A NECESSARY CONDITION FOR THE EXISTENCE OF REGULAR AND SYMMETRICAL PBIB DESIGNS OF T_3 TYPE

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1. Introduction

A necessary condition for the existence of regular and symmetrical PBIB designs in terms of the Hasse-Minkowski p-invariant was obtained for group divisible designs by R. C. Bose and W. S. Connor [1], for designs of L_2 type by S. S. Shrikhand [2], for designs of rectangular type by M. N. Vartak [3], and finally for designs of triangular type by J. Ogawa [4] respectively.

The purpose of this paper is to give a similar necessary condition for the existence of regular symmetrical PBIB designs of T_3 type in terms of the Hasse-Minkowski p-invariant. To this end we consider the proper space related to the design of T_3 type (section 3) and prove lemmas which are necessary in the later argument (section 4), and then the necessary condition for existence of the designs is given with some examples of impossible designs (section 5). Finally a numerical example of the analysis of design of T_3 type is given (section 6).

2. Preliminaries

Let π be a PBIB design, with three associate classes and with parameters

$$v,\ b,\ r,\ p^i_{jk},\ n_i,\ \lambda_i,\ (i, j, k=1, 2, 3),\ k. \tag{2.1}$$

These parameters are not all independent but they are connected by the relations

$$\begin{aligned} &bk=vr, \quad \textstyle\sum_{i=1}^3 n_i=v-1, \quad \sum_{i=1}^3 n_i\lambda_i=r(k-1), \quad p^i_{ju}=p^i_{uj}, \\ &n_i p^i_{ju}=n_j p^j_{iu}=n_u p^u_{ij}, \quad \textstyle\sum_{u=1}^3 p^i_{ju}=n_j-\delta_{ij}, \quad (i, j, u=1, 2, 3), \end{aligned} \tag{2.2}$$

where $\delta_{ij}=0$ or 1 according as $i\neq j$ or $i=j$ respectively.

The association of T_3 type is defined as follows: The number of element is $v=n(n-1)(n-2)/6$, where n is a positive integer. We have an array of $n(n-1)(n-2)/6$ treatments on the cubic form with the following properties: (a) if the tth treatment lies in the ith row, jth column and uth layer, it is denoted by $t=(i, j, u)$, (b) $t=(i, j, u)$ represents the same treatment irrespective of the order of i, j and u, (c) the position in the principal diagonal part is blank, i.e., $(i, j, i)=(i, i, j)=(j, i, i)=\phi$. For two treatments $t=(i, j, k)$, $t'=(i', j', k')$, we find the following relations:

two treatments are first associates for $i\neq i'$, $j=j'$, $k=k'$,
two treatments are second associates for $i\neq i'$, $j\neq j'$, $k=k'$,
two treatments are third associates for $i\neq i'$, $j\neq j'$, $k\neq k'$,

where each index takes on a value of 1 through n.

In this association, the parameters of association are as follows:

(2.3) $$n_1=3(n-3),\ n_2=3(n-3)(n-4)/2,\ n_3=(n-3)(n-4)(n-5)/6\,,$$

(2.4)
$$p^1_{ij}=\left\|\begin{matrix} n-2 & 2(n-4) & 0 \\ 2(n-4) & (n-4)^2 & (n-4)(n-5)/2 \\ 0 & (n-4)(n-5)/2 & (n-4)(n-5)(n-6)/6 \end{matrix}\right\|$$
$$p^2_{ij}=\left\|\begin{matrix} 4 & 2(n-4) & (n-5) \\ 2(n-4) & (n-5)(n+2)/2 & (n-5)(n-6) \\ (n-5) & (n-5)(n-6) & (n-5)(n-6)(n-7)/6 \end{matrix}\right\|$$
$$p^3_{ij}=\left\|\begin{matrix} 0 & 9 & 3(n-6) \\ 9 & 9(n-6) & 3(n-6)(n-7)/2 \\ 3(n-6) & 3(n-6)(n-7)/2 & (n-6)(n-7)(n-8)/6 \end{matrix}\right\|.$$

PBIB design of T_3 type is an arrangement of v treatments with the association of T_3 type being allocated to b blocks of size k each in such a way that (1) each block contains k different treatments, (2) each treatment occurs in r blocks and (3) any two treatments occur together in λ_i blocks, if they are the ith associates. If the incidence matrix of this design is denoted by N, then it is also well known that

(2.5) $$NN'=rB_0+\lambda_1B_1+\lambda_2B_2+\lambda_3B_3\,,$$

where B_i is the ith association matrix.

3. Some properties of proper space related to regular and symmetrical PBIB design of T_3 type

According to L. C. A. Corsten [7] we conceive $P=NN'$ as the matrix of the linear transformation of a vector space A consisting of vectors $\boldsymbol{x}=(x_1, x_2, \cdots, x_v)'$ into itself, where the coordinate x_t corresponds to the tth treatment. From (2.5) the tth coordinate y_t in $\boldsymbol{y}=P\boldsymbol{x}$ is equal to $rx_t+\lambda_1S_1+\lambda_2S_2+\lambda_3S_3$ where $S_j(j=1, 2, 3)$ represents the sum of the coordinates in $\boldsymbol{x}$ corresponding to the jth associates of treatment t. $\boldsymbol{s}'=(1, 1, \cdots, 1)$ is a proper vector of P with the characteristic root $r+\lambda_1n_1+\lambda_2n_2+\lambda_3n_3=rk$.

We shall consider the $(v-1)$ dimensional subspace $\bar{A}$ of all vectors in A which are orthogonal to the vector $\boldsymbol{s}$. For every vector $\boldsymbol{x}$ in $\bar{A}$, we have the following relation

$$x_t+S_1+S_2+S_3=0 . \tag{3.1}$$

If we denote the treatment by (i, j, k) with the restriction $i<j<k$, then we may construct a set of n vectors $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ and a set of $n(n-1)/2$ vectors $\boldsymbol{d}_{12}, \boldsymbol{d}_{13}, \cdots, \boldsymbol{d}_{1n}, \cdots, \boldsymbol{d}_{2n}, \cdots, \boldsymbol{d}_{n-1\,n}$ in the vector space A in the following way: (a) the coordinates x_t of the vector $\boldsymbol{c}_p$ corresponding to such treatments $t=(i, j, k)$ as $1\leqq i=p<j<k\leqq n$, or $1\leqq i<j=p<k\leqq n$, or $1\leqq i<j<k=p\leqq n$ are unity and the other coordinates of the vector are zero, (b) the coordinates x_t of the vector $\boldsymbol{d}_{pq}$ corresponding to such treatments $t=(i, j, k)$ as $1\leqq i=p<j=q<k\leqq n$, or $1\leqq i=p<j<k=q\leqq n$, or $1\leqq i<j=p<k=q\leqq n$ are unity and the other coordinates of the vector are zero.

