A CONSISTENT ESTIMATOR FOR THE MEAN DEVIATION OF THE PEARSON TYPE DISTRIBUTION

GIITIRO SUZUKI

(Received Feb. 24, 1965)

1. Introduction and Summary

It is natural to consider the sample mean deviation as a sample characteristic corresponding to the population mean deviation. But by definition the sample mean deviation contains moduli, which makes it difficult to treat its sampling distribution, etc. I have not yet seen any literature concerning the sampling distribution of the sample mean deviation other than Godwin's [3], in which he has treated the normal case and the result of which is too complicated as the sample size n increases.

As for the Pearson type distribution, it can be easily shown that the mean deviation (more generally the absolute central moment of any odd order) is expressed as a function of mean and three central moments from the 2nd order to the 4th order (cf. [5]). In this paper we consider the statistic obtained by replacing these four moments in this function with the corresponding sample characteristics and show that its sampling distribution is asymptotically normal and, consequently, it is a consistent estimator for the mean deviation.

2. The relation between the mean deviation and the central moments

We shall consider the Pearson distribution, of which the probability density function satisfies the differential equation

$$f'(x)=\frac{a+x}{b_0+b_1x+b_2x^2}f(x)\,. \tag{2.1}$$

We put

$$\begin{aligned} \mu&=\int_{-\infty}^{\infty} xf(x)dx \\ \mu_\nu&=\int_{-\infty}^{\infty} (x-\mu)^\nu f(x)dx \qquad (\nu=0,\ 1,\ 2,\ \cdots)\,. \end{aligned} \tag{2.2}$$

Then we have immediately from (2.1)

$$[(b_0+b_1\mu+b_2\mu^2)+(b_1+2b_2\mu)(x-\mu)+b_2(x-\mu)^2]f'(x) = [(a+\mu)+(x-\mu)]f(x) \tag{2.3}$$

$$[(b_0+b_1\mu+b_2\mu^2)(x-\mu)^\nu+(b_1+2b_2\mu)(x-\mu)^{\nu+1}+b_2(x-\mu)^{\nu+2}]f'(x) = [(a+\mu)(x-\mu)^\nu+(x-\mu)^{\nu+1}]f(x) \qquad (\nu=1,\,2,\,\cdots,\,7)\,. \tag{2.4}$$

Integrating the left hand sides by parts, we can find, assuming that the integrals exist and that $\lim_{x\to\pm\infty}(x-\mu)^9 f(x)=0$, the following relations:

$$\begin{aligned} &-(b_1+2b_2\mu)=a+\mu \\ &-\nu(b_0+b_1\mu+b_2\mu^2)\mu_{\nu-1}-(\nu+1)(b_1+2b_2\mu)\mu_\nu-(\nu+2)b_2\mu_{\nu+1} \\ &\qquad =(a+\mu)\mu_\nu+\mu_{\nu+1} \qquad (\nu=1,\,2,\,\cdots,\,7). \end{aligned} \tag{2.5}$$

Using the matrix notations the above relations can be rewritten as

$$\boldsymbol{M}\boldsymbol{p}=-\boldsymbol{\mu}\,, \tag{2.5$'$}$$

where

$$\boldsymbol{M}=\begin{bmatrix} 1 & 0 & 1 & 2\mu \\ 0 & 1 & \mu & \mu^2+3\mu_2 \\ 0 & 0 & 2\mu_2 & 4(\mu\mu_2+\mu_3) \\ 0 & 3\mu_2 & 3(\mu\mu_2+\mu_3) & 3\mu^2\mu_2+6\mu\mu_3+5\mu_4 \end{bmatrix}, \tag{2.6}$$

$$\boldsymbol{p}=\begin{bmatrix} a \\ b_0 \\ b_1 \\ b_2 \end{bmatrix}, \qquad \boldsymbol{\mu}=\begin{bmatrix} \mu \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}.$$

Therefore, these four constants in the equation (2.1) can be expressed in terms of the mean μ and the three central moments μ_2, μ_3, μ_4 (assuming that the matrix $\boldsymbol{M}$ is non-singular):

$$\boldsymbol{p}=-\boldsymbol{M}^{-1}\boldsymbol{\mu}\,, \tag{2.7}$$

where

(2.8)

$$\boldsymbol{M}^{-1}=\begin{bmatrix} 1 & \dfrac{-12\mu_2\mu_3}{M} & \dfrac{9\mu_2^2-5\mu_4}{M} & \dfrac{4\mu_3}{M} \\ 0 & \dfrac{-2(3\mu^2\mu_2^2+6\mu\mu_2\mu_3+6\mu_3^2-5\mu_2\mu_4)}{M} & \dfrac{9\mu\mu_2^2-3\mu^2\mu_3+9\mu_2\mu_3-5\mu\mu_4}{M} & \dfrac{2(\mu^2\mu_2-3\mu_2^2+2\mu\mu_3)}{M} \\ 0 & \dfrac{12\mu_2(\mu\mu_2+\mu_3)}{M} & \dfrac{-9\mu_2^2+6\mu\mu_3+5\mu_4}{M} & \dfrac{-4(\mu\mu_2+\mu_3)}{M} \\ 0 & \dfrac{-6\mu_2^2}{M} & \dfrac{-3\mu_3}{M} & \dfrac{2\mu_2}{M} \end{bmatrix}$$

$$M=-2(9\mu_2^3+6\mu_3^2-5\mu_2\mu_4)\,. \tag{2.9}$$

Then we obtain the relations between parameters and the four moments which reduce, in case $\mu=0$, to Kendall [4]'s (6.4).

