ON SOME ASYMPTOTIC PROPERTIES OF A CLASS OF NON-PARAMETRIC TESTS BASED ON THE NUMBER OF RARE EXCEEDANCES*)

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1. Introduction and summary

Non-parametric tests for various problems relating to two or more independent samples, based on the number of rare exceedances, are due to Mosteller [4], Wilks-Rosenbaum [8] [9], Kamat [3], Haga [2], among others. Now, in non-parametric theory, the class of parent distributions are usually of quite general mathematical form, and for this reason, we require that for a broad family of parent distributions, the proposed test (*s*) must be consistent, i. e., the power of the test (*s*) should approach one, as the sample sizes increase. But, unfortunately, the class of tests referred to above fails to be consistent for the entire family of continuous cumulative distribution functions (cdf's). In this paper, the consistency, asymptotic power and some other related properties of these tests are studied in detail.

Firstly, broad families of cdf's are traced here, for which these tests will be consistent or not. In fact, each of the above tests are shown to be inconsistent for various common types of parent cdf's. Secondly, for different classes of alternative specifications and different families of parent cdf's, the asymptotic distributions of these tests criteria are obtained in some simple forms, whence their power properties studied. Thirdly, the scale-tests by Wilks-Rosenbaum and by Kamat are based on the assumed identity of the associated location parameters and cdf's. A comparative study of the power of these two tests for the parent cdf's being normal and the sample sizes being less than 10, is due to Sukhatme [13]. Now, in the general case, we desire to test for the identity of the scale parameters, without assuming the homogeneity of the locations (cf. Sukhatme [12]), and hence the modifications necessary to make such tests invariant under any change of the locations, and the regularity conditions needed to render such modified tests asymptotically distribution-free, are studied here. Similarly, the location tests

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by Wilks-Rosenbaum or by Haga are based on the assumed identity of the associated scale parameters, and the possibility of using them for testing the homogeneity of locations without presuming the homogeneity of the scales, has been thoroughly explored here. Mosteller's k-sample alippage test has also been studied from the points of consistency, asymptotic distribution and some other related properties. Finally, some aspects of the class of tests referred to above are contrasted with those of a different class of non-parametric tests, which may be said, to be based on the number of normal exceedances or on certain U-statistics.

2. Consistency of the class of tests

Instead of studying separately the consistency (or inconsistency) of each of the tests referred to earlier (in section 1), we would consider first the case of the following test, and later, append a theorem on that of the others.

The K-test. Let $X_1, \cdots, X_m$ be a random sample of m units drawn from a universe with a continuous cdf $F_1(x)$, and let $Y_1, \cdots, Y_n$ be a second random sample drawn independently from a second universe with a continuous cdf $F_2(x)$. Let $X_{(r)}$ and $X_{(m-r+1)}$ be respectively the rth smallest and the rth largest observation in the first sample and let, in the second sample k (and k') observations have values greater than $X_{(m-r+1)}$ (and less than $X_{(r)}$). Then the proposed test is based on the values of k (and k') and would be termed the K-test (and K'-test).

Let us first consider the null distribution of K (i. e., when $F_1(x) \equiv F_2(x)$), as this will be required subsequently in the proofs of the theorems to appear. This is easily shown to be

$$p(k|H_0) = r\binom{m}{r}\binom{n}{k}\Big/(r+k)\binom{m+n}{r+k}, \text{ for } k=0, 1, 2, \cdots; \tag{2.1}$$

thus putting $\lambda = n/(m+n)$, and taking m, n both large, subject to a given λ, $0<\lambda<1$, (2.1) reduces to

$$\binom{k+r-1}{r-1}\lambda^k(1-\lambda)^r, \text{ for } k=0, 1, \cdots; \tag{2.2}$$

which is a negative binomial distribution with the parameters (r, λ). Thus, for any given level of significance α, $0<\alpha<1$, the critical values of k are finite and have finite asymptotic limits.

Now, it would be shown later on that the asymptotic properties of the tests depend on the behaviour of the sample extreme values $X_{(r)}$ and $X_{(m-r+1)}$. Also, as regards the asymptotic behaviour of the sample extreme values, the class of parent cdf's of the continuous type may be broadly divided into three types, namely (i) exponential type, (ii)

Cauchy type, and (iii) distributions having a finite extremity (cf. Gumbel [1]). Accordingly, we have considered here each of the above types of parent cdf's separately, and the consistency (or inconsistency) of the tests has been established in the respective cases. Now, if we use the K-test using the righthand side tail of the values of k as the critical region, the test would be termed the R.H. K-test, and a similar convention follows for the L.H. K-test.

(i) *Exponential type of cdf's.*

For this family of cdf's, the cdf F together with its first two derivatives, is continuous everywhere and vanishes at the extremity of the range, which extends to infinity. Writing then

$$\phi_2(x)=[1-F(x)]/F'(x) \quad \text{and} \quad C_2(x)=\frac{d}{dx}\left\{-\phi_2(x)\right\}, \tag{2.3}$$

we may characterise this type of cdf's by

$$\begin{aligned} &\text{(a)} \quad \lim_{x=\infty} F'(x)=0=\lim_{x=\infty} F''(x), \\ &\text{(b)} \quad \lim_{x=\infty} c_2(x)=0 . \end{aligned} \tag{2.4}$$

A similar case follows with the lefthand tail of the distribution, provided the range extends to $-\infty$, on the left. Further, we have classified the exponential type of cdf's into three sub-types (cf. Sen [10]) as follows :

$$\begin{aligned} &\text{(a) Convex exponential type:} && \lim_{x=\infty} \phi_2(x)=0 , \\ &\text{(b) Simple exponential type:} && \lim_{x=\infty} \phi_2(x)=d : 0<d<\infty , \\ &\text{(c) Concave exponential type:} && \lim_{x=\infty} \phi_2(x)=\infty. \end{aligned} \tag{2.5}$$

Then we have the following theorems.

THEOREM 2.1. *For $F_2(x)=F_1(x-d)$, with $H_0: d=0$, and for F_1 belonging to the family of exponential type of cdf's, the R.H. K-test will be consistent for the set of alternatives $H: d>0$, only for the convex type. A similar result holds for the L. H. K-test also.*

PROOF. Writing $k_0(m, n)$ as the critical value of k (R.H. K-test), it follows from (2.1) and (2.2) that for a given $\lambda=n/(m+n): 0<\lambda<1$ and for all $m>m_0$, we can select two integers k_1 and k_2, such that $k_1\leqq k_0(m, n)\leqq k_2$ for all $m\geqq m_0$. It then follows from (2.1), using the simple relation between the incomplete binomial sum and the incomplete Beta integral (cf. Rao [7], p. 33) that for $m\geqq m_0$, and a given λ,

$$P\{k\geqq k_0(m, n)\mid H\}\geqq P\{k\geqq k_2 \mid H\} \tag{2.6}$$

$$=\frac{\sqrt{m+1}}{\sqrt{r}\ \sqrt{m-r+1}}\int_0^1 [F_1(x)]^{m-r}[1-F_1(x)]^{r-1}$$

$$\times\left\{\frac{\sqrt{n+1}}{\sqrt{k_2}\ \sqrt{n-k_2+1}}\int_{F_2(x)}^1 y^{n-k_2}(1-y)^{k_2-1}dy\right\}dF_1(x)\,.$$

