NOTE ON THE MULTIVARIATE BURR'S DISTRIBUTION

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1. Introduction

In a paper by I. W. Burr [1], the distribution having a simple algebraic cumulative distribution function was introduced. The function is

$$(1)\qquad F(x)=\begin{cases} 1-\dfrac{1}{(1+x^c)^k} & x\geqq 0 \\ 0 & x<0\,, \end{cases}$$

where c, $k\geqq 1$ are real numbers. The probability density function of this distribution is

$$(2)\qquad f(x)=\frac{kcx^{c-1}}{(1+x^c)^{k+1}}\,.$$

Recently, the Weibull distribution has become to be used widely as the time-to-failure distribution. The probability density function of this distribution is

$$(3)\qquad w(x;\, b,\, \theta)=\begin{cases} \theta\, bx^{b-1}e^{-\theta x^b} & x>0 \\ 0 & x\leqq 0\,, \end{cases}$$

where $\theta>0$, $b>0$.

If θ is a random variable and has a probability density function $g(\theta)$, the resulted time-to-failure distribution has the density function

$$(4)\qquad f(x;\, b)=\int w(x;\, b,\, \theta)g(\theta)d\theta\,.$$

If we assume that $g(\theta)$ is

$$(5)\qquad g(\theta;\, p,\, \alpha)=\begin{cases} \dfrac{\alpha^p}{\Gamma(p)}\theta^{p-1}e^{-\alpha\theta} & \theta>0 \\ 0 & \theta\leqq 0\,, \end{cases}$$

then,

$$(6)\qquad f(x;\, b,\, p,\, \alpha)=\int_0^\infty w(x;\, b,\, \theta)g(\theta;\, p,\, \alpha)d\theta$$

$$=\frac{\alpha^p b x^{b-1}}{\Gamma(p)}\int_0^\infty \theta^p e^{-\theta(x^b+\alpha)}d\theta$$

$$=pb\alpha^p\,\frac{x^{b-1}}{(x^b+\alpha)^{p+1}}$$

where $b,\, p\geqq 1$.

Letting $b=c$, $\alpha=1$, $p=k$, (6) agrees with (2). That is, Burr's distribution [2] is a compound Weibull distribution with a gamma-distribution as a compounder.

It is easy to introduce multivariate Burr's distribution by using this fact. It will be shown in section 3 that these distributions have some nice properties as multivariate distributions.

2. Definition of multivariate Burr's distribution

Let $b_1, \cdots, b_n$ be real numbers not smaller than 1 and $\rho_1, \cdots, \rho_n$ positive numbers. As is easily checked, we have the following equation:

$$(7)\qquad \int_0^\infty \Big(\prod_{k=1}^n b_k\rho_k\theta x_k^{b_k-1}e^{-\rho_k\theta x^b{}_k}\Big)\frac{\alpha^p}{\Gamma(p)}e^{-\alpha\theta}\theta^{p-1}d\theta$$

$$=\alpha^p\frac{\Gamma(p+n)}{\Gamma(p)}\Big(\prod_1^n b_k\rho_k\Big)\frac{\prod_1^n x_k^{b_k-1}}{\Big(\alpha+\sum_1^n \rho_k x_k^{b_k}\Big)^{p+n}}\,.$$

Putting $r_k=\frac{\rho_k}{\alpha}$ in the right side of (7), we define

$$(8)\qquad f(x_1, \cdots, x_n;\, b_1, \cdots, b_n;\, r_1, \cdots, r_n;\, p)$$

$$=\begin{cases}\dfrac{\Gamma(p+n)}{\Gamma(p)}\Big(\prod_1^n b_k r_k\Big)\dfrac{\prod_1^n x_k^{b_k-1}}{\Big(1+\sum_1^n r_k x_k^{b_k}\Big)^{p+n}} & x_i>0\ (i=1, \cdots, n)\\[2ex] 0 & \text{otherwise.}\end{cases}$$

The equation (7) implies that $f(x_1, \cdots, x_n;\, b_1, \cdots, b_n;\, r_1, \cdots, r_n;\, p)$ is the n-dimensional probability density function of compound Weibull (n dimensional direct product) distribution with the gamma distribution as a compounder. We may call this distribution "multivariate Burr's distribution."