LEMMA 3.1. *$\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ are linearly independent and $\boldsymbol{d}_{12}, \boldsymbol{d}_{13}, \cdots, \boldsymbol{d}_{n-1\,n}$ are linearly independent, too.*

If $a_1\boldsymbol{c}_1+a_2\boldsymbol{c}_2+ \cdots +a_n\boldsymbol{c}_n=\boldsymbol{O}$, where $\boldsymbol{O}$ is the zero vector, then

$$\begin{gathered} a_1+a_2+a_3=0 \\ a_1+a_2+a_4=0 \\ \cdots\cdots\cdots\cdots \\ a_{n-1}+a_{n-2}+a_n=0 , \end{gathered}$$

therefore $a_1=a_2= \cdots =a_n=0$, consequently $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ are linearly independent. If $a_{12}\boldsymbol{d}_{12}+a_{13}\boldsymbol{d}_{13}+ \cdots +a_{n-1\,n}\boldsymbol{d}_{n-1\,n}=0$, then $a_{12}+a_{13}+a_{23}=0$, $a_{12}+a_{14}+a_{24}=0, \cdots, a_{n-2\,n-1}+a_{n-2\,n}+a_{n-1\,n}=0$, therefore $a_{12}=a_{13}= \cdots =a_{n-1\,n}=0$, consequently $\boldsymbol{d}_{12}, \boldsymbol{d}_{13}, \cdots, \boldsymbol{d}_{n-1\,n}$ are linearly independent.

Let the n dimensional linear subspace of A spanned by these n linearly independent vectors $\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n$ be called A_1^* and the $n(n-1)/2$ dimensional linear subspace of A spanned by these $n(n-1)/2$ linearly independent vectors $\boldsymbol{d}_{12}, \boldsymbol{d}_{13}, \cdots, \boldsymbol{d}_{n-1\,n}$ be called A_2^*. Since $\boldsymbol{c}_1+\boldsymbol{c}_2+ \cdots +\boldsymbol{c}_n=3\boldsymbol{s}$, $2\boldsymbol{c}_p=\sum_{i=1}^{p-1}\boldsymbol{d}_{ip}+\sum_{j=p+1}^{n}\boldsymbol{d}_{pj}$, the space A_1^* contains the one-dimensional space spanned by $\boldsymbol{s}$, and the space A_2^* contains the space A_1^*. Moreover we consider $(n-1)$ dimensional subspace A_1^{**} of the space A_1^* orthogonal to $\boldsymbol{s}$ and $n(n-3)/2$ dimensional subspace A_2^{**} of the space A_2^* orthogonal to the vectors $\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n$.

Now, consider the inner product of any vector in A_1^{**}, $\gamma_1 c_1+\gamma_2 c_2+\cdots+\gamma_n c_n$ say, and s, and the inner product of any vector in A_2^{**}, $\sum_{p<q}\cdot\gamma_{pq}d_{pq}$ say, and s and moreover the inner product of c_r and $\sum_{p<q}\gamma_{pq}d_{pq}$; then the first, the second and the third are respectively equal to

$$(3.2)\quad (\textstyle\sum_{p=1}^{n}\gamma_p c_p,\ s)=\sum_{p=1}^{n}\gamma_p(c_p,\ s)=(\gamma_1+\gamma_2+\cdots+\gamma_n)(n-1)(n-2)/2=0,$$

$$(3.3)\quad (\textstyle\sum_{p<q}\gamma_{pq}d_{pq},\ s)=\sum_{p<q}\gamma_{pq}(d_{pq},\ s)=(n-2)\sum_{p<q}\gamma_{pq}=0,$$

$$\begin{aligned}(3.4)\quad (\textstyle\sum_{p<q}\gamma_{pq}d_{pq},\ c_r)&=\textstyle\sum_{i=1}^{r-1}\gamma_{ir}(d_{ir},\ c_r)+\sum_{j=r+1}^{n}\gamma_{rj}(d_{rj},\ c_r)+\sum_{i<j\neq r}\gamma_{ij}(d_{ij},\ c_r)\\ &=(n-2)(\textstyle\sum_{i=1}^{r-1}\gamma_{ir}+\sum_{j=r+1}^{n}\gamma_{rj})+\{\sum_{p<q}\gamma_{pq}-(\sum_{i=1}^{r-1}\gamma_{ir}+\sum_{j=r+1}^{n}\gamma_{rj})\}\\ &=(n-3)(\textstyle\sum_{i=1}^{r-1}\gamma_{ir}+\sum_{j=r+1}^{n}\gamma_{rj})=0.\end{aligned}$$

We further note that, if the treatment t occurs in the pth row, qth column and rth layer of the association scheme, the coordinate x_t of any vector $x=\sum_{p=1}^{n}\gamma_p c_p$ in A_1^{**} and the coordinate x_t' of any vector $x'=\sum_{p<q}\gamma_{pq}d_{pq}$ in A_2^{**} corresponding to the pth row, qth column and rth layer of the association scheme are equal to $\gamma_p+\gamma_q+\gamma_r$ and $\gamma_{pq}+\gamma_{qr}+\gamma_{pr}$ respectively. That is to say, the inner product $(x,\ e)$ and $(x',\ e)$ is equal to $\gamma_p+\gamma_q+\gamma_r$ and $\gamma_{pq}+\gamma_{qr}+\gamma_{pr}$ respectively, where the vector e has been constructed in such a way that we write unity in the position of which the corresponding indices occur in the place of the association scheme being occupied by the treatment $t=(p,\ q,\ r)$; write zero everywhere-else.