Now, it has been shown by the author (Sen [10]) that on defining the characteristic rth largest value $x_{m,r}$ by $F_1(x_{m,r})=(m-r)/m$, that for the entire family of exponential type of cdf's, $x_{m,r}\to\infty$ as $m\to\infty$, while $|X_{(m-r+1)}-x_{m,r}|\xrightarrow{P}0$, only for the convex type. Further from (2.3) and (2.5), we have

$$\text{(2.7)}\qquad \lim_{m=\infty}\left[\frac{m}{r}F_1'(x_{m,r})\right]=\lim_{m=\infty}[1/\phi_2(x_{m,r})]=\lim_{m=\infty}[1/\phi_2(x)]$$

$$=\infty,\ \text{for convex type,}$$

$$<c_1<\infty,\ \text{for non-convex type.}$$

Let us first consider the case of convex exponential type of cdf's, and put $\delta_m=\left[\frac{m}{r}F_1'(x_{m,r})\right]^{-\delta_1}$, where $0<\delta_1<1$; $a_m^{(1)}=m[1-F_1(x_{m,r}+\delta_m)]$, and $a_m^{(2)}=m[1-F_1(x_{m,r}-\delta_m)]$. It then follows from (2.7) and some simple computations that $\delta_m\to 0$, $a_m^{(1)}\to 0$, and $a_m^{(2)}\to\infty$ as $m\to\infty$, and writing $Z=m[1-F_1(x)]$ and $W_{z.d}=n[1-F_2(x)]=n[1-F_1(x-d)]$, in (2.6), the r.h.s. of (2.6) can be written as

$$\text{(2.8)}\qquad \frac{1}{\sqrt{r}}\int_{a_m^{(1)}}^{a_m^{(2)}} e^{-z}z^{r-1}\left[\frac{1}{\sqrt{k_2}}\int_0^{W_{z.d}} e^{-w}w^{k_2-1}dw\right]dz+\eta_m^{(1)}$$

where $\eta_m^{(1)}\to 0$ as $m\to\infty$. Again, we get following some simple steps that

$$\text{(2.9)}\qquad W_{z.d}=\rho z e^{d/\phi_2(x-\theta d)}\,,\quad 0<\theta<1\,;\quad \rho=\lambda/(1-\lambda)$$

and hence, it can be shown with little effort that for all $a_m^{(1)}\leqq z\leqq a_m^{(2)}$, $\lim_{m=\infty} W_{z.d}=\infty$. Hence, from (2.8), it readily follows that

$$\lim_{m=\infty} P\{k\geqq k_0(m,\ n)\,|\,H\colon d>0\}=1\,.$$

Hence, the consistency of the R.H. K-test for convex exponential type of cdf's. Again, to establish the inconsistency of the test for the other two exponential type of cdf's, we have

$$\text{(2.10)}\qquad P\{k\geqq k_0(m,\ n)\,|\,H\colon d>0\}\leqq P\{k\geqq k_1\,|\,H\colon d>0\}$$

$$=\frac{\sqrt{m+1}}{\sqrt{r}\ \sqrt{m-r+1}}\int_0^1 [F_1(x)]^{m-r}[1-F_1(x)]^{r-1}$$

$$\times\left\{\frac{\sqrt{n+1}}{\sqrt{k_1}\ \sqrt{n-k_1+1}}\int_{F_2(x)}^{1} y^{n-k_1}(1-y)^{k_1-1}dy\right\}dF_1(x)\,.$$

Let us now select a sequence of values $\{x_m^*\}$, such that $F_1(x_m^*)=(m-r_1)/m$, r_1 being any positive quantity chosen arbitrarily. It then follows readily that $P\{x_{(m-r+1)}>x_m^*\}>0$, and this can be made to converge to any limit $p^*: 0<p^*<1$, (with $m\to\infty$) by a proper choice of r_1. Defining then Z and $W_{z.d}$ as before, we have for all $z\leqq r_1$

$$W_{z.d}=\rho z e^{d/\phi_2(x-\theta d)}\,,\qquad 0<\theta<1$$

and hence by (2.7), it follows readily that for all $z\leqq r_1$

$$\lim_{m=\infty} W_{z.d}\leqq \rho z e^{d/c_1}\leqq \rho r_1 e^{d/c_1}\leqq c_2<\infty\,. \tag{2.11}$$

Thus, we have

$$\operatorname*{Sup}_{z\leqq r_1}\left\{\lim_{m=\infty}\frac{\sqrt{n+1}}{\sqrt{k_1}\ \sqrt{n-k_1+1}}\int_{F_2(x)}^{1} y^{n-k_1}(1-y)^{k_1-1}dy\right\} \tag{2.12}$$

$$\leqq\frac{1}{\sqrt{k_1}}\int_0^{c_2} e^{-w}w^{k_1-1}dw=\hat{p}<1\,.$$

Thus, from (2.10) and (2.12), we get following simple steps that

$$\lim_{m=\infty} P\{k\geqq k_0(m,\,n)\,|\,H: d>0\}\leqq\lim_{m=\infty} P\{k\geqq k_1\,|\,H: d>0\}$$

$$\leqq p^*\hat{p}+(1-p^*)<1\,.$$

Hence, the inconsistency of the R.H. K-test for non-convex exponential type of cdf's.

The case with the L.H. K-test follows precisely on the same line. Hence, the theorem.

THEOREM 2.2. *If $F_2(x)=F_1(\nu x)$ with $H_0: \nu=1$ and against the set of alternatives $H: \nu<1$ (or $\nu>1$), then the R.H. K-test (or the L.H. K-test) will be consistent for the entire family of exponential type of cdf's.*

PROOF. As in the proof of the preceding theorem, we define $z=m[1-F_1(x)]$ and $W_{z.\nu}=n[1-F_2(x)]=n[1-F_1(\nu x)]$; we have then

$$W_{z.\nu}=\rho z\exp\{(1-\nu)x/\phi_2(x-\theta(1-\nu)x)\},\qquad 0<\theta<1\,. \tag{2.13}$$

It is also known that for the entire family of exponential type of cdf's, $x/\phi_2(x)\to\infty$ as $x\to\infty$ (cf. Gumbel [1], p. 126), and hence, it can be shown with little effort that in probability one,

$$X_{(m-r+1)}/\phi_2\{(1-\theta(1-\nu))X_{(m-r+1)}\}$$

can be made arbitrarily large. Thus, from (2.13), we get that $W_{z.\nu}$ can be made arbitrarily large (if m is taken large) in probability one. Hence, proceeding precisely on the same line as in (2.6) through (2.9), we get that

$$\lim_{m=\infty} P\{k \geqq k_0(m, n) \mid H: \nu<1\}=1.$$

A similar proof also applies to the L.H. K-test.

Hence, the theorem.

(ii) *Cauchy type of cdf's.*

For this type of cdf also, the cdf F and its first two derivatives are continuous and vanish at the extremities of the range, which extends to infinity. But here the cdf F is characterised as follows:

$$\text{(2.14)} \qquad \lim_{x=\infty} c_2(x)=-1/c_2\,, \quad c_2>0, \text{ i.e., } \lim_{x=\infty} x^{c_2}[1-F(x)]=A<\infty,$$

where $c_2(x)$ has been defined in (2.3). A similar characterisation also applies to the lower extremity.

THEOREM 2.3. *If $F_2(x)=F_1(x-d)$(or $F_1(\nu x)$) with $H_0: d=0$ (or $\nu=1$) against the set of alternatives $H: d>0$ (or $\nu<1$), then the R.H. K-test will be inconsistent for any $F_1(x)$ belonging to the family of Cauchy type of cdf's. A similar result holds for the L.H. K-test.*

PROOF. Defining Z and $W_{z.d}$ (or $W_{z.\nu}$) as in the proofs of theorems 2.1 and 2.2, we have by virtue of (2.14) that for all $Z \leqq r_1$ (r_1 being defined as in the proof of theorem 2.1)

$$\text{(2.15)} \qquad \begin{aligned} &\sup_{z \leqq r_1} \{\lim_{m=\infty} W_{z.d}\} \leqq \rho r_1 < \infty, \\ &\sup_{z \leqq r_1} \{\lim_{m=\infty} W_{z.\nu}\} \leqq \rho r_1/\nu^{c_2} < \infty\,, \end{aligned}$$

and hence, proceeding precisely as in (2.10) through (2.12), the theorem follows.

Thus for Cauchy type of cdf's, the K-test will be inconsistent for both location and scale variations.

(iii) *Distribution having a finite extremity.*

THEOREM 2.4 *If $F_1(x)$ has a finite upper end-point a, i.e., $E_1(a)=1$, then the R.H. K-test (or L.H. K-test) is consistent against the set of alternatives that $F_2(a)<1$ (or $F_2(a-d)=1$ for some $d>0$), provided both $F_1(x)$ and $F_2(x)$ are strictly monotonic at a (or $a-d$). A similar result holds for the lower extremity also, provided the same is finite.*

The proof is simple and is left to the reader. In the connection, it is worth studying the following consequences of theorem 2.4.

(a) If we let $F_2(x)=F_1(x-d)$, then for $d>0$, $F_2(a)<1$, and $d<0$, $F_2(a-d')=1$, for some $d'>0$. Consequently the K-test will be consistent for any shift in location.