3. Properties of the distribution

THEOREM 1. *Any marginal distribution of multivariate Burr's distribution is also (multivariate) Burr's distribution:*

$$\int\cdots\int f(x_1, \cdots, x_s, y_1, \cdots, y_t; b_1, \cdots, b_s, c_1, \cdots, c_t; r_1, \cdots, r_s, q_1, \cdots, q_t; p)dy_1 \cdots dy_t = f(x_1, \cdots, x_s; b_1, \cdots, b_s; r_1, \cdots, r_s; p). \tag{9}$$

PROOF. Since

$$\int_0^\infty \frac{x^c}{(\beta+x)^a}dx = \frac{\Gamma(c+1)\Gamma(a-c-1)}{\beta^{a-c-1}\Gamma(a)} \tag{10}$$

where $\beta>0$, $c\geqq 0$, $a-c\geqq 2$,

$$\begin{aligned}\int_0^\infty f(x_1, \cdots, x_n; b_1, \cdots, b_n; r_1, \cdots, r_n; p)dx_n &= \frac{\Gamma(p+n)}{\Gamma(p)}\left(\prod_1^n b_k r_k\right)\prod_1^{n-1} x_k^{b_k-1}\int_0^\infty \frac{x_n^{b_n-1}}{\{(1+\sum_1^{n-1} r_k x_k^{b_k}) + r_n x_n^{b_n}\}^{p+n}}dx_n \\ &= \frac{\Gamma(p+n-1)}{\Gamma(p)}\left(\prod_1^{n-1} b_k r_k\right)\frac{\prod_1^{n-1} x_k^{b_k-1}}{(1+\sum_1^{n-1} r_k x_k^{b_k})^{p+n-1}}.\end{aligned} \tag{11}$$

The general cases follow from (11).

THEOREM 2. *Any conditional distribution of multivariate Burr's distribution is also (multivariate) Burr's distribution:*

$$\begin{aligned} &f(x_1, \cdots, x_s \mid x_{s+1}=\xi_{s+1}, \cdots, x_n=\xi_n) \\ &= f\left(x_1, \cdots, x_s; b_1, \cdots, b_s; \frac{r_1}{1+\sum_{s+1}^n r_k\xi_k^{b_k}}, \cdots, \frac{r_s}{1+\sum_{s+1}^n r_k\xi_k^{b_k}}; p+n-s\right).\end{aligned} \tag{12}$$

PROOF.

$$f(x_1, \cdots, x_s \mid \xi_{s+1}, \cdots, \xi_n) = \frac{f(x_1, \cdots, x_s, \xi_{s+1}, \cdots, \xi_n)}{\int\cdots\int f(x_1, \cdots, x_s, \xi_{s+1}, \cdots, \xi_n)dx_1 \cdots dx_s}. \tag{13}$$

Applying (8) to the right side of (13) and using formula (9), we can obtain (12).

If $p > \max_{i=1,2} \frac{2}{b_i}$, then the correlation coefficient ρ_{x_1, x_2} of x_1 and x_2 is given by

$$\rho_{x_1, x_2} = \frac{\Gamma(p)\Gamma\left(\frac{1}{b_1}+1\right)\Gamma\left(\frac{1}{b_2}+1\right)\left\{\Gamma(p)\Gamma\left(p-\frac{1}{b_1}-\frac{1}{b_2}\right)-\Gamma\left(p-\frac{1}{b_1}\right)\Gamma\left(p-\frac{1}{b_2}\right)\right\}}{\prod_{i=1}^{2}\sqrt{\Gamma(p)\Gamma\left(\frac{2}{b_i}+1\right)\Gamma\left(p-\frac{2}{b_i}\right)-\Gamma^2\left(p-\frac{1}{b_i}\right)\Gamma^2\left(\frac{1}{b_i}+1\right)}}.$$

For example, if we take $p=b_1=b_2=2$, we get $\rho=\frac{\pi}{4+\pi}$.

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REFERENCES

[1] I. W. Burr, "Cumulative frequency functions," *Ann. Math. Statist.* 13 (1942), 215-232.
[2] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*. Vol. 1, (three volume edition), (1958), 173-174.