THEOREM 3.1. *If the conditions*

(3.41) $$\beta(\mu)t\to\theta(>0) \quad \text{and} \quad \zeta(\mu)t\to 0, \quad \text{as } (t\to\infty)_k\,,$$

are satisfied, then it holds that

(3.42) $$P_n(\mu, t)\to e^{-\theta}\frac{\theta^n}{n!}\,, \text{ as } (t\to\infty)_k\,.$$

for every non-negative integer n.

PROOF. By (3.40),

$$P_0(\mu, t)=e^{-\beta(\mu)t}+G_0(\mu, t)\,,$$

where

$$G_0(\mu, t)=e^{-\beta(\mu)t}\int_0^t Q_0(\mu, x)e^{\beta(\mu)x}dx\,:$$

Then, from (3.39) it readily follows that

(3.43) $$|G_0(\mu, t)|\leqq\zeta(\mu)t, \text{ uniformly in positive } t,$$

for every small μ.

Suppose that

(3.44) $$P_{n-1}(\mu,\ t)=e^{-\beta(\mu)t}\frac{[\beta(\mu)t]^{n-1}}{(n-1)\,!}+G_{n-1}(\mu,\ t)\ ,$$

and

(3.45) $$|\,G_{n-1}(\mu,\ t)\,|\leqq\zeta(\mu)t\sum_{i=0}^{n-1}(\beta(\mu)t)^i\ ,\ \text{uniformly in } t(>0)$$

for every small μ, $0<\mu<\mu_0$, say. Then, by (3.40) we get

(3.46) $$P_n(\mu,\ t)=e^{-\beta(\mu)t}\frac{[\beta(\mu)t]^n}{n\,!}+G_n(\mu,\ t)\ ,$$

where

(3.47) $$G_n(\mu,\ t)=e^{-\beta(\mu)t}\left[\beta(\mu)\int_0^t G_{n-1}(\mu,\ x)e^{\beta(\mu)x}dx+\int_0^t Q_n(\mu,\ x)e^{\beta(\mu)x}dx\right]\ .$$

Hence, by (3.39) and (3.45) it holds that

(3.48) $$|\,G_n(\mu,\ t)\,|\leqq\zeta(\mu)t\sum_{i=0}^{n}(\beta(\mu)t)^i\ ,\ \text{uniformly in } t(>0)\ ,$$

for every μ, $0<\mu<\mu_0$.

Thus, by the mathematical induction, we are sure that (3.46) and (3.48) hold true for every non-negative integer n, from which the theorem follows.

COROLLARY 3.1. *If the life-distribution has a finite mean value* α, *then it holds that*

(3.49) $$P_n(\mu,\ t)\to e^{-\xi}\frac{\xi^n}{n\,!}\ ,\ \text{as } (t\to\infty)_k\ ,$$

where $\xi=\alpha^{k-1}\lambda/(k-1)\,!$, λ *being the same as in section 1.*

PROOF. From the definition of $\zeta(\mu)$, it is easy to see that the condition of this corollary implies the second condition of (3.41).

By the definition (3.19), it holds that

$$\beta(\mu)t\to\lambda\int_0^\infty\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{k-1}(1-F(u_1+\ \cdots\ +u_i))du_1\cdots du_{k-1}\ ,\ \text{as } (t\to\infty)_k\ ,$$

due to the Lebesgue convergence theorem. Hence, from (2.7) it follows that $\beta(\mu)t\to\xi$ as $(t\to\infty)_k$. The corollary, now, follows from the preceding theorem.

Since

$$W_k(\mu,\ t)=\sum_{n=1}^{\infty}P_n(\mu,\ t)=1-P_0(\mu,\ t)\ ,$$

the following is a direct consequence of this corollary.

COROLLARY 3.2. *If the life-distribution has a finite mean value* α, *then, it holds that*

$$W_k(\mu, t) \to 1 - e^{-\xi}, \text{ as } (t \to \infty)_k . \tag{3.50}$$

Basing upon this limiting formula, we can state the following, which is the main result of this section.

THEOREM 3.2. *If the life-distribution has a finite mean value* α, *then the Bouman-Velden-Yamamoto asymptotic evaluation formula (1.2) holds true, in the case when* $k \geqq 2$.

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DEPT. OF MECH., SCHOOL OF SCI. AND ENG., WASEDA UNIVERSITY

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