(b) If we let $F_2(x)=F_1(\nu x)$ with $H_0: \nu=1$, it then follows similarly that the K-test will be consistent for any $\nu\neq 1$, provided that $a\neq 0$.

The consistency (or inconsistency) of the K'-test follows precisely on the same line. We will now consider the case of the tests referred to earlier.

THEOREM 2.5. *The tests by Wilks-Rosenbaum, Kamat or Haga will be consistent (or not) according as either of our (or both) K-test and K′-test is consistent (or not).*

PROOF. Let k' and k be respectively the number of observations in the second sample lying below $X_{(1)}$ and above $X_{(m)}$. Then the Wilks-Rosenbaum scale test is based on $W=k+k'$, and the critical region is demarcated as $W\geqq W_0$ (or $W\leqq W_0$) for the alternative hypothesis $H: \nu< 1$ (or $\nu>1$), in the set up $F_2(x)=F_1(\nu x)$. Now writing $x=X_{(1)}$, $y=X_{(m)}$, $z_1=mF_1(x)$, $Z_2=m[1-F_1(y)]$ and $V_{z_1z_2\cdot\nu}=n[F_2(x)+1-F_2(y)]$, the distribution of W, after some simplification, reduces asymptotically to

$$(2.16)\quad P\{W\geqq W_0\,|\,H:\nu\}=\int_0^\infty\int_0^\infty e^{-(z_1+z_2)}\left\{\frac{1}{\sqrt{W_0}}\int_0^{V_{z_1z_2\cdot\nu}} e^{-w}w^{W_0-1}dw\right\}dz_1\,dz_2\,.$$

Hence, proceeding precisely on the same line as in theorem 2.1, we are to show that this test will be consistent, if for all $a_m^{(1)}\leqq Z_1$, $Z_2\leqq a_m^{(2)}$ (where $a_m^{(1)}$, $a_m^{(2)}$ are defined earlier, in the proof of theorem 2.1) $V_{z_1z_2\cdot\nu}\to\infty$ as $m\to\infty$. Since

$$(2.17)\qquad V_{z_1z_2\cdot\nu}=nF_2(x)+n[1-F_2(y)]\,,$$

it will tend to ∞ with $m\to\infty$, provided at least one of $nF_2(x)$ and $n[1-F_2(y)]\to\infty$ with $m\to\infty$, i.e., at least one of the K'-test and K-test is consistent. Hence, Wilks-Rosenbaum's scale test will be consistent, if at least one of our K-test and K'-test is consistent. Again, if both the K-test and K'-test are inconsistent, we have for all Z_1, $Z_2\leqq r_1$ (for any suitably chosen positive quantity r_1), $nF_2(X)$ and $n[1-F_2(y)]$ converging to some finite limits as $m\to\infty$, and consequently, from (2.17) we note that for all Z_1, $Z_2\leqq r_1$, $V_{Z_1Z_2\cdot\nu}$ converges to some finite limit as $m\to\infty$. Hence, from (2.16) we directly get that $\lim_{m=\infty} P\{W\geqq W_0\,|\,H:\nu<1\}<1$, i.e., the Wilks-Rosenbaum's test is also inconsistent. The case with the left hand sided (critical region) test with W follows precisely on the same line.

Now let l' and l be the number of observations in the first sample lying below $Y_{(1)}$ and above $Y_{(n)}$ respectively. Then Kamat's test may be put into an equivalent form

$$d=k+k'-(l+l') \tag{2.18}$$

and the critical region may be demarcated as $|d|\geqq d_0$.

Now, out of k, k', l and l', two and only two are positive integers, and the remaining two must be equal to zero. Thus, we may have the following cases: (i) $d=k+k'$, (ii) $d=k-l'$, (iii) $d=k'-l$ and (iv) $d=-(l+l')$, (where k, k', l, l' are all positive in the cited cases), and in either case, proceeding more or less on the same line as in Wilks-Rosenbaum's test, we arrive at the same conclusion, viz., Kamat's test will be consistent if either of the K-test and K'-test is so, and if both these tests are inconsistent, so will be Kamat's test.

Finally, Haga's two-sample location test is based on

$$S=(k-k')-(l-l'), \tag{2.19}$$

and the critical region is given by $|S|\geqq S_0$. As in Kamat's test we have either of the four cases: (i) $S=k-k'$, (ii) $S=k+l'$, (iii) $S=-(k'+l)$ and (iv) $S=l'-l$ (where in each case k, k', l, l' are positive), and then proceeding more or less on the same line as in Wilks-Rosenbaum's test, it can be readily proved that Haga's test will be consistent if at least one of our k-test and k'-test is so (for location), while if both these tests are inconsistent, so will be Haga's test.

Hence, the theorem.

Theorem 2.1 to 2.5 clearly give an idea about the class of parent cdf's, for which the class of tests referred to above will be consistent or not.

3. Asymptotic distributions of the test criteria

We will now consider different families of alternative specifications relating to changes in location or scale parameters, and the asymptotic distributions of the class of tests referred to earlier, will be deduced for such alternatives. Obviously, families of alternative specifications will be related with the sample sizes, in such a way that the tests have a power function different from zero or unity. As for the Cauchy type of cdf's, all these tests are shown to be inconsistent, this type of cdf's will not be considered here. Similarly for location tests, non-convex exponential type of cdf's will not be considered. Here also, we will consider first the case of k and later pass on to that of others.

(i) *Exponential type of cdf's.*

THEOREM 3.1 *There exists a monotonically non-decreasing function*

$N=N(m, n)$, such that for the class of alternatives $F_2(x)=F_1(x-d/N)$ or $F_1(x-xd/N)$, with a real and finite d, and for n, m both large, subject to $n/(m+n)=\lambda$, $0<\lambda<1$; the asymptotic distribution of k is given as

$$p(k\mid d)=\binom{k+r-1}{r-1}\lambda_d^k(1-\lambda_d)^r, \quad \text{for } k=0, 1, \cdots ;$$

where $\lambda_d=\lambda e^d/\{1+\lambda(e^d-1)\}$, and where
(i) for $F_2(x)=F_1(x-d/N)$ with F_1 belonging to the family of convex exponential type of cdf's,

$$N=\frac{m}{r}F_1'(x_{m,r}) \text{ with } F_1(x_{m,r})=(m-r)/m ; \quad \text{and}$$

(ii) for $F_2(x)=F_1(x-xd/N)$ with F_1 belonging to the family of non-concave exponential type of cdf's,

$$N=\frac{m}{r}x_{m,r}F_1'(x_{m,r})$$

(and for concave exponential type of cdf's, some additional restrictions appear to be necessary, namely, instead of $\lim_{x=\infty} x/\phi_2(x)=\infty$ (cf. Gumbel, [1], p. 126), $x/\phi_2(x)$ is monotonic for $x\geqq x_0$, and $xc_2(x)/\phi_2(x)$ is bounded, even proceeding to the limit $x\to\infty$).

PROOF. The distribution of k is given by

$$(3.1)\quad p(k\mid d)=r\binom{m}{r}\int_0^1[F_1(x)]^{m-r}[1-F_1(x)]^{r-1}\binom{n}{k}[F_2(x)]^{n-k}[1-F_2(x)]^k dF_1(x).$$

Now for the class of exponential type of cdf's, we have from (2.3) and (2.7) that

$$(3.2)\qquad 1-F_1(x)=e^{-\phi_2(x)}, \text{ where } \phi_2'(x)=1/\phi_2(x)=F_1'(x)/[1-F_1(x)].$$

Thus for the convex exponential type of cdf's, we have (i) $\lim_{x=\infty}\phi_2'(x)=\infty$, and (ii) $\lim_{x=\infty}C_2(x)=0$. Hence, writing $z=m[1-F_1(x)]$ and $w=n[1-F_2(n)]=n[1-F_1(x-d/N)]$, (where $x=X_{(m-r+1)}$) and noting that by definition $N=\phi_2'(x_{m,r})$, we get after some simple but somewhat lengthy algebraic manipulations that $w\xrightarrow{P}\rho z e^d$, where $\rho=\lambda/(1-\lambda)$. Hence, from (3.1), we get that $p(k|d)$ asymptotically reduces to

$$(3.3)\qquad \frac{1}{\sqrt{r}}\frac{1}{k!}\int_0^\infty e^{-z}z^{r-1}e^{-z\rho e^d}(z\rho e^d)^k dz=\binom{r+k-1}{r-1}\lambda d^k(1-\lambda d)^r,$$

for $k=0, 1, 2, \cdots$;

where $\qquad \lambda d=\lambda e^d/\{1+e^d(1-\lambda)\}$.