Next we consider the mean deviation of $f(x)$ defined by

$$d=\int_{-\infty}^{\infty}|x-\mu|f(x)dx=-2\int_{-\infty}^{\mu}(x-\mu)f(x)dx\,. \tag{2.10}$$

Integrating the both sides of (2.3) over $(-\infty, \mu)$ and using the relation (2.5), we obtain

$$(b_0+b_1\mu+b_2\mu^2)f(\mu)=(1+2b_2)\int_{-\infty}^{\mu}(x-\mu)f(x)dx\,. \tag{2.11}$$

Furthermore, from (2.5)

$$(1+3b_2)\mu_2=-(b_0+b_1\mu+b_2\mu^2)\,. \tag{2.12}$$

Combining (2.10)∼(2.12) we obtain the relation

$$d=\frac{1+3b_2}{1+2b_2}2\mu_2 f(\mu)=\frac{1+3b_2}{1+2b_2}2\mu_2 Ce^{g(\mu)}\,, \tag{2.13}$$

where $g(x)=g(x;\, a,\, b_0,\, b_1,\, b_2)$ is an indefinite integral of $\dfrac{a+x}{b_0+b_1x+b_2x^2}$ and c is such a positive constant as

$$c=c(a,\, b_0,\, b_1,\, b_2)=\left[\int_{-\infty}^{\infty}e^{g(x)}dx\right]^{-1}\,. \tag{2.14}$$

Thus we summarize:

LEMMA 1. *The mean deviation of the Pearson distribution can be expressed by (2.13). Furthermore, these four constants are given by (2.7) in terms of the mean* μ *and the three central moments* μ_2, μ_3, μ_4.

3. The limiting distribution of the sample central moment vector

Let $\boldsymbol{m}$ be the sample characteristic vector corresponding to $\boldsymbol{\mu}$:

$$\boldsymbol{m}=\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}=\begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} X_i \\ \frac{1}{n}\sum_{i=1}^{n}(X_i-m_1)^2 \\ \frac{1}{n}\sum_{i=1}^{n}(X_i-m_1)^3 \\ \frac{1}{n}\sum_{i=1}^{n}(X_i-m_1)^4 \end{bmatrix}, \tag{3.1}$$

which converges in probability to $\boldsymbol{\mu}$. By elementary calculation we have

$$E\{\sqrt{n}\,(\boldsymbol{m}-\boldsymbol{\mu})\}=\boldsymbol{o}+\boldsymbol{e}_n\,,\qquad V\{\sqrt{n}\,(\boldsymbol{m}-\boldsymbol{\mu})\}=\boldsymbol{D}+\boldsymbol{D}_n \tag{3.2}$$

where $\boldsymbol{e}_n$ and $\boldsymbol{D}_n$ are a vector and a,matrix, whose elements are $o\left(\frac{1}{\sqrt{n}}\right)$, $o\left(\frac{1}{n}\right)$, respectively, and the matrix $\boldsymbol{D}$ is given by

(3.3)

$$\boldsymbol{D}=\begin{bmatrix} \mu_2 & \mu_3 & \mu_4-3\mu_2^2 & \mu_5-4\mu_2\mu_3 \\ \mu_3 & \mu_4-\mu_2^2 & \mu_5-4\mu_2\mu_3 & \mu_6-\mu_2\mu_4-4\mu_3^2 \\ \mu_4-3\mu_2^2 & \mu_5-4\mu_2\mu_3 & \mu_6-6\mu_2\mu_4-\mu_3^2+9\mu_2^3 & \mu_7-3\mu_2\mu_5-5\mu_3\mu_4+12\mu_2^2\mu_3 \\ \mu_5-4\mu_2\mu_3 & \mu_6-\mu_2\mu_4-4\mu_3^2 & \mu_7-3\mu_2\mu_5-5\mu_3\mu_4+12\mu_2^2\mu_3 & \mu_8-8\mu_3\mu_5-\mu_4^2+16\mu_2\mu_3^2 \end{bmatrix}$$

(see, for example, Cramér [2] p. 350).

For the later purpose, we shall prove the following lemma:

LEMMA 2. *The sample characteristic vector* $\boldsymbol{m}$ *converges in probability to* $\boldsymbol{\mu}$ *and the limiting distribution of* $\sqrt{n}\,(\boldsymbol{m}-\boldsymbol{\mu})$ *is* $N(\boldsymbol{o},\,\boldsymbol{D})$.