Similarly, consider the following inner products of the vectors

$$\begin{aligned}(3.5)\quad (d,\ x)&=(\gamma_p+\gamma_q)(n-2)+(\gamma_1+\gamma_2+\cdots+\gamma_n)-(\gamma_p+\gamma_q)\\ &\quad+(\gamma_q+\gamma_r)(n-2)+(\gamma_1+\gamma_2+\cdots+\gamma_n)-(\gamma_q+\gamma_r)\\ &\quad+(\gamma_p+\gamma_r)(n-2)+(\gamma_1+\gamma_2+\cdots+\gamma_n)-(\gamma_p+\gamma_r)\\ &=2(\gamma_p+\gamma_q+\gamma_r)(n-3)=2(n-3)x_t=3x_t+S_1,\end{aligned}$$

$$\begin{aligned}(3.6)\quad (c,\ x)&=\gamma_p(n-1)(n-2)/2+(\gamma_1+\gamma_2+\cdots+\gamma_n)(n-2)-\gamma_p(n-2)\\ &\quad+\gamma_q(n-1)(n-2)/2+(\gamma_1+\gamma_2+\cdots+\gamma_n)(n-2)-\gamma_q(n-2)\\ &\quad+\gamma_r(n-1)(n-2)/2+(\gamma_1+\gamma_2+\cdots+\gamma_n)(n-2)-\gamma_r(n-2)\\ &=(\gamma_p+\gamma_q+\gamma_r)(n-2)(n-3)/2=x_t(n-2)(n-3)/2\\ &=3x_t+2S_1+S_2,\end{aligned}$$

$$\begin{aligned}(3.7)\quad (d,\ x')&=\gamma_{pq}(n-2)+\textstyle\sum_{i=1}^{p-1}\gamma_{ip}+\sum_{j=p+1}^{n}\gamma_{pj}+\sum_{i=1}^{q-1}\gamma_{iq}\\ &\quad+\textstyle\sum_{j=q+1}^{n}\gamma_{qj}-2\gamma_{pq}+\gamma_{qr}(n-2)+\sum_{i=1}^{q-1}\gamma_{iq}+\sum_{j=q+1}^{n}\gamma_{qj}\\ &\quad+\textstyle\sum_{i=1}^{r-1}\gamma_{ir}+\sum_{j=r+1}^{n}\gamma_{rj}-2\gamma_{qr}+\gamma_{pr}(n-2)+\sum_{i=1}^{p-1}\gamma_{ip}\end{aligned}$$

$$+\sum_{j=p+1}^{n}\gamma_{pj}+\sum_{i=1}^{r-1}\gamma_{ir}+\sum_{j=r+1}^{n}\gamma_{rj}-2\gamma_{pr}$$
$$=(n-4)(\gamma_{pq}+\gamma_{qr}+\gamma_{pr})=(n-4)x'_t=3x'_t+S'_1\,,$$

(3.8) $$(\boldsymbol{c},\,\boldsymbol{x}')=0=3x'_t+2S'_1+S'_2\,,$$

where $$\boldsymbol{d}=\boldsymbol{d}_{pq}+\boldsymbol{d}_{qr}+\boldsymbol{d}_{pr}\,,\qquad \boldsymbol{c}=\boldsymbol{c}_p+\boldsymbol{c}_q+\boldsymbol{c}_r\,.$$

Then, from (3.1), (3.5) and (3.6) we get the following result,

(3.9) $$S_1=(2n-9)x_t,\quad S_2=(n-4)(n-9)x_t/2,\quad S_3=-(n-4)(n-5)x_t/2,$$

and from (3.1), (3.7) and (3.8) we get

(3.10) $$S_1=(n-7)x'_t\,,\ S_2=-(2n-11)x'_t\,,\ S_3=(n-5)x'_t\,.$$

The relation (3.8) follows from the fact that every vector of A_2^{**} is orthogonal to the given n basis vectors of A_1^{**}.

Finally we consider the orthocomplement of A_2^* with respect to A and call this A_3. The dimension of A_3 is $n(n-1)(n-2)/6-n(n-1)/2=n(n-1)(n-5)/6$. Let $\boldsymbol{x}''$ be any vector in A_3. We get the following inner products of the vectors.

(3.11) $$(\boldsymbol{d},\,\boldsymbol{x}'')=0=3x''_t+S_1\,,$$
$$(\boldsymbol{c},\,\boldsymbol{x}'')=0=3x''_t+2S_1+S_2\,.$$

Hence $S_1=-3x''_t$, $S_2=3x''_t$, $S_3=-x''_t$ for all vectors in A_3.

Now it follows from the previous paragraph that the coordinate y_t of $P\boldsymbol{x}$ where $\boldsymbol{x}$ is restricted to A_1^{**}, is $\{r+(2n-9)\lambda_1+(n-4)(n-9)\lambda_2/2-(n-4)(n-5)\lambda_3/2\}\,x_t$. Therefore A_1^{**} is a proper space of NN' with proper value $\rho_1=r+(2n-9)\lambda_1+(n-4)(n-9)\lambda_2/2-(n-4)(n-5)\lambda_3/2$. Similarly A_2^{**} is a proper space of NN' with proper value $\rho_2=r+(n-7)\lambda_1-(2n-11)\lambda_2+(n-5)\lambda_3$, and A_3 also is a proper space of NN' with proper value $\rho_3=r-3\lambda_1+3\lambda_2-\lambda_3$.

It is quite easy to find the Gramian P_1 of the given basis of A_1^*, the union of the proper space A_1^{**} and the proper space spanned by $\boldsymbol{s}$, and the Gramian P_2 of the given basis of A_2^*, the union of the proper spaces A_1^{**}, A_2^{**} and the proper space spanned by $\boldsymbol{s}$. In order to write down P_1 and P_2, we only need the inner products of the vectors $\boldsymbol{c}_1$, $\boldsymbol{c}_2,\cdots,\boldsymbol{c}_n$ and $\boldsymbol{d}_{12}$, $\boldsymbol{d}_{13},\cdots,\boldsymbol{d}_{n-1\,n}$, respectively. Then they are written down as follows:

(3.12) $$P_1=\begin{vmatrix} (n-1)(n-2)/2 & n-2 & \cdots\cdots & n-2 \\ n-2 & (n-1)(n-2)/2 & \cdots\cdots & n-2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ n-2 & n-2 & \cdots\cdots & (n-1)(n-2)/2 \end{vmatrix},$$

$$
(3.13)\qquad P_2=\left|\begin{array}{c|c|c|c|l}
n-2 \quad \cdots 1 & 1 \ \cdots \ 1 & 0\cdots\cdots 0 & 0\cdots\cdots & \\
\ddots & 1\ 0 \ \cdots 0 & 1\cdots\cdots 1 & 0\cdots\cdots & \\
 & \ddots & 1 \quad \cdots 0 & 1\cdots\cdots & \} n-1 \\
 & & \ddots & 1\ 0\cdots & \\
 & & & \ddots & \\
1\cdots \quad n-2 & 0\cdots \quad 1 & 0\cdots \quad 1 & 0\cdots & \\ \hline
1\ 1 \quad \cdots 0 & n-2\ \cdots 1 & 1\cdots \quad 1 & 0\cdots\cdots & \\
\vdots\ \ddots & \ddots & 1 \quad \cdots 0 & 1\cdots\cdots & \\
\vdots \quad \ddots & & \ddots & 1 & \} n-2 \\
 & & & \ddots & \\
1\ 0\cdots \quad 1 & 1\cdots\ n-2 & 0\cdots \quad 1 & 0\cdots & \\ \hline
0\ 1\ 1 \quad \cdots 0 & 1\ 1\ \cdots 0 & n-2\ \cdots 1 & 1\cdots & \\
\vdots\ \vdots\ \ddots & \vdots\ \ddots & \ddots & 1 & \} n-3 \\
 & & & \ddots & \\
0\ 1\ 0 \quad \cdots 1 & 1\ 0\cdots\ 1 & 1\cdots\ n-2 & & \\ \hline
0\ 0\ 1\ 1\ \cdots 0 & 0\ 1\ 1\ 0 & 1\ 1\ 0 & n-2\ \cdots & \\
\vdots\ \vdots\ \vdots\ \vdots\ \ddots & \vdots\ \vdots\ \vdots\ \vdots & \vdots\ \ddots & \ddots & \} n-4 \\
\underbrace{\qquad\qquad}_{n-1} & \underbrace{\qquad\quad}_{n-2} & \underbrace{\qquad\quad}_{n-3} & \underbrace{\qquad\quad}_{n-4}\cdots & \ \vdots
\end{array}\right| .
$$

4. A necessary theorem and lemmas

Hasse's Theorem [5]. The necessary and sufficient conditions for two positive-definite, rational and symmetric matrices A and B of the same order to be rationally congruent are that, in the first place, the square-free parts of the determinants of both matrices are the same, and in the second, the Hasse-Minkowski p-invariants of both matrices coincide with each order for all primes p including p_∞.

If we denote the n leading principal minor determinants of A by

$$(4.1)\qquad D_1,\ D_2,\ \cdots,\ D_{n-1},\ D_n=|A|$$

and let $D_0=1$, then [5] the Hasse-Minkowski p-invariant of A is given by

$$(4.2)\qquad C_p(A)=(-1,\ -1)_p\prod_{i=0}^{n-1}(D_{i+1},\ -D_i)_p$$

for each prime p, where the symbol $(a,\ b)_p$ denotes the extended Hilbert symbol of the norm residue [5], which is defined by

$$(4.3)\qquad (a,\ b)_p=\begin{cases} 1 & \text{if } ax^2+by^2=1 \text{ has a } p\text{-adic solution,} \\ -1 & \text{otherwise.}\end{cases}$$

Now we shall list some useful properties of $C_p(A)$ as lemmas.

LEMMA 4.1. *If A and B are rational and symmetrical and if*

$$U=\begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix},$$

then

$$C_p(U)=(-1,\ -1)_p(|A|,\ |B|)_pC_p(A)C_p(B)\,. \tag{4.4}$$

LEMMA 4.2. *If A, B and C are rational and symmetric and if*

$$U=\begin{Vmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{Vmatrix},$$

then

$$C_p(U)=C_p(A)C_p(B)C_p(C)(|A|,\ |B|)_p(|B|,\ |C|)_p(|A|,\ |C|)_p\,. \tag{4.5}$$

LEMMA 4.3. *If A is $n\times n$ matrix, then*

$$C_p(\rho_A)=(-1,\ \rho)_p^{n(n+1)/2}(\rho,\ |A|)_p^{n+1}C_p(A)\,. \tag{4.6}$$

LEMMA 4.4. *If the $n-1$ rational vectors $\boldsymbol{b}_2, \boldsymbol{b}_3, \cdots, \boldsymbol{b}_n$ of dimensionality n are linearly independent and are orthogonal to $\boldsymbol{s}$, then the Gramian of the set, i.e.,*

$$U=\begin{Vmatrix} \boldsymbol{b}'_2 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{b}'_n \end{Vmatrix} \|\boldsymbol{b}_2 \cdots \boldsymbol{b}_n\|$$

has the p-invariant $C_p(U)=(-1,\ -1)_p$.

LEMMA 4.5. *So long as we restrict ourselves to rational vectors, the p-invariant of a vector set, i.e., the p-invariant of the Gramian of the set is uniquely determined by the linear subspace generated by the vectors of the set.*

We shall summarize the necessary properties of Hilbert's symbol [5] and some of the fundamental properties of the Legendre symbol (q/p) of the quadratic residue. For reference we reproduce the results of B. W. Jones [5].

LEMMA 4.6.

1. $(\alpha,\ \beta)_\infty=1$ *unless α and β are both negative.*
2. $(\alpha,\ \beta)_p=(\beta,\ \alpha)_p$.
3. $(\alpha\rho^2,\ \beta\rho^2)_p=(\alpha,\ \beta)_p$.
4. $(\alpha,\ -\alpha)_p=1$.
5. *If $\alpha=p^a\alpha_1$, $\beta=p^b\beta_1$ with α_1 and β_1 units, then*

a. *if p is odd,* $(\alpha,\beta)_p=\left(\frac{-1}{p}\right)^{ab}\left(\frac{\alpha_1}{p}\right)^{b}\left(\frac{\beta_1}{p}\right)^{a}$

b. *if* $p=2$, $(\alpha,\beta)_2=\left(\frac{2}{\alpha_1}\right)^{b}\left(\frac{2}{\beta_1}\right)^{a}(-1)^{(\alpha_1-1)(\beta_1-1)/4}$.