For the two-sample scale problem i.e., $F_2(x)=F_1(x-xd/N)$ with $N=x_{m,r}\ \psi_2'(x_{m,r})$, we have for the convex exponential type of cdf's, essentially by the same technique that $w \xrightarrow{P} z\rho e^d$ and hence as in (3.3), we arrive at the desired result.

Also for simple exponential type of cdf's, we have (i) $\lim_{x=\infty} c_2(x)=0$, but (ii) $\lim_{x=\infty}\psi_2'(x)=1/d_2$, $(d_2<\infty)$. Thus, selecting two sequences of values $\{C_{i,m}\}i=1, 2$, with $1-C_{1,m}=C_{2,m}-1=x_{m,r}^{-\delta_1}$, $0<\delta_1<1$, it readily follows that

$$\lim_{m=\infty} P\{X_{(m-r+1)}\in[c_{1,m}x_{m,r},\ c_{2,m}x_{m,r}]\}=1\,. \tag{3.4}$$

And for all x lying within $(C_{1,m}x_{m,r},\ C_{2,m}x_{m,r})$, we have

$$\psi_2(x-xd/N)=\psi_2(x)-d+\eta_N \tag{3.5}$$

where $N=x_{m,r}\psi_2'(x_{m,r})$ and $\eta_N\to 0$ as $m\to\infty$. Thus, we have $w\xrightarrow{P} z\rho e^d$ and hence we arrive at (3.3).

Now for concave exponential type of cdf, we have (i) $\lim_{x=\infty} c_2(x)=0$, (ii) $\lim_{x=\infty}\psi_2'(x)=0$, but (iii) $\lim_{x=\infty} x\psi_2'(x)=\infty$. Also, as to our assumption, we have for all $x\geqq x_0$, $x\psi_2'(x)$ non-decreasing with $\lim_{x=\infty} x\psi_2'(x)c_2(x)\leqq c<\infty$ (and this additional restriction is found to be satisfied by the majority of the referred type of distributions). In this case, we select two sequences of values $\{C_{i,m}\}$, $i=1, 2$, as $1-C_{1m}=C_{2m}-1=\{x_{m,r}\psi_2'(x_{m,r})\}^{-\delta_1}$, $0<\delta_1<1$; then it can be shown after some algebraic manipulations (c.f. Sen ([10], p. 305) that

$$\lim_{m=\infty} P\{X_{(m-r+1)}\in[c_{1,m}x_{m,r},\ c_{2,m}x_{m,r}])\}=1\,. \tag{3.6}$$

Also,

$$\begin{aligned} w&=\rho m e^{-\psi_2(x-xd/N)}\,; \qquad N=x_{m,r}\psi_2'(x_{m,r}) \\ &=\rho m e^{-\psi_2(x)+dx\psi_2'(\xi)/x_{m,r}\psi_2'(x_{m,r})} \\ &\qquad (\xi=x-\lambda_1 xd,\ 0<\lambda_1<1)\,. \end{aligned} \tag{3.7}$$

Also, it can be shown after some lengthy computations that for all x lying within the interval referred to in (3.6),

$$x\psi_2'(\xi)/x_{m,r}\psi_2'(x_{m,r})=1+\eta_m \tag{3.8}$$

where $\eta_m\to 0$ as $m\to\infty$. Thus from (3.7) and (3.8), we get that $w\xrightarrow{P}\rho z e^d$ and hence we arrive at (3.3).

Hence, the theorem.

(ii) *Distributions having a finite end-point.*

THEOREM 3.2 *If $F_2(x)=F_1(x-d/n_1)$ (or $F_1(x[1-d/n_1])$), where a ($<\infty$) is the upper end-point of the cdf, and d is real and finite, then the distribution of k asymptotically reduces to*

$$p(k\,|\,d)=e^{-\rho\Delta}\sum_{s=0}^{k}\frac{\Delta^s}{s\,!}\binom{r+k-s-1}{r-1}\lambda^k(1-\lambda)^{r-s}\ ,\ for\ k=0,\ 1,\ \cdots;$$

where $\rho=\lambda/(1-\lambda)$ and $f_1(a)=F_1'(a)$ and $\Delta=df_1(a)$ (or $adf_1(a)$).

Further, if we have for some positive integer p,

$$1-F_1(a)=F_1'(a)=\cdots=F_1^{(p)}(a)=0 \quad and \quad f_1^{(p)}(a)=F_1^{(p+1)}(a)\neq 0\ ,$$

then for

$$F_2(x)=F_1(x-dm^{-1/(p+1)})\ (or\ F_1(x-xdm^{-1/(p+1)}))\ ,$$

the distribution of k asymptotically reduces to

$$\frac{\rho^k}{\Gamma r\,k\,!}\int_0^\infty\left\{e^{-y(1+\rho)-(-1)^p\rho\sum_{s=1}^{p+1}\binom{p+1}{s}(\Delta/c_p)^s y^{1-s/(p+1)}}\right\}$$
$$\cdot\{y^{k+r-1}[1+(-1)^p\textstyle\sum_{s=1}^{p+1}\binom{p+1}{s}(\Delta/cp)^s y^{-s/(p+1)}]^k\}dy$$

where $c_p=\{(p+1)\,!/\,|f_1^{(p)}(a)\,|\,\}^{1/(p+1)}$ and $\Delta=d$ (or ad).

PROOF. Here also we write $z=m[1-F_1(x)]$ and $w=n[1-F_2(x)]$. Then for $f(a)>0$, we have after some simplifications

$$w=\rho m[1-F_1(x-d/n_1)]\xrightarrow{P}\rho(z+\Delta)\ \text{where}\ \Delta=df_1(a)\ . \tag{3.9}$$

Substituting (3.9) in (3.1), we get the asymptotic distribution of k, as sketched in the theorem. Also in the case $F_2(x)=F_1(x-dx/n_1)$ with $f_1(a)>0$, we get similarly that $w\xrightarrow{P}\rho(z+\Delta)$ where $\Delta=adf_1(a)$, and hence, the same asymptotic distribution also applies to k.

Hence, the first part of the theorem.

For the second part of the theorem, we note that for $F_2(x)=F_1(x-dm^{-1/(p+1)})$, we have under the stated regularity conditions,

$$\begin{aligned} w&=n[1-F_2(x)]=\rho z+\rho m[F_1(x)-F_2(x)]\\ &\xrightarrow{P}\rho z+(-1)^p\textstyle\sum_{s=1}^{p+1}\binom{p+1}{s}y^{1-s/(p+1)}(d/C_p)^s,\qquad y=m[1-F_1(x)]\ ;\end{aligned}\tag{3.10}$$

where C_p is defined in the theorem. Thus, from (3.1) and (3.10), we arrive at the desired result. In the case of $F_2(x)=F_1(x[1-dm^{-1/(p+1)}])$ we have

$$x-\frac{xd}{N}=x-\frac{da}{N}+\frac{d(a-x)}{N}\qquad \text{where}\ N=m^{1/(p+1)}$$

and as $(a-x) \xrightarrow{P} 0$, we get similarly that (3.10) also holds true, in this case, and hence, we arrive at the same result.

Hence the theorem.

(iii) *Asymptotic distributions of other test criteria.*

We have so far sketched the asymptotic distribution of k. The case of k' follows precisely on the same line. Thus for exponential type of cdf's, we are to define ${}_{m,r}x$ by $F_1({}_{m,r}x)=r/m$, $\psi_1'(x)=F_1'(x)/F_1(x)$, and $N'=\psi_1'({}_{m,r}x)$ or $|{}_{m,r}x\psi_1'({}_{m,r}x)|$ according as $F_2(x)=F_1(x-d/N')$ or $F_1(x-dx/N')$, ψ_1' being the first derivative of ψ_1. Now, for the asymptotic distribution of a test criterion, which is based both on k and k', we require that $\lim_{m=\infty} N'/N=\theta$, $0<\theta<\infty$; as otherwise for any given sequence of alternatives, either of k and k' will degenerate to zero or will be asymptotically greater than any preassigned finite quantity, so that the asymptotic distribution of the given test criterion will cease to be of any interest. Now, for distributions symmetrical about their medians, obviously $N=N'$, while for asymmetrical distributions N'/N will tend to a finite limit, for all r, only under very restrictive regularity conditions. Thus, for the class of exponential type of cdf's, we will consider only the symmetric ones for the asymptotic distribution of Wilks-Rosenbaum, Kamat, and Haga's test criteria. Similarly, for distributions having finite end-points, we will consider only the case, where the two terminal contacts are of the same order.