Proof. The first assertion is clear from the relation (3.2). To prove the second assertion, we divide this random vector into two portions;

$$\sqrt{n}\,(\boldsymbol{m}-\boldsymbol{\mu})=\sqrt{n}\,(\tilde{\boldsymbol{m}}-\boldsymbol{\mu})+\sqrt{n}\,(\boldsymbol{m}-\tilde{\boldsymbol{m}}) \tag{3.4}$$

where

$$\tilde{\boldsymbol{m}}=\begin{bmatrix} m_1 \\ \tilde{m}_2 \\ \tilde{m}_3 \\ \tilde{m}_4 \end{bmatrix}=\begin{bmatrix} \frac{1}{n}\sum X_i \\ \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \\ \frac{1}{n}\sum (X_i-\mu)[(X_i-\mu)^2-3\mu_2] \\ \frac{1}{n}\sum (X_i-\mu)[(X_i-\mu)^3-4\mu_3] \end{bmatrix}. \tag{3.5}$$

Since

$$\sqrt{n}\,(\tilde{\boldsymbol{m}}-\boldsymbol{\mu})=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{y}_i \tag{3.6}$$

where

$$(3.7) \qquad \boldsymbol{y}_i = \begin{pmatrix} x_i - \mu \\ (X_i-\mu)^2 - \mu_2 \\ (X_i-\mu)[(X_i-\mu)^2 - 3\mu_2] - \mu_3 \\ (X_i-\mu)[(X_i-\mu)^3 - 4\mu_3] - \mu_4 \end{pmatrix}$$

and $E\{\boldsymbol{y}_i\}=\boldsymbol{o}$, $V\{\boldsymbol{y}_i\}=\boldsymbol{D}$, the asymptotic distribution of $\sqrt{n}(\tilde{\boldsymbol{m}}-\boldsymbol{\mu})$ is the 4-variate normal $N(\boldsymbol{o}, \boldsymbol{D})$ by the multivariate central limit theorem (for example, see Anderson [1] p. 74).

On the other hand we can prove that each element of $V\{\sqrt{n}(\boldsymbol{m}-\tilde{\boldsymbol{m}}\}$ is at most $o\left(\frac{1}{\sqrt{n}}\right)$; therefore $\sqrt{n}(\boldsymbol{m}-\tilde{\boldsymbol{m}})$ converges in probability to $\boldsymbol{o}$. Thus by the convergence theorem we conclude that $\sqrt{n}(\boldsymbol{m}-\boldsymbol{\mu})$ has the same limiting distribution as $\sqrt{n}(\tilde{\boldsymbol{m}}-\boldsymbol{\mu})$, namely $N(\boldsymbol{o}, \boldsymbol{D})$.

4. A consistent estimator and its asymptotic distribution

We put

$$(4.1) \qquad H(\boldsymbol{m}) = H(m_1, m_2, m_3, m_4) = \frac{1+3p_4}{1+2p_4} 2p_5\{c(p_1, p_2, p_3, p_4)\exp[g(p_6; p_1, p_2, p_3, p_4)]\}$$

where

(4.2)

$$\boldsymbol{p}=\boldsymbol{p}(\boldsymbol{m}) = \begin{bmatrix} p_1(m_1, m_2, m_3, m_4) \\ p_2(m_1, m_2, m_3, m_4) \\ p_3(m_1, m_2, m_3, m_4) \\ p_4(m_2, m_3, m_4) \\ p_5(m_2) \\ p_6(m_1) \end{bmatrix} = \begin{bmatrix} -m_1 + m_3 I_1 I_2 \\ I_1[m_1(2m_1m_2+m_3)I_2 + (m_1^2+m_2)I_3] \\ -I_1[(4m_1m_2+m_3)I_2 + 2m_1I_3] \\ I_1[2m_2I_2+I_3] \\ m_2 \\ m_1 \end{bmatrix}$$

$$(4.3) \qquad I_1 = I_1(m_2, m_3, m_4) = \frac{1}{2(9m_2^3+6m_3^2-5m_2m_4)}$$

$$I_2 = I_2(m_2, m_4) = 3m_2^2 + m_4$$

$$I_3 = I_3(m_2, m_3, m_4) = 3m_3^2 - 4m_2m_4 \, .$$

By the lemma 1, we have

$$(4.4) \qquad H(\boldsymbol{\mu}) = \frac{1+3b_2}{1+2b_2} 2\mu_2 c e^{g(\mu)} = d \, .$$

We shall show that the statistic $H(\boldsymbol{m})$ defined by (4.1) is a consistent estimator for the mean deviation d and that the sampling distribution of $\sqrt{n}(H(\boldsymbol{m})-H(\boldsymbol{\mu}))$ is asymptotically normal.