5'. *If p is prime to* $2\alpha\beta$, $(\alpha,\beta)_p=1$, *for p finite,* α *and* β *in* $R(p)$.

6. $(\alpha,\beta)_p(\alpha,\gamma)_p=(\alpha,\beta\gamma)_p$.

7. $(\alpha,\alpha)_p=(\alpha,-1)_p$.

8. $(\alpha\rho,\beta\rho)_p=(\alpha,\beta)_p(\rho,-\alpha\beta)_p$.

9. *If* β *is a non-square in* $F(p)$ *and* $c=1$ *or* -1, *there is for each prime p an integer* α *such that* $(\alpha,\beta)_p=c$. *If, further, b as defined in property 5 is odd, may be taken prime to p.*

10. *If a and b are non-zero rational numbers*

$$\prod(a,b)_p=1,$$

the product extending over all primes p including $p=\infty$.

11. $(a,b)_p=(-ab,a+b)_p$.

Now, from section 3 and the elements of linear associative algebra [6], we conclude that there exist four mutually orthogonal and symmetric matrices $B_0^*=(1/v)G_v$, B_1^*, B_2^* and B_3^* with respective ranks, 1, $n-1$, $n(n-3)/2$ and $n(n-1)(n-5)/6$, such that

$$NN'=\rho_0B_0^*+\rho_1B_1^*+\rho_2B_2^*+\rho_3B_3^* \tag{4.7}$$

where

$$\rho_0=r+3(n-3)\lambda_1+3(n-3)(n-4)\lambda_2/2+(n-3)(n-4)(n-5)\lambda_3/6,$$

and the column vectors of B_1^*, B_2^* and B_3^* respectively generate the proper spaces A_1^{**}, A_2^{**} and A_3 of NN'.

Let us assume, without any loss of generality, that

$$\boldsymbol{b}_1^0,\ \boldsymbol{b}_2^1,\ \cdots,\ \boldsymbol{b}_n^1,\ \boldsymbol{b}_{n+1}^2,\ \cdots,\ \boldsymbol{b}_{n(n-1)/2}^2,\ \cdots,\ \boldsymbol{b}_v^3$$

are linearly independent, and let us put

$$S=\|\,\boldsymbol{b}_1^0\boldsymbol{b}_2^1\cdots\boldsymbol{b}_n^1\boldsymbol{b}_{n+1}^2\cdots\boldsymbol{b}_{n(n-1)/2}^2\cdots\boldsymbol{b}_v^3\,\|.$$

Then S is a non-singular $v\times v$ matrix with rational elements.

Further let

$$Q_1=\left\|\begin{matrix}\boldsymbol{b}_2^{1'}\\ \vdots\\ \boldsymbol{b}_n^{1'}\end{matrix}\right\|\,\|\boldsymbol{b}_2^1\cdots\boldsymbol{b}_n^1\|,\qquad Q_2=\left\|\begin{matrix}\boldsymbol{b}_{n+1}^{2'}\\ \vdots\\ \boldsymbol{b}_{n(n-1)/2}^{2'}\end{matrix}\right\|\,\|\boldsymbol{b}_{n+1}^2\cdots\boldsymbol{b}_{n(n-1)/2}^2\|$$

$$Q_3=\left\|\begin{matrix}\boldsymbol{b}_{n(n-1)/2+1}^{3'}\\ \vdots\\ \boldsymbol{b}_v^{3'}\end{matrix}\right\|\,\|\boldsymbol{b}_{n(n-1)/2+1}^3\cdots\boldsymbol{b}_v^3\|.$$

Then

$$S'NN'S=\begin{Vmatrix} \rho_0/v & & & 0 \\ & \rho_1 Q_1 & & \\ 0 & & \rho_2 Q_2 & \\ & & & \rho_3 Q_3 \end{Vmatrix}$$

or

$$NN'\sim\begin{Vmatrix} rk/v & & & 0 \\ & \rho_1 Q_1 & & \\ 0 & & \rho_2 Q_2 & \\ & & & \rho_3 Q_3 \end{Vmatrix}. \tag{4.8}$$

Since

$$S'S=\begin{Vmatrix} 1/v & & & 0 \\ & Q_1 & & \\ 0 & & Q_2 & \\ & & & Q_3 \end{Vmatrix},$$

we get

$$v|Q_1||Q_2||Q_3|\sim 1\,. \tag{4.9}$$

Next, the Gramian P_1 and P_2 as defined in (3.12) and (3.13) have the following determinants:

$$P_1=\frac{3}{2}(n-1)(n-2)\{(n-2)(n-3)/2\}^{n-1} \tag{4.10}$$

$$P_2=3(n-2)\underbrace{\begin{vmatrix} n-3 & \cdots & 1 \\ & \ddots & \\ 1 & \cdots & n-3 \end{vmatrix}}_{n-2}\underbrace{\begin{vmatrix} n-3 & -1 & \cdots & -1 \\ -1 & n-3 & & \\ \vdots & & \ddots & \\ -1 & 1 & \cdots & n-3 \end{vmatrix}}_{n-2}\underbrace{\begin{vmatrix} n-2 & -2 & \cdots & -2 \\ -1 & n-3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \cdots & n-3 \end{vmatrix}}_{n-3}$$

$$\cdot\underbrace{\begin{vmatrix} n-1 & -3 & \cdots & -3 \\ -1 & n-3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \cdots & n-3 \end{vmatrix}}_{n-4}\underbrace{\begin{vmatrix} n & -4 & \cdots & -4 \\ -1 & n-3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \cdots & n-3 \end{vmatrix}}_{n-5}\underbrace{\begin{vmatrix} n+1 & -5 & \cdots & -5 \\ -1 & n-3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \cdots & n-3 \end{vmatrix}}_{n-6}\cdots$$

Since

$$\begin{vmatrix} n-(5-i) & -(i-1) & \cdots & -(i-1) \\ -1 & n-3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \cdots & n-3 \end{vmatrix}=\begin{vmatrix} 2(n-3) & -2(n-3) & \cdots & -2(n-3) \\ -1 & n-3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & & n-3 \end{vmatrix}$$

$$=2(n-3)(n-4)^{n-i-1}\ (i=2,3,\cdots,n-1)\,,$$

then

$$P_2=3(n-2)\{2(n-3)(n-4)^{n-3}\}(n-4)^{1+2+3+\cdots+(n-3)}\{2(n-3)\}^{n-2} \tag{4.11}$$
$$=3(n-2)\{2(n-3)\}^{n-1}(n-4)^{n(n-3)/2}\,.$$