With these, we get after some simple algebraic manipulations that for the Wilks-Rosenbaum's test criterion w, we have asymptotically

$$p(w\,|\,H)=\sum_{k=0}^{w} p(k\,|\,\underset{r=1}{H})\cdot p(k'=w-k\,|\,\underset{r=1}{H})\,, \tag{3.11}$$

and hence, for exponential type of cdf's, we have for $H: F_2(x)=F_1(x-dx/N)$,

$$p(w\,|\,H)=(w+1)\lambda_d^w(1-\lambda_d)^2, \quad \text{for } w=0, 1, \cdots; \tag{3.12}$$

where

$$\lambda_d=\lambda e^d/\{1+e^d(1-\lambda)\};\ \ \lambda=n/(m+n),\ \ 0<\lambda<1\,.$$

Thus Wilks-Rosenbaum's test is power-equivalent to our K-test with $r=2$. The distribution of w for distributions having finite end-points (say a and b) may be shown to be asymptotically equal to

$$p(w\,|\,H)=e^{-\rho(\Delta+\Delta')}\sum_{s=0}^{w}\frac{(\Delta+\Delta')^s}{s\,!}(w-s+1)\lambda^w(1-\lambda)^{1-s}\,, \quad w=0, 1, \cdots; \tag{3.13}$$

where

$$\Delta = adf(a) \text{ and } \Delta' = -bdf(b),\ b<a, \quad \text{and} \quad \rho = \lambda/(1-\lambda).$$

With l and l' defined as in (2.18), we may put Kamat's test criterion in an equivalent form: $d=k+k'-(l+l')$. Consequently, d may assume the value $D(\geqq 0)$ in either of the following way:

$$\text{(i)} \quad \left.\begin{matrix} k=j,\ k'=D-j \\ l=l'=0 \end{matrix}\right\} j=1, \cdots, D-1, \qquad D\geqq 2,$$

$$\text{(ii)} \quad \left.\begin{matrix} k=D+j,\ l'=j \\ k'=l=0 \end{matrix}\right\} j=1, \cdots, \left[\frac{m+n}{2}-D\right],\ D\geqq 0,$$

$$\text{(iii)} \quad \left.\begin{matrix} k'=D+j,\ l=j \\ k=l'=0 \end{matrix}\right\} j=1, \cdots, \left[\frac{m+n}{2}-D\right],\ D\geqq 0.$$

For negative D, the values of (k, k') and (l, l') are to be interchanged. Now, with simple but somewhat lengthy computations, it can be shown that for symmetrical cdf's of the exponential type, and for the class of scale-alternatives sketched in theorem 3.1, we have

$$\text{(3.14)} \quad p(D\,|\,H) = \begin{cases} (1-\lambda_d)^{-D}\lambda_d^2\{2(1-\lambda_d)^2-(1-Q_d)(1+D)\}/(1-Q_d) & (D<0) \\ 2Q_d^2/(1-Q_d) & (D=0) \\ \lambda_d^D(1-\lambda_d)^2\{2\lambda_d^2-(1-Q_d)(1-D)\}/(1-Q_d) & (D>0) \end{cases}$$

where

$$\lambda_d = \lambda e^d/\{1+e^d(1-\lambda)\}, \quad \lambda = n/(m+n) \quad \text{and} \quad Q_d = \lambda_d(1-\lambda_d).$$

This asymptotic distribution reduces to the null one, when $d=0$, and the two forms are similar to each other. In the case of distributions with finite extremities, the asymptotic distribution of D can not be put in such a simple form. However, it can be readily shown that even under the class of alternative hypotheses, sketched in theorem 3.2, (i) k and k', (ii) l and l', (iii) k and l', and (iv) k' and l are asymptotically independent, and hence, the distribution of D can be obtained, for large samples, as the convolution of the distributions of (i) k and k', (ii) k and l', (iii) k' and l and (iv) l and l'. This, however, fails to be any simple and compact one.

Finally, Haga's two-sample location test criterion may be written as

$$s = k-k'-(l-l').$$

Thus s may assume the value $S(\geqq 0)$ in the following way

$$\text{(i)} \quad \left.\begin{matrix} k=s+j,\ k'=j \\ l=l'=0 \end{matrix}\right\} \quad j=1, \cdots, [(n-s)/2];$$

$$\text{(ii)} \quad \left.\begin{matrix} l'=s+j,\ l=j \\ k=k'=0 \end{matrix}\right\} \quad j=1, \cdots, [(m-s)/2];$$

$$\text{(iii)}\quad \left.\begin{array}{l} k=j,\ l'=s-j \\ k'=l=0 \end{array}\right\}\ j=1,\ \cdots,\ s-1,\ s\geqq 2\,.$$

In this case also, we have for symmetric exponential type of cdf's, the distribution of s, asymptotically reducing to

$$(3.15)\qquad p(S\,|\,H)=\begin{cases} \lambda_d^2(1-\lambda_d)^2\left\{\dfrac{1}{1-\lambda_d^2}+\dfrac{1}{1-(1-\lambda_d)^2}\right\} & (S=0) \\ \dfrac{\{(1-\lambda_d)^2+\lambda_d\}\lambda_d(1-\lambda_d)}{(1-2\lambda_d)}\left\{\dfrac{(1-\lambda_d)^s}{2-\lambda_d}-\dfrac{\lambda_d^s}{1+\lambda_d}\right\} & (S>0) \\ \dfrac{\{\lambda_d^2+(1-\lambda_d)\}\lambda_d(1-\lambda_d)}{(2\lambda_d-1)}\left\{\dfrac{\lambda_d^{-s}}{1+\lambda_d}-\dfrac{(1-\lambda_d)^{-s}}{2-\lambda_d}\right\} & (S<0)\,, \end{cases}$$

where

$$\lambda_d=\lambda e^d/\{1+e^d(1-\lambda)\},\ \lambda=n/(m+n);\qquad H\colon F_2(x)=F_1(x-d/N)\,,$$

N being defined as in theorem 3.1. In the case of distributions having finite extremities, here also the distribution of S will not appear asymptotically in a simple and compact form, and have to be expressed as a convolution of the asymptotic distributions of (i) k, k', (ii) k, l', (iii) k', l and (iv) l, l'.

Thus, a study of the asymptotic distribution of (k, k') gives a basis of the same for all the remaining tests, referred to above.

4. On studentization of the class of tests

The location tests considered so far are based on the assumed identity of the associated scale parameters and similarly the scale tests are based on the postulated identity of the locations. In this section, we will test for the homogeneity of either of the parameters without presuming the homogeneity of the other ones. Thus, for our purpose, we let

$$(4.1)\qquad F_i(x)=F([x-\mu_i]/\delta_i)\ \text{for}\ i=1,\ 2\,;$$

where μ_i and δ_i are respectively the location and scale parameters of the ith population, for $i=1, 2$. Thus on writing $U_i=(X_i-\mu_1)/\delta_1$, $i=1, \cdots, m$ and $V_i=(Y_i-\mu_2)/\delta_2$, $i=1, \cdots, n$, it follows that U's and V's have the common cdf $F(u)$.

Now to test for the homogeneity of δ_1 and δ_2 without assuming the locations μ_1 and μ_2 to be the same, we propose to base the tests on the observations centered at their respective sample medians, and similarly for testing $H_0\colon \mu_1=\mu_2$ without assuming $\delta_1=\delta_2$, the tests are based on the variables studentized by their respective scale parameters, as estimated from the samples. The regularity conditions needed for these studentized tests to be asymptotically distribution-free, are studied here. Here also, we consider in detail the case of our K-test and later append a discussion on that of the others.