First, we can easily calculate the partial derivative matrix of $\boldsymbol{p}$ with respect to $\boldsymbol{m}$ as follows:

$$(4.5) \qquad \left(\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{m}}\right)' = \left(\frac{\partial \boldsymbol{p}}{\partial m_1}, \frac{\partial \boldsymbol{p}}{\partial m_2}, \frac{\partial \boldsymbol{p}}{\partial m_3}, \frac{\partial \boldsymbol{p}}{\partial m_4}\right)$$

$$= \begin{bmatrix} -1 & 2I_4 & I_7 & 2m_3I_9 \\ -p_3 & -2(m_1^2I_5 - m_1I_4 + I_6) & m_1^2I_8 + m_1I_7 - 3m_2I_8 & (m_1^2m_2 + 2m_1m_3 - 3m_2^2)I_9 \\ -2p_4 & 4m_1I_5 - I_4 & -2(m_1I_8 + I_7) & 2(m_1m_2 + m_3)I_9 \\ 0 & -2I_5 & I_8 & m_2I_9 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where

$$I_4 = m_3[(9m_2^2 - 5m_4)I_2 - 12m_2I_3]I_1^2 = [(33m_2^2 - 5m_4)(m_1 + p_1) - 12m_2m_3p_4]I_1$$

$$I_5 = 3(m_3^2I_2 + 2m_2^2I_3)I_1^2 = 3[m_3(m_1 + p_1) + 2m_2^2I_1I_3]I_1$$

$$I_6 = 6[m_2(3m_3^2 - 2m_2m_4)I_2 + (3m_3^2 - m_2m_4)I_3]I_1^2$$

$$(4.6) \qquad = -6[3m_2m_3(m_1 + p_1) - (3m_3^2 - m_2m_4)p_4]I_1$$

$$I_7 = -2(9m_2^3 - 6m_3^2 - 5m_2m_4)I_2I_1^2 = [24m_3(m_1 + p_1) + I_2]I_1$$

$$I_8 = 12m_2m_3I_2I_1^2 = 12m_2(m_1 + p_1)I_1$$

$$I_9 = -6(4m_2^3 + m_3^2)I_1^2 = -2(2p_4 - I_1I_3)I_1 .$$

Next, when we consider H as the function of parameter vector $\boldsymbol{p}$, which has, at least in a neighborhood of $\boldsymbol{p}_0 = \boldsymbol{p}(\boldsymbol{\mu}) = (a, b_0, b_1, b_2, \mu_2, \mu)'$, continuous first and second partial derivatives, and furthermore each $p_i = p_i(\boldsymbol{m})$ is a partially differentiable function of $\boldsymbol{m}$, we can calculate the partial derivatives of H with respect to each $m_\nu (\nu = 1, 2, 3, 4)$ by

$$(4.7) \qquad \frac{\partial H}{\partial m_\nu} = \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)'\left(\frac{\partial \boldsymbol{p}'}{\partial m_\nu}\right) = \sum_{j=1}^{6} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial m_\nu}$$

$$= H\left[\sum_{j=1}^{4}\left\{h_j(m_1;\, p_1, p_2, p_3, p_4) - \frac{c_j(p_1, p_2, p_3, p_4)}{c(p_1, p_2, p_3, p_4)}\right\}\frac{\partial p_j}{\partial m_\nu}\right.$$

$$\left. + \frac{1}{(1+2p_4)(1+3p_4)}\frac{\partial p_4}{\partial m_\nu} + \frac{1}{m_2}\frac{\partial m_2}{\partial m_\nu} + \frac{p_1 + m_1}{p_2 + p_3m_1 + p_4m_1^2}\frac{\partial m_1}{\partial m_\nu}\right],$$

where each $h_j(x;\, p_1, p_2, p_3, p_4)$ is an indefinite integral of

$$\frac{\partial g'(x;\, p_1, p_2, p_3, p_4)}{\partial p_j}=\frac{\partial}{\partial p_j}\left(\frac{p_1+x}{p_2+p_3x+p_4x^2}\right)$$

and the constants $c_j(p_1, p_2, p_3, p_4)$ are given by

$$\begin{aligned}(4.8)\qquad -c_j(p_1, p_2, p_3, p_4) &= \frac{\partial}{\partial p_j}c(p_1, p_2, p_3, p_4)\\ &= -c^2(p_1, p_2, p_3, p_4)\int_{-\infty}^{\infty} h_j(x;\, p_1, p_2, p_3, p_4)\exp\,[g(x;\, p_1, p_2, p_3, p_4)]dx\,,\end{aligned}$$

which are also partially differentiable with respect to $\boldsymbol{m}$. Then we can use the following Anderson's lemma ([1], p. 74, Th. 4, 2. 3), which was essentially proved in Cramér [2] (p. 366).