Since

$$\begin{vmatrix} 1/v & 0 \\ 0 & Q_1 \end{vmatrix} \sim P_1\,, \qquad \begin{vmatrix} 1/v & 0 & 0 \\ 0 & Q_1 & 0 \\ 0 & 0 & Q_2 \end{vmatrix} \sim P_2\,,$$

it will be shown that

$$|Q_1|\sim n\{(n-2)(n-3)/2\}^{n-1} \tag{4.12}$$

$$|Q_2|\sim 2(n-1)(n-2)^{n-1}(n-4)^{n(n-3)/2} \tag{4.13}$$

$$|Q_3|\sim 3\cdot 2^{n-1}(n-2)(n-3)^{n-1}(n-4)^{n(n-3)/2}\,. \tag{4.14}$$

From lemmas 4.2 and 4.4 it follows that

$$C_p\left(\begin{Vmatrix} Q_1 & & \\ & Q_2 & \\ & & Q_3 \end{Vmatrix}\right) \tag{4.15}$$
$$=C_p(Q_1)C_p(Q_2)C_p(Q_3)(|Q_1|,|Q_2|)_p(|Q_3|,|Q_2|)_p(|Q_3|,|Q_1|)_p=(-1,-1)_p\,.$$

Moreover, the following relations can be shown from lemma 4.3.

$$C_p(\rho_1Q_1)=(-1,\rho_1)_p^{n(n+1)/2}(\rho_1,|Q_1|)_p^{n}C_p(Q_1)\,, \tag{4.16}$$

$$C_p(\rho_2Q_2)=(-1,\rho_2)_p^{n(n-1)(n-2)(n-3)/8}(\rho_2,|Q_2|)_p^{n(n-3)/2+1}C_p(Q_2)\,, \tag{4.17}$$

$$C_p(\rho_3Q_3)=(-1,\rho_3)_p^{n(n-1)(n-5)(n-2)(n^2-4n-3)/72}(\rho_3,|Q_3|)_p^{n(n-1)(n-5)/6+1}C_p(Q_3)\,. \tag{4.18}$$

5. Necessary conditions for the existence of regular symmetrical PBIB design of T_3 type

In this section, we shall show the non-existence of certain regular symmetrical PBIB designs of T_3 type. If the design is symmetrical, i.e., $v=b$ and $r=k$, then the incidence matrix N is a square matrix with elements 0 and 1, hence in the regular case NN' must be a perfect square. Thus first of all

$$\rho_1^{n-1}\rho_2^{n(n-3)/2}\rho_3^{n(n-1)(n-5)/6}\sim 1\,, \tag{5.1}$$

and from (4.1) and lemmas 4.2 and 4.4 it follows that

$$(5.2)\qquad C_p\left(\begin{Vmatrix} \rho_1 Q_1 & & \\ & \rho_2 Q_2 & \\ & & \rho_3 Q_3 \end{Vmatrix}\right)$$
$$=C_p(\rho_1 Q_1)C_p(\rho_2 Q_2)C_p(\rho_3 Q_3)(|\rho_1 Q_1|, |\rho_2 Q_2|)_p(|\rho_1 Q_1|, |\rho_3 Q_3|)_p(|\rho_2 Q_2|, |\rho_3 Q_3|)_p$$
$$=(-1, -1)_p .$$

Then, we get from lemma 4.4 and relations (4.15), (4.16), (4.17) and (4.18)

$$(5.3)\qquad (\rho_1, \rho_2)_p^{n(n-1)(n-3)/2}(|Q_1|, \rho_2)_p^{n(n-3)/2}(\rho_1, |Q_2|)_p^{n-1}$$
$$\cdot(\rho_1, \rho_3)_p^{n(n-1)^2(n-5)/6}(|Q_1|, \rho_3)_p^{n(n-1)(n-5)/6}(\rho_1, |Q_3|)_p^{n-1}$$
$$\cdot(\rho_2, \rho_3)_p^{n^2(n-1)(n-3)(n-5)/12}(|Q_2|, \rho_3)_p^{n(n-1)(n-5)/6}$$
$$\cdot(\rho_2, |Q_3|)_p^{n(n-3)/2}(-1, \rho_1)_p^{n(n-1)/2}(-1, \rho_2)_p^{n(n-1)(n-2)(n-3)/8}$$
$$\cdot(-1, \rho_3)_p^{n(n-1)(n-5)(n-2)(n^2-4n-3)/72}(\rho_1, |Q_1|)_p^{n}$$
$$\cdot(\rho_2, |Q_2|)_p^{n(n-3)/2+1}(\rho_3, |Q_3|)_p^{n(n-1)(n-5)/6+1}=1.$$

From (4.9) and (5.1) it follows that

$$(5.4)\qquad (-1, \rho_1)_p^{n(n-1)/2}(-1, \rho_2)_p^{n(n-1)(n-2)(n-3)/8}$$
$$\cdot(-1, \rho_3)_p^{n(n-1)(n-5)(n-2)(n^2-4n-3)/72}(\rho_1, |Q_1|)_p(\rho_2, |Q_2|)_p$$
$$\cdot(\rho_3, |Q_3|)_p(\rho_1, \rho_2)_p^{n(n-1)(n-3)/2}(-1, \rho_3)_p^{n(n-1)(n-5)/6}=1.$$

Substituting (4.12), (4.13) and (4.14) into (5.4), consequently we get the following theorem.