Modified K-test (Scale). We define μ_1 and μ_2 as respectively the medians of the first and second populations and let $\tilde{X}$ and $\tilde{Y}$ be their sample counter-parts. We then write $\hat{X}_i = X_i - \tilde{X}$, $i=1, \cdots, m$, $\hat{X}^{(1)} = X_{(m-r+1)} - \tilde{X}$ and $\hat{Y}_i = Y_i - \tilde{Y}$ for $i=1, \cdots, n$. Then the modified K-test, called the K_s-test is based on the number of $\hat{Y}$'s (say, k_s) larger than that of $\hat{X}^{(1)}$. Now for small samples, the distribution of k_m will be somewhat involved and depend appreciably on the unknown $F(u)$. However, for large samples, it will be shown here that under certain regularity conditions, k_s has asymptotically a nonparametric distribution.

Let us write $\tilde{U} = (\tilde{X} - \mu_1)/\delta_1$, $\tilde{V} = (\tilde{Y} - \mu_2)/\delta_2$, $\nu = \delta_1/\delta_2$ and $U^{(1)} = (X_{(m-r+1)} - \mu_1)/\delta_1$. Then we can write k_s equivalently as the number V_i's ($i=1, \cdots, n$) having values greater than that of

$$U_s^{(1)} = \tilde{V} + \nu(U^{(1)} - \tilde{U}) \tag{4.2}$$

where under $H_0 : \delta_1 = \delta_2$, i.e., $\nu = 1$.

Asymptotic null distribution of k_s. Let $\tilde{X} = X_{(a)}$ and $\tilde{Y} = Y_{(b)}$. Here we will consider only values of $k_s \leqq n-b$, as it can be shown that under $H_0 : \nu = 1$, $P\{k_s > n-b\}$ converges to zero, exponentially with $m, n \to \infty$. Thus for any $k_s = k(\leqq n-b)$, $p_s(k) = P\{k_s = k\}$ is given by

(4.3)

$$a\binom{m}{a}\int\cdots\int_A [F(\tilde{U})]^{a-1}[1-F(\tilde{U})]^{m-a} b\binom{n}{b}[F(\tilde{V})]^{b-1}[1-F(\tilde{V})]^{n-b}$$

$$\cdot r\binom{m-a}{r}\left[\frac{F(U^{(1)})-F(\tilde{U})}{1-F(\tilde{U})}\right]^{m-a-r}\left[\frac{1-F(U^{(1)})}{1-F(\tilde{U})}\right]^{r-1}$$

$$\cdot \binom{n-b}{k}\left[\frac{F(U_s^{(1)})-F(\tilde{V})}{1-F(\tilde{V})}\right]^{n-b-k}\left[\frac{1-F(U_s^{(1)})}{1-F(\tilde{V})}\right]^{k} dF(\tilde{U})dF(U^{(1)})dF(\tilde{V}),$$

where $A : -\infty \leqq \tilde{U} < U^{(1)} \leqq \infty$, $-\infty \leqq \tilde{V} \leqq \infty$.

Let us now divide the domain A into two disjoint domains R and R^*, where R is defined by the simultaneous inequalities $|\tilde{U}| < \varepsilon/2$, $|\tilde{V}| < \varepsilon/2$. It can then be shown with little effort that for $m \geqq m_0(\varepsilon, \delta)$ and a given value of $\lambda = n/(m+n)$, $0 < \lambda < 1$, $P(R) > 1-\delta$. Also we let $\rho = \lambda/(1-\lambda)$ and

$$Z=m[1-F(U^{(1)})],\ Z_s=(m-a)[1-F(U^{(1)})]/[1-F(\tilde{U})]\,, \tag{4.4}$$
$$W=n[1-F(U_s^{(1)})],\ \text{and}\ W_s=(n-b)[1-F(U_s^{(1)})]/[1-F(\tilde{V})]\,.$$

Then by simple algebraic manipulations we get that $Z_s \xrightarrow{P} Z$ and $W_s \xrightarrow{P} W$. If we can show that under H_0, $W \xrightarrow{P} \rho Z$, then we would get directly from (4.3) that the distribution of k_s will not depend on the unknown $(\tilde{U}, \tilde{V})$ and hence will reduce asymptotically to the form given in (2.2). If on the other hand W does not converge in probability to Z, then $W=\rho Z+h(\tilde{U}, \tilde{V})$, where $h(\tilde{U}, \tilde{V})$ will depend on $(\tilde{U}, \tilde{V})$, apart from its dependence on $F(U)$. If we substitute this expression in for W in (4.3), we get that the asymptotic distribution of k_s will depend on the parameters of the sampling distribution of $\tilde{U}$, $\tilde{V}$, i.e., on $f(0)$ as well as on the unknown cdf $F(u)$. Thus, the test will not generally be asymptotically distribution-free. Thus, for our purpose, we require to study whether $W-\rho Z \xrightarrow{P} 0$ or not.

Now

$$\rho^{-1}W-Z=m[F(U^{(1)})-F(U_s^{(1)})]\,. \tag{4.5}$$

Let us first consider the case where the cdf $F(u)$ has a finite extremity a, i.e., $F(a)=1$, having a terminal contact of the tth order (i.e., $F^{(t)}(a)\neq 0$, while the lower order derivatives all vanish at $u=a$), for some $t\geqq 1$. Then on writing $\alpha_m=m^{1/t}(U^{(1)}-a)$ and $\beta_m=n^{1/t}(\tilde{U}-\tilde{V})$, we get from (4.5) that

$$\rho^{-1}W-Z=\frac{F^{(t)}(a-\xi)}{t\,!}\sum_{s=1}^{t}(-1)^s\binom{t}{s}\alpha_m^{t-s}\beta_m^s\,, \tag{4.6}$$

where $\xi \xrightarrow{P} 0$.

Now for a terminal contact of tth order α_m has a non-degenerate limiting distribution (cf. Sen [10]). Thus (4.6) will converge in probability to zero, only if $\beta_m \xrightarrow{P} 0$. If now $F^{(1)}(0)>0$, then $\sqrt{m}\cdot \tilde{U}$ and $\sqrt{n}\cdot \tilde{V}$ both have a limiting normal distribution with zero mean and a finite variance. Thus, in this case, $\beta_m \xrightarrow{P} 0$, only if $t>2$. Again, if at the population median, $F(u)-1/2$ and its first $(t'-1)$ derivatives vanish while $F^{(t')}(0)\neq 0$, for some $t'\geqq 1$, then $m^{\delta}\cdot \tilde{U}$ and $n^{\delta}\cdot \tilde{V}$ will have a limiting distribution, only for some $\delta\leqq 1/2t'$ (cf. Sen [10]). Hence, we have the following theorem.

THEOREM 4.1. *For distributions having a finite extremity a, the K_s-test will be asymptotically distribution-free, if the order of terminal contact (t) and the order of contact at the median (t') satisfy*

$$0<2t'<t \ ; \ t>2 .$$

It is thus seen that for distributions having a finite extremity, the K_s-test is distribution-free only under somewhat restrictive regularity conditions.

Now for *exponential type* of distributions, we have

$$\rho^{-1}W-Z=m(\tilde{U}-\tilde{V})f(U^{(1)}+\theta(\tilde{V}-\tilde{U})) \qquad 0<\theta<1 . \tag{4.7}$$

Now for any absolutely continuous cdf F, $m^{\delta}(\tilde{U}-\tilde{V})$ has asymptotically a non-degenerate distribution for some $\delta: 0<\delta\leqq 1/2$, and thus $\tilde{U}-\tilde{V}\xrightarrow{P}0$. Hence, if we can show that $m^{1-\delta}f(U^{(1)})\xrightarrow{P}0$ for any $\delta>0$, it would then follow that (4.7) $\xrightarrow{P}0$, i.e., the K_s-test is asymptotically distribution-free.