LEMMA 3. *Let $U(n)$ be an m-component random vector and $\boldsymbol{b}$ a fixed vector. Assume that $U(n)$ converges stochastically to $\boldsymbol{b}$ and $\sqrt{n}\,(U(n)-\boldsymbol{b})$ is asymptotically distributed according to $N(\boldsymbol{o}, \boldsymbol{T})$. Let $w=f(\boldsymbol{u})$ be a real-valued function of a vector $\boldsymbol{u}$ with the first and second derivatives in a neighborhood of $\boldsymbol{u}=\boldsymbol{b}$. Let $\left.\frac{\partial f(\boldsymbol{u})}{\partial u_i}\right|_{\boldsymbol{u}=\boldsymbol{b}}$ be the i-th component of $\boldsymbol{\phi}_b$. Then the limiting distribution of $\sqrt{n}\,[f(V(n))-f(\boldsymbol{b})]$ is $N(\boldsymbol{o}, \boldsymbol{\phi}_b'\boldsymbol{T}\boldsymbol{\phi}_b)$.*

Combining the above lemma with our lemmas 1 and 2, we obtain

THEOREM. *Let $x_1, x_2, \cdots, x_n$ be the sample from a population having probability density function $f(x)$ defined by (2.1). Let $\boldsymbol{m}$ be the sample characteristic vector defined by (3.1) and define the statistic $H(\boldsymbol{m})$ by (4.1). If we assume the existence of the 8th order moment, then the sampling distribution of $\sqrt{n}\,(H(\boldsymbol{m})-d)$, where d is defined by (2.13), is asymptotically $N(o, \boldsymbol{\phi}'\boldsymbol{D}\boldsymbol{\phi})$, where $\boldsymbol{D}$ is given by (3.3) and the ν-th element of $\boldsymbol{\phi}$ by*

$$\begin{aligned}(4.9)\qquad \phi_\nu &= \left[\frac{\partial H(\boldsymbol{m})}{\partial m_\nu}\right]_{\boldsymbol{m}=\boldsymbol{\mu}}\\ &= d\left[\sum_{j=1}^{4}\left(h_j(\mu)-\frac{c_j}{c}\right)N_{j\nu}+\frac{N_{4\nu}}{(1+2b_2)(1+3b_2)}\right.\\ &\qquad\left.+\frac{N_{5\nu}}{\mu_2}+\frac{a+\mu}{b_0+b_1\mu+b_2\mu^2}N_{6\nu}\right],\end{aligned}$$

where $h_j(x)=h_j(x;\, a, b_0, b_1, b_2)$ is an indefinite integral of

$$\left[\frac{\partial g'(x;\, p_1, p_2, p_3, p_4)}{\partial p_j}\right](p_1, p_2, p_3, p_4)'=(a, b_0, b_1, b_2)'$$

and c, c_j are given by (2.14) and (4.8) replacing p_1, p_2, p_3, p_4 with $a, b_0,$

b_1, b_2, respectively, and $N_{j\nu}=\left[\frac{\partial p_j}{\partial m_\nu}\right]_{m=\mu}$ *are obtained by (4.5) replacing* $\boldsymbol{m}$ *with* $\boldsymbol{\mu}$.

Consequently the statistic $H(\boldsymbol{m})$ is a consistent estimator for the mean deviation d.

5. Some special cases

In this section we shall calculate the vector $\boldsymbol{\phi}$ given by (4.9) in such cases as at least one element of the parameter vector $\boldsymbol{p}$ vanishes.

(i) $b_1=b_2=0$, $b_0<0$ (normal case). We may consider, in this case, the statistic $H(\boldsymbol{m})$ given by (4.1) letting $p_3=p_4=0 (p_2<0)$. Then

$$g(x;\ p_1,\ p_2)=\frac{1}{2p_2}(x+p_1)^2, \qquad c(p_1,\ p_2)=\frac{1}{\sqrt{-2\pi p_2}}$$

and using the relation (2.5) we obtain

$$h_1(\mu)=h_1(\mu;\ a,\ b_0)=0,\ h_2(\mu)=h_2(\mu;\ a,\ b_0)=0$$

$$\frac{c_1}{c}=\frac{c_1(a,\ b_0)}{c(a,\ b_0)}=0\,, \quad \frac{c_2}{c}=\frac{c_2(a,\ b_0)}{c(a,\ b_0)}=-\frac{1}{2\mu_2}\ .$$

Furthermore, since $N_{11}=N_{22}=-1$, $N_{61}=N_{52}=1$ and other $N_{j\nu}$'s vanish,

$$\phi_2=\frac{d}{2\mu_2}\,,\ \phi_\nu=0\ \ (\nu\neq 2)\,,$$

$$\boldsymbol{\phi}'\boldsymbol{D}\boldsymbol{\phi}=\left(\frac{d}{2\mu_2}\right)^2(\mu_4-\mu_2^2)=\frac{d^2}{2}\ .$$