THEOREM. *Necessary conditions for the existence of regular symmetrical* PBIB *design of T_3 type are*

(i) $\rho_1^{n-1}\rho_2^{n(n-3)/2}\rho_3^{n(n-1)(n-5)/6}\sim 1$,

(ii) $O_p=(-1, \rho_1)_p^{n(n-1)/2}(-1, \rho_2)_p^{n(n-1)(n-2)(n-3)/8}(-1, \rho_3)_p^{n(n-1)(n-5)(n^3-6n^2+5n+18)/72}$
$$\cdot(\rho_1, \rho_2)_p^{n(n-1)(n-3)/2}(\rho_1, n)_p(\rho_1, (n-2)(n-3)/2)_p^{n-1}$$
$$\cdot(\rho_2, 2(n-1))_p(\rho_2, n-2)_p^{n-1}(\rho_2, n-4)_p^{n(n-3)/2}$$
$$\cdot(\rho_3, 3(n-2))_p(\rho_3, 2(n-3))_p^{n-1}(\rho_3, n-4)_p^{n(n-3)/2}=1$$

for all primes p.

Examples of non-existent PBIB designs of T_3 type.

(1) $n=7;\ v=b=35,\ r=k=8,\ \lambda_1=1,\ \lambda_2=2,\ \lambda_3=2,$

$O_p=(-1, 1)_p(-1, 6)_p(-1, 9)_p(1, 7)_p(6, 12)_p(9, 15)_p \quad O_3=(2/3)=-1.$

Hence this design is impossible.

(2) $n=7$; $v=b=35$, $r=k=11$, $\lambda_1=4$, $\lambda_2=3$, $\lambda_3=2$,

$$O_p=(-1,\ 16)_p(-1,\ 6)_p(-1,\ 6)_p(16,\ 7)_p(6,\ 3)_p(6,\ 15)_p$$
$$O_3=(5/3)=(2/3)=-1.$$

Hence this design is impossible.

(3) $n=7$; $v=b=35$, $r=k=12$, $\lambda_1=4$, $\lambda_2=4$, $\lambda_3=3$,

$$O_p=(-1,\ 11)_p(-1,\ 6)_p(-1,\ 9)_p(11,\ 7)_p(6,\ 3)_p(9,\ 15)_p \quad O_3=(2/3)=-1.$$

Hence this design is impossible.

(4) $n=19$; $v=b=969$, $r=k=57$, $\lambda_1=9$, $\lambda_2=3$, $\lambda_3=3$,

$$O_p=(-1,\ 228)_p(-1,\ 36)_p(228,\ 19)_p(36,\ 51)_p$$
$$O_{19}=(3,\ 19)_{19}=(3/19)=-(19/3)=-1.$$

Hence this design is impossible.

6. A numerical example

Table 6.1 gives the yield which are produced by a set of three workers and lay-out of a design of the T_3 type with treatments (sets of three workers) indicated by sets of capital letters within brackets.

The parameters of the design are: $v=35$, $n=7$, $b=35$, $r=k=4$, $\lambda_1=1$, $\lambda_2=0$, $\lambda_3=0$.

$$B_1^*=2\{3(n-3)B_0+(2n-9)B_1+(n-9)B_2-9B_3\}/\{n(n-2)(n-3)\}$$
$$=(12B_0+5B_1-2B_2-9B_3)/70,$$

$$B_2^*=\{3(n-3)(n-4)B_0+(n-4)(n-7)B_1-(4n-22)B_2+18B_3\}/$$
$$\{(n-1)(n-2)(n-4)\}=(6B_0-B_2+3B_3)/15,$$

$$B_3^*=\{(n-3)(n-4)(n-5)B_0-(n-4)(n-5)B_1+2(n-5)B_2-6B_3\}/$$
$$\{(n-2)(n-3)(n-4)\}=(12B_0-3B_1+2B_2-3B_3)/30,$$

where B_1^*, B_2^*, B_3^* are idempotents of the association algebra which is the linear closure of association matrices B_0, B_1, B_2 and B_3.

The computational details of analysis of variance are given in Table 6.2 and Table 6.3 (see also Ogawa & Ishii [8]).

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Table 6.1

(The yields in this Table are not actual samples but conceivable data.)

Block number	1	2	3	4	5	6	7	8	9	10
	12.5 (ABC)	11.4 (ABC)	12.0 (ABC)	12.5 (ABD)	12.1 (ABD)	17.2 (ABE)	18.2 (ACD)	17.6 (ACD)	12.4 (ACE)	20.1 (ADE)
	12.8 (ABD)	10.3 (ABE)	11.3 (ABF)	11.2 (ABE)	10.5 (ABF)	15.2 (ABF)	17.1 (ACE)	14.3 (ACF)	10.4 (ACF)	19.8 (ADF)
	13.0 (ACD)	13.1 (ACE)	12.1 (ACF)	14.5 (ADE)	13.5 (ADF)	13.6 (AEF)	20.3 (ADE)	16.9 (ADF)	12.5 (AEF)	13.4 (AEF)
	10.4 (BCD)	12.0 (BCE)	14.0 (BCF)	17.1 (BDE)	12.0 (BDF)	12.4 (BEF)	21.4 (CDE)	14.5 (CDF)	9.8 (CEF)	13.6 (DEF)
Total	48.7	46.8	49.4	55.3	48.1	58.4	77.0	63.3	45.1	66.9
Block number	11	12	13	14	15	16	17	18	19	20
	12.1 (ABC)	16.4 (ACD)	12.6 (ABD)	15.1 (ABE)	10.2 (ABF)	13.6 (ACE)	11.4 (ACF)	19.1 (ADE)	17.4 (ADF)	15.3 (AEF)
	15.2 (ABG)	15.2 (ACG)	13.1 (ABG)	16.2 (ABG)	9.1 (ABG)	14.3 (ACG)	12.0 (AFG)	21.4 (ADG)	16.6 (ADG)	14.0 (AEG)
	14.7 (ACG)	20.0 (ADG)	17.4 (ADG)	14.3 (AEG)	11.0 (BFG)	10.2 (AEG)	16.2 (ACG)	16.3 (AEG)	12.6 (AFG)	13.8 (AFG)
	16.4 (BCG)	15.0 (CDG)	15.3 (BDG)	13.2 (BEG)	10.4 (AFG)	12.5 (CEG)	12.4 (CFG)	18.0 (DEG)	15.4 (DFG)	12.3 (EFG)
Total	58.4	66.6	58.4	58.8	40.7	50.6	52.0	74.8	62.0	55.4