Let us now borrow the notations in section 2, particularly the ones defined in (2.3) through (2.7). Let $\{r_m\}$ be a sequence of values with $\lim_{m=\infty} r_m=\infty$ but $\lim_{m=\infty}\frac{1}{m}r_m=0$. Then as in theorem 2.1, $P\{Z\leqq r_m\}\to 1$ as $m\to\infty$ where $Z=m[1-F(X_{(m-r+1)})]$. Also we write $r_m=m[1-F(\hat{U}_m)]$ and thus $\frac{1}{m}r_m=e^{-\psi_2(\hat{U}_m)}$, and hence $\hat{U}_m\to\infty$ as $m\to\infty$. Also, we have $\psi_2(\hat{U}_m)=\log(m/r_m)$ and as $\psi_2(x)$ is a strictly monotonically increasing function of x with $\lim_{\lambda=\infty}\psi_2(x)=\infty$, we can always select the sequence of values $\{r_m\}$ such that $r_m\leqq C\hat{U}_m$. Thus, if we can show that for any $0<\delta<1$, $m^{1-\delta}f(\hat{U}_m)\to 0$ as $m\to\infty$, then the same can be shown to be true for all $U\geqq\hat{U}_m$, as $f(u)$ can be made monotonically non-decreasing for all $U\geqq U_1$, some finite positive quantity, and as $U^{(1)}$ can be made greater than $\hat{U}_m$ (and hence U_1) with probability approaching to unity.
Now

$$m^{1-\delta}f(\hat{U}_m)=r_m^{1-\delta}\psi_2'(\hat{U}_m)e^{-\delta\psi_2(\hat{U}_m)} . \tag{4.8}$$

Thus, to prove that (4.8) converges to zero as $m\to\infty$, it is sufficient to prove that for any positive l,

$$\lim_{x=\infty} x^{l}\psi_2'(x)e^{-\delta\psi_2(x)}=0 \qquad \text{for any } 0<\delta<1 \tag{4.9}$$

and (4.9) can easily be proved by applying L' Hospital's rule and using the fundamental properties of $\psi_2(x)$, $\psi_2'(x)$ and $C_2(x)$ discussed in (2.3) through (2.7). Hence, in this case $W-\rho Z\xrightarrow{P}0$.

In the case of *Cauchy type* of distributions, we can similarly express $\rho^{-1}w-Z$ as in (4.7) and then using the fundamental property, viz., $\lim_{x=\infty} xf(x)/[1-F(x)]=c_2>0$, it would follow precisely on the same way that $\rho^{-1}W-Z \xrightarrow{P} 0$. Thus, we arrive at the following theorem.

THEOREM 4.2. *For distributions belonging to the family of exponential type or Cauchy type of cdf's, the K_s-test will be asymptotically distribution-free.*

By an adaptation of the techniques used in sections 2 and 3, it follows that theorems 4.1 and 4.2 are also applicable to the studentized form of Wilks-Rosenbaum and Kamat's tests.

Studentized K-test (-location). With the setup in (4.1), here we would like to test for the identity of μ_1 and μ_2 without necessarily assuming $\delta_1=\delta_2$. We then define

$$\delta_i=\xi_{p_2}^{(i)}-\xi_{p_1}^{(i)}, \quad 0<p_1<p_2<1 \quad \text{and for } i=1,\ 2; \tag{4.10}$$

where $\xi_p^{(i)}$ is the pth quantile of the ith cdf, for $i=1, 2$. Let now $X^{(1)}$, $X^{(2)}$ (and $Y^{(1)}$, $Y^{(2)}$) be respectively the p_1th and p_2th sample quantiles in the first (and the second) sample, and we define $\hat{\delta}_1=X^{(2)}-X^{(1)}$, $\hat{\delta}_2=Y^{(2)}-Y^{(1)}$. It then follows from the well-known properties of the sample quantiles that under very mild restrictive regularity conditions on the cdf's, $\hat{\delta}_i \xrightarrow{P} \delta_i$, for $i=1, 2$. Let us also write $U=(X-\mu_1)\delta_1$, $V=(Y-\mu_2)/\delta_2$, $U^{(i)}=(X^{(i)}-\mu_1)/\delta_1$ for $i=1, 2$, $V^{(i)}=(Y^{(i)}-\mu_2)/\delta_2$ for $i=1, 2$; and $U^{(3)}=(X_{(m-r+1)}-\mu_1)/\delta_1$. Also let $\tilde{X}$ be the first sample median. Then studentized variables are written as $\hat{X}_i=(X_i-\tilde{X})/\hat{\delta}_1$ for $i=1, \cdots, m$ and $\hat{Y}_i=(Y_i-\tilde{X})/\hat{\delta}_2$ for $i=1, \cdots, n$. It is thus seen that any difference of μ_1 and μ_2 will make $\hat{X}$ and $\hat{Y}$ stochastically different from each other. Let us now write $\hat{X}_{(m-r+1)}=(X_{(m-r+1)}-\tilde{X})/\hat{\delta}_1$ and then base our studentized K-test on K_l, the number of $\hat{Y}$ values greater than $\hat{X}_{(m-r+1)}$. Writing $\gamma_m=(\delta_1\hat{\delta}_2)/(\hat{\delta}_1\delta_2)$ and $\nu=\delta_2/\delta_1$, we can equivalently write k_l as the number V-values in the second sample with values greater than that of (under $H_0:\mu_1=\mu_2$)

$$U_l^{(3)}=\gamma_m U^{(3)}+\tilde{U}(\nu-\gamma_m), \quad \text{where} \quad \tilde{U}=(\tilde{X}-\mu_1)/\delta_1\,. \tag{4.11}$$

Proceeding then precisely on the same line as in the case of the modified K-test and avoiding the details of the proofs thereby, we arrive at the following two theorems.

THEOREM 4.3. *For any distribution having a finite extremity a, the studentized K-test will be asymptotically distribution-free, if the order of contact at the terminal (t), at the median (t_0) and at the p_1-th and p_2-th quantiles (t_1, t_2) satisfy: $0<2t_0$, $(2t_1, 2t_2)<t$, $t>2$.*

THEOREM 4.4. *For any distribution of the exponential type or the*

Cauchy type, the studentized K-test is asymptotically distribution-free.

Now Rosenbaum's two-sample location test and the test by Haga are based on the number of exceedances k_l, k'_l, etc. which are analogous to k, k', etc., where the original variables are replaced by the studentized observations, it follows that the same theorems (i.e., 4.3 and 4.4) also apply to these two tests.

5. On Mosteller's c-sample slippage test

It may be noted that some of the tests considered in the earlier sections may be extended to the case of c-samples. However, for the present, we will not enter into the discussion of them, and only consider here some aspects of a c-sample slippage test proposed by Mosteller [4] and later extend to the general case of unequal samples by Mosteller and Tukey [5]. The null distribution of the test criterion as well as its asymptotic form have been studied by them, but nothing is known precisely about the consistency or the power properties of the test.

Let r_i be the number of observations in the ith sample with values greater than that of the maximum of the remaining $(c-1)$ samples pooled, for $i=1, \cdots, c$. The test is then based upon $r=\max(r_1, \cdots, r_c)$ where it may be noted that one and only one of $r_1, \cdots, r_c$ is positive, while the rest are all zero.

Consistency of the test. Let $F_i(x)$ denote the cdf of the ith population for $i=1, \cdots, c$, and we let

$$F_i(x)=F(x-d_i) \text{ or } F(x[1-d_i]) \text{ for } i=1, \cdots, c\,; \tag{5.1}$$

where $d_1, \cdots, d_c$ are all real and finite (and in the case of $F(x[1-d_i])$, $d_i<1$). By an adaptation of the same technique as has been employed in the theorems 2.1 to 2.5, we have the following.

THEOREM 5.1. *Mosteller's c-sample slippage test will be consistent under the same set of regularity conditions, as have been required with our K-test, provided further*

$$d_j=max\,(d_1, \cdots, d_c)>(d_1, \cdots, d_{j-1}, d_{j+1}, \cdots, d_c)\,.$$

But, if two or more of $(d_1, \cdots, d_c)$ *be simultaneously equal to* d_j, *the largest among* $d_1, \cdots, d_c$, *the test fails to be consistent.*

The proof is not sketched here, but is available with the author (Sen [10]).

Asymptotic non-null distribution of r. The technique by Mosteller or by Mosteller and Tukey will not be applicable here, as in such a case all possible permutations of the observations are not equally likely. Here we have applied the techniques used in theorems 3.1 and 3.2 and deduced the expression for the non-null distribution of r for different families of alternative specifications, chosen in such a way that the power lies between 0 and 1.