The statistic $H(\boldsymbol{m})$ in this case can be expressed by

$$\sqrt{\frac{2}{\pi}m_2}\,,$$

and we obtain

COROLLARY 1. *The statistic*

$$\sqrt{\frac{2}{\pi}m_2}=\sqrt{\frac{2}{\pi^n}\sum_{i=1}^{n}(x_i-\bar{x})^2} \tag{5.1}$$

from $N(\mu,\ \sigma^2)$ is a consistent estimator for the mean deviation

(5.2) $$d=\sqrt{\frac{2}{\pi}}\sigma$$

and its variance is given by

(5.3) $$\frac{d^2}{2n}+o\left(\frac{1}{n}\right)=\frac{\sigma^2}{n\pi}+o\left(\frac{1}{n}\right).$$

Remark. The variance of the sample mean deviation

$$\delta=\frac{1}{n}\sum_{i=1}^{n}|x_i-\bar{x}|$$

is given by

$$\frac{\sigma^2}{n}\left(1-\frac{2}{\pi}\right)+o\left(\frac{1}{n}\right)$$

(Kendall [4], p. 215), which is larger than (5.3) for large n.

(ii) $b_2=0$, $b_1<0$. In this case

$$g(x;\, p_1,\, p_2,\, p_3)=\frac{p_1p_3-p_2}{p_3^2}\log(-p_2-p_3x)+\frac{p_2+p_3x}{p_3^2}\left(x>-\frac{p_2}{p_3}\right)$$

$$c^{-1}(p_1,\, p_2,\, p_3)=(-p_3)^{\frac{2(p_1p_3-p_2)}{p_3^2}+1}\Gamma\left(\frac{p_1p_3-p_2}{p_3^2}+1\right),$$

where

$$\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx\,.$$

Using the relation (2.5) we can calculate

$$h_1(\mu)-\frac{c_1}{c}=\frac{1}{b_1}\left[\log\frac{\mu_2}{b_1^2}-c'\right]$$

$$h_2(\mu)-\frac{c_2}{c}=-\frac{1}{b_1^2}\left[\log\frac{\mu_2}{b_1^2}-c'\right]+\frac{1}{\mu_2}$$

$$h_3(\mu)-\frac{c_3}{c}=\frac{2b_0-ab_1}{b_1^3}\left[\log\frac{\mu_2}{b_1^2}-c'\right]+\frac{2b_0}{\mu_3},$$

where

$$c'=c'(a,\, b_0,\, b_1)=\frac{\Gamma'\left(\dfrac{ab_1-b_0}{b_1^2}+1\right)}{\Gamma\left(\dfrac{ab_1-b_0}{b_1^2}+1\right)}$$

$$\Gamma'(p)=\int_0^\infty x^{p-1}(\log x)e^{-x}dx\,.$$

Furthermore, since

$$N=(N_{j\nu})=\left(\left[\frac{\partial p_j}{\partial m_\nu}\right]_{m=\mu}\right)=\begin{bmatrix} -1 & \dfrac{b_1}{\mu_2} & \dfrac{1}{2\mu_2} & 0 \\ -b_1 & -\left(2+\dfrac{b_0}{\mu_2}\right) & \dfrac{\mu}{2\mu_2} & 0 \\ 0 & -\dfrac{b_1}{\mu_2} & -\dfrac{1}{2\mu_2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

we get

$$\phi' D\phi=\frac{d^2}{\mu_2^2\mu_3^2}[J_1J^2-2J_2J+J_3]$$

where

$$J=\frac{4\mu_2^3}{\mu_3^2}\left[\log\frac{4\mu_2^3}{\mu_3^2}-c'\right]$$

$$J_1=4\mu_2^2\mu_6-12\mu_2\mu_3\mu_5+9\mu_3^2\mu_4-24\mu_2^3\mu_4+35\mu_2^2\mu_3^2+36\mu_2^5$$

$$J_2=2\mu_2^2\mu_6-5\mu_2\mu_3\mu_5+3\mu_3^2\mu_4-12\mu_2^3\mu_4+15\mu_2^2\mu_3^2+18\mu_2^5$$

$$J_3=\mu_2\mu_6-2\mu_2\mu_3\mu_5+\mu_3^2\mu_4-6\mu_2^3\mu_4+6\mu_2^2\mu_3^2+9\mu_2^5$$

$$d=H(\mu)=2\mu_2\frac{(-b_0-b_1\mu)^{(ab_1-b_0)/b_1^2}e^{(b_0+b_1\mu)/b_1^2}}{(-b_1)^{\{2(ab_1-b_0)/b_1^2\}+1}\Gamma\left(\dfrac{ab_1-b_0}{b_1^2}+1\right)}$$

$$=\frac{\mu_3}{\mu_2}\left(\frac{4\mu_2^3}{e\mu_3^2}\right)^{4\mu_2^3/\mu_3^2}\Big/\Gamma\left(\frac{4\mu_2^3}{\mu_3^2}\right).$$

