# AN EXAMPLE OF THE TWO-SIDED WILCOXON TEST WHICH IS NOT UNBIASED

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(Received June 29, 1964)

The purpose of this note is to answer the question stated in Lehmann ([1], p. 240), of whether the two-sided Wilcoxon test is unbiased against the two-sided alternatives or not. Although every one-sided Wilcoxon test is unbiased against the one-sided alternatives, we shall show by an example that the two-sided Wilcoxon test is not necessarily unbiased against some special two-sided translation alternatives.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two random samples drawn from the distributions $F(x)$ and $G(x)$, respectively. These distributions are assumed to be absolutely continuous with respect to the Lebesgue measure. To test the hypothesis $\mathrm{H}: F(x)=G(x)$, consider the following test function $\phi(X_1, \cdots, X_m; Y_1, \cdots, Y_n)$:

$$(1)\quad \phi(X_1, \cdots, X_m; Y_1, \cdots, Y_n)=\begin{cases} 1 & \text{when } X_1, \cdots, X_m < Y_1, \cdots, Y_n \\ & \quad\text{or}\quad X_1, \cdots, X_m > Y_1, \cdots, Y_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that the test $\phi$ is a two-sided Wilcoxon test with the level $\alpha=2(m!\,n!)/(m+n)!$. The power of the test is given by

$$\begin{aligned}(2)\quad \beta &= P\{X_1, \cdots, X_m < Y_1, \cdots, Y_n\}+P\{X_1, \cdots, X_m > Y_1, \cdots, Y_n\} \\ &= P\{\max_{1\leq i\leq m} X_i < \min_{1\leq j\leq n} Y_j\}+P\{\min_{1\leq i\leq m} X_i > \max_{1\leq j\leq n} Y_j\} \\ &= \int_{-\infty}^{\infty} F(x)^m d[1-(1-G(x))^n]+\int_{-\infty}^{\infty} G(x)^n d[1-(1-F(x))^m] \\ &= n\int_{-\infty}^{\infty} F(x)^m (1-G(x))^{n-1} dG(x)+m\int_{-\infty}^{\infty} G(x)^n (1-F(x))^{m-1} dF(x)\,. \end{aligned}$$

Let us now specify the distribution functions $F(x)$ and $G(x)$ as follows:

$$(3)\quad F(x)=\begin{cases} 1-e^{-x} & x\geqq 0 \\ 0 & x<0\,, \end{cases}$$

$$G(x)=F(x-\Delta)\,,$$

and test the hypothesis $H:\Delta=0$ against the two-sided alternatives $K:\Delta\neq 0$.

The power function, $\beta_{m,n}(\Delta)$, of the test $\phi$ for this special case is obtained from (2):

$$\begin{aligned}(4)\qquad \beta_{m,n}(\Delta)&=n\int_{\Delta}^{\infty}(1-e^{-x})^m e^{-n(x-\Delta)}dx+m\int_{\Delta}^{\infty}(1-e^{-(x-\Delta)})^n e^{-mx}dx\\ &=\frac{m\,!\,n\,!}{(m+n)\,!}e^{-m\Delta}+ne^{n\Delta}\int_0^{e^{-\Delta}}t^{n-1}(1-t)^m dt\\ &=\frac{m\,!\,n\,!}{(m+n)\,!}e^{-m\Delta}+n\int_0^1 x^{n-1}(1-e^{-\Delta}x)^m dx\end{aligned}$$

for $\Delta\geqq 0$ and similarly

$$(5)\qquad \beta_{m,n}(\Delta)=\frac{m\,!\,n\,!}{(m+n)\,!}e^{n\Delta}+m\int_0^1 x^{m-1}(1-e^{\Delta}x)^n dx$$

for $\Delta<0$. From (4) and (5) we have

$$(6)\qquad \beta_{m,n}(\Delta)=\beta_{n,m}(-\Delta)\,.$$

Differentiating (4) with respect to $\Delta$, we have

$$\begin{aligned}(7)\qquad h(\Delta)&=e^{\Delta}\beta'_{m,n}(\Delta)\\ &=-\frac{m\,!\,n\,!}{(m+n)\,!}me^{-(m-1)\Delta}+mn\int_0^1 x^n(1-e^{-\Delta}x)^{m-1}dx\,,\end{aligned}$$