Block number	21	22	23	24	25	26	27	28	29	30
	12.0 (B C D)	10.7 (B C D)	15.2 (B C G)	11.9 (B C E)	11.3 (B C E)	16.4 (B C F)	20.1 (B D E)	17.1 (B D F)	14.5 (B E F)	16.5 (B D E)
	10.2 (B C E)	12.1 (B C F)	13.5 (B C D)	12.3 (B C F)	17.1 (B C G)	19.3 (B C G)	18.1 (B D G)	13.3 (B D G)	14.2 (B E G)	14.7 (B D F)
	15.3 (B D E)	10.4 (B D F)	12.3 (B E G)	10.6 (B E F)	16.5 (B E G)	17.1 (C F G)	17.3 (B E G)	16.1 (B F G)	16.1 (B F G)	16.5 (B E F)
	15.0 (C D E)	10.0 (C D F)	15.2 (C D G)	10.1 (C E F)	13.4 (C E G)	15.8 (B F G)	15.2 (D E G)	17.0 (D F G)	15.0 (E F G)	13.4 (D E F)
Total	52.5	43.2	56.2	44.9	58.3	68.6	70.7	63.5	59.8	61.1

Block number	31	32	33	34	35	
	13.4 (C D E)	17.4 (C D E)	15.2 (C D F)	7.5 (D E F)	8.5 (C E F)	
	11.4 (C E F)	16.1 (C D G)	12.3 (C D G)	6.7 (D E G)	7.8 (C D G)	
	12.3 (C D F)	15.3 (C E G)	10.2 (C F G)	7.8 (D F G)	9.3 (C F G)	
	10.7 (D E F)	16.3 (D E G)	14.0 (D F G)	6.5 (E F G)	8.0 (E F G)	Grand total
Total	47.8	65.1	51.7	28.5	33.6	1942.2

Table 6.2

Treatments	T	Q	B_1Q	B_2Q	B_3Q	$70B_1^*Q$	$15B_2^*Q$	$30B_3^*Q$
A B C	48.0	−2.825	−2.075	14.100	−9.200	10.325	−58.650	28.125
A B D	50.0	−2.625	20.575	−7.000	−10.950	183.925	−41.600	−74.375
A B E	53.8	−1.025	−6.675	8.200	−0.500	−57.575	−15.850	25.625
A B F	47.2	−1.950	−6.750	9.875	−1.175	−66.325	−25.100	20.125
A C D	65.2	1.300	20.350	−17.225	−4.425	191.625	11.750	−66.625
A C E	56.2	1.325	−9.125	7.150	0.650	−49.875	2.750	55.625
A C F	48.2	−4.250	1.750	0.875	1.625	−58.625	−21.500	−59.375
A D E	74.0	5.500	15.950	−30.725	9.275	123.725	91.550	−71.125
A D F	67.6	7.525	0.175	−6.500	−1.200	114.975	48.050	80.375
A E F	58.4	−1.650	−15.200	17.275	−0.425	−126.525	−28.450	61.625
A B G	53.6	−0.475	0.775	1.525	−1.825	11.550	−9.850	0.500
A C G	60.4	3.500	−2.850	−2.050	1.400	19.250	27.250	42.250
A D G	75.4	9.950	0.600	−3.500	−7.050	192.850	42.050	131.750
A E G	54.8	−5.100	3.700	0.950	0.450	−48.650	−30.200	−71.750
A F G	48.8	−3.725	6.175	−9.375	6.925	−57.400	7.800	−102.750
B C D	46.6	−3.550	13.450	3.100	−13.000	135.450	−63.400	−37.750
B C E	45.4	−5.225	11.825	−23.125	16.525	−106.050	41.350	−194.000
B C F	54.8	3.275	−16.175	6.125	6.775	−114.800	33.850	79.750
B C G	68.0	7.625	−15.025	1.900	5.500	−36.925	60.350	123.875
B D E	69.0	9.100	−15.500	13.350	−6.950	67.550	20.400	203.250
B D F	54.2	0.225	4.775	−1.825	−3.175	58.800	−6.350	−5.750
B D G	59.0	−3.200	18.300	−7.475	−7.625	136.675	−34.600	−85.375
B E F	54.0	−2.050	−8.550	−2.850	13.450	−182.700	30.900	−45.000
B E G	61.2	−0.700	−8.500	4.125	5.075	−104.825	6.900	10.125
B F G	59.0	0.850	−3.200	−12.375	14.725	−113.575	61.650	−49.125
C D E	67.2	6.600	−7.750	6.450	−5.300	75.250	17.250	131.250
C D F	52.0	0.500	1.200	5.600	−7.300	66.500	−24.500	35.500
C D G	58.6	−1.300	16.550	−8.575	−6.675	144.375	−19.250	−62.375
C E F	39.8	−3.050	−15.250	14.650	3.650	−175.000	−22.000	27.500
C E G	49.0	−2.900	−4.900	4.625	3.175	−97.125	−12.500	−20.375
C F G	49.0	−2.475	0.775	−9.250	10.950	−105.875	27.250	−83.375
D E F	45.2	−5.875	19.375	−21.325	7.825	−1.400	9.550	−194.750
D E G	56.2	−3.575	12.325	−3.000	5.750	76.475	−35.700	−68.625
D F G	54.2	2.775	−3.625	8.600	−7.750	67.725	−15.200	84.625
E F G	41.8	−2.525	−27.475	37.700	−7.700	−173.775	−75.950	150.625

Elements of the vector $\boldsymbol{T}$ are total yields of the treatments and elements of the vector $\boldsymbol{Q}$ are the adjusted yields of the treatments.

Table 6.3

	s.s.	*d.f.*	*m.s.*
Treatments (adjusted for blocks)	206.1391	34	
Main effects of factors $A, B, \cdots, G$	49.3435 $\left(\frac{k}{rk-\rho_1} Q'B_1^* Q\right)$	6 $(n-1)$	8.224
Two factor interactions of $A, B, \cdots, G$	72.1929 $\left(\frac{k}{rk-\rho_2} Q'B_2^* Q\right)$	14 $\left(\frac{n(n-3)}{2}\right)$	5.157
Three factor interactions of $A, B, \cdots, G$	84.6027 $\left(\frac{k}{rk-\rho_3} Q'B_3^* Q\right)$	14 $\left(\frac{n(n-1)(n-5)}{6}\right)$	6.043
Blocks (unadjusted)	994.13	34	
Error	301.44	71	4.246
Total	1295.57	139	