Especially, when $a=-(\nu-2)$, $b_0=0$, $b_1=-2$(χ^2-distribution with ν as the degree of freedom which we denote by $\chi^2(\nu)$), by the relation (2.5) we can calculate

$$J=\frac{\nu}{2}\left[\log\frac{\nu}{2}-c'\right]=\frac{\nu}{2}\left[\log\frac{\nu}{2}-\frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)}\right]$$

$$(5.4)\qquad J_1=768\nu^3(\nu^2+12\nu+45)$$

$$J_2=128\nu^3(3\nu^2+38\nu+72)$$

$$J_3=64\nu^3(3\nu^2+50\nu+96)$$

$$(5.5)\qquad d=4(\nu/2e)^{\nu/2}/\Gamma(\nu/2)\,.$$

Thus we obtain

COROLLARY 2. *The statistic*

$$\frac{m_3}{m_2}\left(\frac{4m_2^3}{em_3^2}\right)^{4m_2^3/m_3^2}\Big/\,\Gamma\left(\frac{4m_2^3}{m_3^2}\right)$$

from $\chi^2(\nu)$ *is a consistent estimator for the mean deviation* d *given by (5.5) and its variance is given by*

$$\frac{1}{n}\frac{4(\nu/2e)^{\nu}}{\nu\Gamma^2(\nu/2)}\left\{3\nu^2(\nu^2+12\nu+45)J'^2-2\nu(3\nu^2+38\nu+72)\,J'+(3\nu^2+50\nu+96)\right\}$$
$$+o\left(\frac{1}{n}\right)$$

where J' *is given by*

$$J'=\log\frac{\nu}{2}-\frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)}\,.$$

(iii) $a=b_1=0,\quad b_0,\ b_2<0$. As before we obtain

$$g(x;\ p_2,\ p_4)=\frac{1}{2p_4}\log(-p_2-p_4x^2)$$

$$c^{-1}(p_2,\ p_4)=(-p_2)^{1/2p_4+1/2}(-p_4)^{-1/2}B\left(\frac{1}{2},\ -\frac{1}{2p_4}-\frac{1}{2}\right)$$

where

$$B(\alpha,\ \beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

and

$$\phi'D\phi=\frac{d^2}{4\mu_2^2\mu_4^2(\mu_4-3\mu_2^2)^2}[J_1J^2+J_2J+J_3]$$

where

$$J=\frac{3\mu_2}{(\mu_4-3\mu_2^2)^2}\left[\log\frac{2\mu_2\mu_4}{5\mu_4-9\mu_2^2}+\frac{B'\left(\frac{1}{2},\ \frac{2\mu_4-3\mu_2^2}{\mu_4-3\mu_2^2}\right)}{B\left(\frac{1}{2},\ \frac{2\mu_4-3\mu_2^2}{\mu_4-3\mu_2^2}\right)}\right]$$

$$B'(\alpha,\ \beta)=\frac{\partial}{\partial\beta}B(\alpha,\ \beta)$$

$$J_1=4\mu_2^2\mu_4^2(\mu_4-3\mu_2^2)^2[\mu_2^2\mu_8-4\mu_2\mu_4\mu_6+4\mu_4^3-\mu_2^2\mu_4^2]$$

$$J_2=2\mu_2\mu_4(\mu_4-3\mu_2^2)6[\mu_2^4\mu_8+2\mu_2\mu_4^2\mu_6-14\mu_2^3\mu_4\mu_6-4\mu_4^4+6\mu_2^2\mu_4^3+4\mu_2^4\mu_4^2]$$

$$J_3=9\mu_2^6\mu_8+6\mu_2^3\mu_4^2\mu_6-6\mu_2^5\mu_4\mu_6+\mu_4^5-3\mu_2^2\mu_4^4+2\mu_2^4\mu_4^3-4\mu_2^6\mu_4^2$$

$$d=\frac{1+3b_2}{1+2b_2}\,2\mu_2\sqrt{\frac{b_2}{b_0}}\Big/B\left(\frac{1}{2},\ -\frac{1}{2b_2}-\frac{1}{2}\right)$$

$$=\frac{2\sqrt{2\mu_2\mu_4(\mu_4-3\mu_2^2)}}{3(\mu_4-\mu_2^2)}\Big/B\left(\frac{1}{2},\ \frac{2\mu_4-3\mu_2^2}{\mu_4-3\mu_2^2}\right).$$

In particular, letting $b_0=-\frac{\nu}{\nu+1}$, $b_2=-\frac{\nu}{\nu+1}$, we obtain

COROLLARY 3. *The statistic*

$$\frac{2\sqrt{2m_2m_4(m_4-3m_2^2)}}{3(m_4-m_2^2)B\left(\frac{1}{2},\ \frac{2m_4-3m_2^2}{m_4-3m_2^2}\right)}$$

from the t-distribution with ν as the degree of freedom is a consistent estimator for the mean deviation

$$d=\frac{2\sqrt{\nu}}{\nu-1}\Big/B\left(\frac{1}{2},\ \frac{\nu}{2}\right)$$

and its variance is given by

$$\frac{1}{n}\ \frac{\nu}{324(\nu-1)^2(\nu-6)(\nu-8)B^2\left(\frac{1}{2},\ \frac{\nu}{2}\right)}$$