and

$$(8)\qquad h(0)=\frac{m\,!\,n\,!}{(m+n)\,!}(n-m)\,.$$

Since $h(\Delta)$ is strictly increasing for $\Delta\geqq 0$ whenever $m>1$, (a) the equation $h(\Delta)=0$ has a unique solution in the interval $(0,\infty)$ when $n<m$, (b) $h(0)=0$, when $m=n>1$, (c) $h(\Delta)>0$ for $\Delta\geqq 0$ when $n>m>1$. In the case that $m=1$, $\beta'_{1,n}(\Delta)=(n-1)e^{-\Delta}/(n+1)$ and hence $\beta'_{1,n}(\Delta)$ is always positive for $\Delta\geqq 0$ when $n>m=1$.

Considering the symmetric property of $\beta_{m,n}(\Delta)$ as in (6), we can conclude that in all cases except when $m=n=1$, $\beta'_{m,n}(\Delta)=0$ has a unique solution $\Delta=\Delta_0$, $\beta_{m,n}(\Delta)$ is strictly decreasing for $\Delta<\Delta_0$ and strictly increasing for $\Delta>\Delta_0$, and further that $\Delta_0>0$, $\Delta_0=0$ and $\Delta_0<0$ according as $m>n$, $m=n>1$ and $n>m$, respectively. In other words, the test $\phi$ defined by (1) is not unbiased against the two-sided alternatives $\Delta\neq 0$ when $m\neq n$ and is unbiased when $m=n>1$. From (4) and (5) it can easily be seen that

$$\lim_{\Delta\to\pm\infty} \beta_{m,n}(\Delta)=1 . \tag{9}$$

We shall give two numerical examples which are obtained from (4) and (5).

(i) In case $m=19$ and $n=1$: $\alpha=0.10$

$$\beta_{19,1}(\Delta)=\begin{cases} \dfrac{1}{20}\left[e^{-19\Delta}+e^{\Delta}\left\{1-(1-e^{-\Delta})^{20}\right\}\right] & (\Delta\geqq 0) \\ 1-\dfrac{9}{10}e^{\Delta} & (\Delta<0) . \end{cases}$$

(ii) In case $m=n=3$: $\alpha=0.10$

$$\beta_{3,3}(\Delta)=\begin{cases} 1-\dfrac{9}{4}e^{-\Delta}+\dfrac{9}{5}e^{-2\Delta}-\dfrac{9}{20}e^{-3\Delta} & (\Delta\geqq 0) \\ 1-\dfrac{9}{4}e^{\Delta}+\dfrac{9}{5}e^{2\Delta}-\dfrac{9}{20}e^{3\Delta} & (\Delta<0) . \end{cases}$$

The following figure shows the power function $\beta_{m,n}(\Delta)$ for the cases (i) and (ii). In both cases the level of the test $\phi$ is equal to 0.10. In case of (i) the power function attains the minimum value 0.062 at about $\Delta=0.2$ and is less than 0.10 for $0<\Delta<0.7$.

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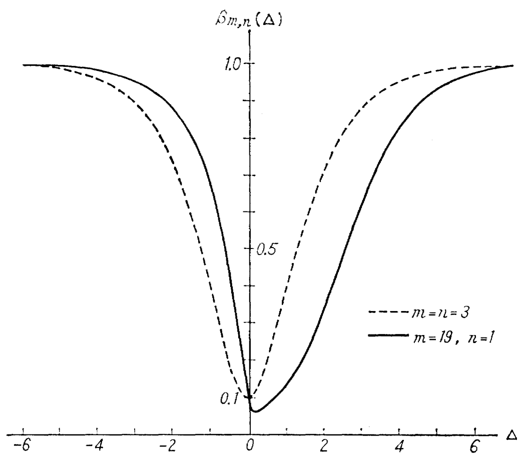


Fig. 1. The power of the test

## REFERENCE

[1] E. L. Lehmann, *Testing Statistical Hypotheses*, John Wiley and Sons Inc., 1959.