THEOREM 5.2. *There exists a monotonically non-decreasing function* $n=n(N)$, $N=n_1+\cdots+n_c$, *such that for the class of alternatives*

$$F_i(x)=F(x-d_i/n) \text{ or } F(x-xd_i/n) \text{ for } i=1, \cdots, c$$

with real and finite $(d_1, \cdots, d_c)$ *and for large* N, *subject to* $n_i/N=\rho_i$ *for* $i=1, \cdots, c$, *the distribution of* r, *for* $F(x)$ *belonging to the family of exponential type of cdf's, comes out as*

$$p(r \mid d)=\frac{\sum_{i=1}^{c}(\rho_i e^{d_i})^r}{(\sum_{i=1}^{c}\rho_i e^{d_i})^r}-\frac{\sum_{i=1}^{c}(\rho_i e^{d_i})^{r+1}}{(\sum_{i=1}^{c}\rho_i e^{d_i})^{r+1}}, \quad \text{for } r=1, 2, 3, \cdots;$$

where (i) for the family of alternatives $F_i(x)=F(x-d_i/n)$ *for* $i=1, \cdots, c$, *with* $F(x)$ *belonging to the family of convex exponential type,*

$$n=Nf(\hat{X}_N) \text{ where } F(\hat{X}_N)=(N-1)/N, \quad f=F',$$

and (ii) for the family of alternatives $F_i(x)=F(x-xd_i/n)$ *for* $i=1, \cdots, c$ *with* $F(x)$ *belonging to the entire class of exponential type of cdf's,*

$$n=N\cdot\hat{X}_N\cdot f(\hat{X}_N)\,.$$

Again, if $F(x)$ *has a finite extremity* a, *with* $f(a)=F^{(1)}(a)>0$, *then the asymptotic distribution of* r *is given as*

$$p(r \mid d)=e^{-\sum_{k=1}^{c}\rho_k \Delta_k}\Big[\sum_{i=1}^{c}\rho_i^r(1-\rho_i)\sum_{s=0}^{r}\Delta_i^s/s!\Big], \quad \text{for } r=1, 2, \cdots;$$

where $\Delta_i=f(a)d_i$ *or* $af(a)d_i$ *according as* $F_i(x)=F(x-d_i/N)$ *or* $F(x-xd_i/N)$ *for* $i=1, \cdots, c$.

The proof of the theorem is not considered here, as it follows more or less on the same line as in theorems 3.1 and 3.2. The case of Cauchy type of cdfs has also not been considered here as for this class of distributions, the test fails to be consistent. Also, we have not considered distributions having a finite extremity with a contact of order greater than one. In this case, of course, we can find out the distribution of r, as in the case with a first order terminal contact, but the form of the asymptotic distribution will be quite cumbrous. These are therefore not reproduced here, however, they are available with the author (Sen, [10]).

6. Rare exceedances Vs. normal exceedances

Here some aspects of the non-parametric tests based on the number of rare exceedances are contrasted with those based on the normal exceedances (e.g. median tests, Wilcoxon's test, etc.). The following points are intended to be discussed here.

(i) *Consistency of the tests.* The tests based on normal exceedances, have been shown by various authors, to be consistent under very mild restrictions on the parent cdf. The continuity or possibly the absolute continuity of the cdfs, appears to be sufficient in most of the cases. On the otherhand tests based on rare exceedances, have been shown here to be consistent under more restrictive regularity conditions.

(ii) *Asymptotic distribution.* The tests based on normal exceedances have mostly some limiting distribution (under a family of alternative specifications including the null one) of known and standard forms, and as detailed tables for them are available, the power of the test, at least for large samples, can readily be traced. On the otherhand, tests studied here, have some discrete limiting distributions (e.g., negative Binomial etc.) or some convolutes of them. The power of the tests thus can not readily be traced from standard tables, as are available otherwise. Further, the difficulty usually arises with the convolutions, as in most of the cases, they are quite tedious and mostly no simple algebraic form can be attached to them, particularly when the null hypotheses is not true.

(iii) *Studentization of the tests.* It has been shown by the author (Sen [11]) that the studentized form of location tests based on normal exceedances will be consistent and asymptotically distribution-free provided the distribution is absolutely continuous and have a non-zero density at the population median. Similarly, the modified form of scale tests based on normal exceedances will be consistent and asymptotically distribution-free, if the cdf is symmetrical and absolutely continuous. The test based on rare exceedances, when thus studentized or modified, have been shown here to be consistent and asymptotically distribution-free, only for certain type of cdfs. Thus, in some cases, the former class of tests are suitable, while in some other ones, the tests studied here will be suitable.

(iv) *Asymptotic power.* As the two classes of tests have asymtotic distributions entirely different from each other, the usual definition of asymptotic power-efficiency (cf. Pitman [6]) will not apply here. And, in fact, any comparison of the power functions, requires the tracing of the corresponding non-null distributions, which in term depends appreciably on the parent cdf, the alternative specifications and the sample sizes. Thus, in general, it is not possible to draw some conclusions valid for certain broad families of cdfs. However, the following appears to be wellworth consideration.

Let $n_1, \cdots, n_c$ be the sample sizes of the c samples and let $N=n_1+\cdots+n_c$. Let now T_N and T_N^* be two tests with critical regions $C(T)$ and $C(T^*)$ respectively, so that $P\{T \in C(T) \mid H_0\}=P\{T^* \in C(T^*) \mid H_0\}=\alpha$, the level of significance, and we are to compare $P\{T \in C(T) \mid H\}$ with $P\{T^* \in C(T^*) \mid H\}$.

Let now $\psi(N)$ and $\psi^*(N)$ be two non-decreasing function of N, such that for the set of alternatives that for the location problem with $E_i(x)=F(x+\lambda_i/\psi(N))$ for $i=1, \cdots, c$; where $\lambda_1, \cdots, \lambda_c$ are all real and finite, we have

$$\lim_{N=\infty} P\{T \in C(T) \mid \lambda\}=P_\lambda,\ 0<P_\lambda<1\,, \tag{6.1}$$

and for $F_i(x)=F(x+\lambda_i/\psi^*(N))$,

$$\lim_{N=\infty} P\{T^* \in C(T^*) \mid \lambda\}=P_\lambda^*,\ 0<P_\lambda^*<1\,. \tag{6.2}$$

(In the case of scale problem, λ_i is replaced by $x\lambda_i$). If now $\lim_{N=\infty}\psi(N)/\psi^*(N)=0$, we get from (6.1) and (6.2) that T^* will be asymptotically more power efficient and the power efficiency of T with respect T^* is asymptotically equal to zero, (as $\lim_{N=\infty} P\{T \in C(T) \mid \lambda/\psi^*(N)\}=0$). Similarly, if $\lim_{N=\infty}\psi(N)/\psi^*(N)=\infty$, T will be asymptotically more power-efficient. But, if $\psi(N)/\psi^*(N)\to d<\infty$ as $N\to\infty$, we cannot definitely say whether T or T^* will be more power-efficient and the result will evidently depend on P_λ and P_λ^*, i.e., on the parent cdfs. In this case, numerical evaluation appears to be the only avenue.

Now, as regards the tests based on normal exceedances, it is known that $\psi(N)=\sqrt{N}$. So we are only to study $\psi^*(N)$ for the class of tests proposed here. Now, for exponential type of cdfs, $\psi^*(N)=Nf(\hat{X}_N)$ or $N\hat{X}_N f(\hat{X}_N)$ (where $F(\hat{X}_N)=(N-1)/N$) according as the set of alternatives relation to location or scale changes. In any case, it follows from (4.8) and (4.9) with little computations, that $\psi^*(N)/\psi(N)\to 0$ as $N\to\infty$. Consequently, the class of tests proposed here, are asymptotically less power-efficient than the ones based on normal exceedances. In the case of Cauchy type of cdfs, the tests studied here, are inconsistent, while the tests based on normal exceedances, will have as before $\psi(N)=\sqrt{N}$. Hence in this case also, they are definitely better.

Finally, in the case of cdfs having a finite endpoint a, with $f(a)>0$, it follows then from theorem 3.2 that $\psi^*(N)=N$, and hence $\psi(N)/\psi^*(N)\to 0$ as $N\to\infty$. Consequently, in this case, the tests based on the number of rare exceedances will be more (power-) efficient. While, if for some $t>1$, $1-F(a)=\cdots=F^{(t-1)}(a)=0$ and $F^{(t)}(a)\neq 0$, it then follows from theorem 3.2 that $\psi^*(N)=N^{1/t}$. Thus, for any $t>2$, $\psi(N)/\psi^*(N)\to\infty$ as $N\to\infty$, i.e., the tests based on the normal exceedances are more (power-) efficient. Finally, for $t=2$, we have $\psi^*(N)=\sqrt{N}$, and hence, in such a case, nothing can be said about the relative performance of the two classes of tests, without actual tabulations pertaining to different alternative specifications, even for the large samples.

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