$$\times\ [7776(\nu-2)^2(\nu-4)AJ'^2+77(\nu-2)BJ'+36C]+o\left(\frac{1}{n}\right)$$

where

$$J'=\log\frac{\nu}{\nu+1}+\frac{B'\left(\frac{1}{2},\ \frac{\nu}{2}\right)}{B\left(\frac{1}{2},\ \frac{\nu}{2}\right)}$$

$$A=\nu^3-4\nu^2+23\nu-20$$

$$B=9\nu^4+369\nu^3-6354\nu^2+20096\nu-19776$$

$$C=35\nu^4-463\nu^3+2245\nu^2-4318\nu+2832.$$

6. Concluding remarks

(i) More generally, since any order absolute central moment can be expressed as a function of mean and three central moments ([5]), we can obtain by the similar method both the consistent estimator for the absolute central moment and its asymptotic variance. For example, since the 3rd order absolute central moment d_3 can be expressed by

$$d_3=\mu_3-4\mu_3F(\mu)+\frac{4[(1+4b_2)(a+\mu)\mu_3+(1+3b_2)^2\mu_2^2]}{(1+2b_2)(1+4b_2)}f(\mu)\,, \tag{6.1}$$

we obtain as a consistent estimator for this

$$\begin{aligned} H_3(m)&=m_3-4m_3c(p_1,\ p_2,\ p_3,\ p_4)\int_{-\infty}^{m_1}\exp\,[g(x;\ p_1,\ p_2,\ p_3,\ p_4)]dx\\ &+\frac{4[(1+4p_4)(p_1+m_1)m_3+(1+3p_4)^2m_2^2]}{(1+2p_4)(1+4p_4)}c(p_1,\ p_2,\ p_3,\ p_4)\\ &\cdot\exp\,[g(m_1;\ p_1,\ p_2,\ p_3,\ p_4)]\,, \end{aligned} \tag{6.2}$$

where

$$g(x;\ p_1,\ p_2,\ p_3,\ p_4)=\int_{-\infty}^{x}\frac{p_1+t}{p_2+p_3t+p_4t^2}dt$$

$$c(p_1,\ p_2,\ p_3,\ p_4)=\left\{\int_{-\infty}^{\infty}\exp\,[g(x;\ p_1,\ p_2,\ p_3,\ p_4)]dx\right\}^{-1}$$

and $p_1,\ p_2,\ p_3,\ p_4$ are the functions of the sample central moment vector $\boldsymbol{m}$ given by (4.2). Moreover the asymptotic variance of this statistic can be calculated along the same line as explained in section 4. Especially, we have

COROLLARY 4. *The statistic*

$$2\sqrt{\frac{2}{\pi}}\,m_2^{3/2}=2\sqrt{\frac{2}{\pi}}\left\{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{x})^2\right\}^{3/2}$$

from $N(\mu, \sigma^2)$ is a consistent estimator for the 3rd order absolute central moment

$$d_3=\int_{-\infty}^{\infty}|x-\mu|^3 f(x)dx=2\sqrt{\frac{2}{\pi}}\,\sigma^3$$

and its variance is given by

$$\frac{36\sigma^6}{n}+o\left(\frac{1}{n}\right).$$

(ii) The statistic $H(m)$ given by (4.1) is not necessarily unbiased. But when we can find such constant c_n depending on the sample size n that

$$c_n=\frac{d}{E\{H(m)\}},$$

we can define

$$\tilde{H}(m)=c_n H(m)$$

which is an unbiased estimator for d. Furthermore the asymptotic variance of $\tilde{H}(m)$ is the same as one of $H(m)$ because $c_n\to 1$ as $n\to\infty$. The same is true for the statistic $H_3(m)$ given by (6.2). Particularly, the statistic defined by (5.1) is not an unbiased estimator for d. On the other hand, we have

COROLLARY 5. *The statistic*

$$\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\sqrt{\frac{n}{\pi}m_2}$$

from $N(\mu, \sigma^2)$ is an unbiased estimator for the mean deviation and its variance is given by (5.3).

7. Acknowledgments

The author had the chance to read the draft of the interesting paper [6] which stimulated his attention to the problem in the present paper and he is heartily thankful to Professor A. R. Kamat of Gokhal Institute of Politics and Economics.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, John Wiley, 1958.

[2] H. Cramér, *Mathematical Methods of Statistics*, Princeton Univ. Press, 1946.

[3] H. J. Godwin, "On the distribution of the estimate of mean deviation obtained from samples from a normal population," *Biometrika*, 33 (1943), 254-256.

[4] M. G. Kendall, *The Advanced Theory of Statistics*, Vol. 1 (1948), London.

[5] G. Suzuki, "On the absolute central moments I, II," *Proc. Inst. Stat. Math.* (In Japanese) 12 (1964), 225-241; 13 (1965), 63-70.

[6] A. R. Kamat, "A property of the mean deviation (of some discrete distributions)," *Ann. Inst. Stat. Math.*, 17 (1965), 287-293.