CERTAIN CLASS OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS

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1. Introduction and summary

A characteristic function (abbr. ch. f.) $\varphi(t)$ is said to be stable if to every $a, b>0$, there corresponds $c>0$ such that the equation

$$\varphi(ct)=\varphi(at)\varphi(bt) \tag{1}$$

holds.

The most familiar example of stable ch. f. is the ch. f. $\varphi(t)=e^{(-\sigma^2/2)t^2}$ of the normal distribution with mean zero.

Non-normal stable ch. f. is written as either

$$\varphi(t)=e^{i\gamma t-\beta|t|} \qquad \text{(the Cauchy distribution)}$$

or

$$\varphi(t)=\exp\left[-C\,|t|^{\alpha}\left(\cos\gamma-i\frac{t}{|t|}\sin\gamma\right)\right]$$

where $\qquad 0<\alpha<2\,, \qquad \alpha\neq 1\,.$

We are interested in the ch. f. $\varphi(t)$ for which

$$\varphi(t)=\varphi(at)\varphi(bt) \tag{2}$$

holds, where a and b are given non-zero constants. The author considered this problem in the previous paper [4], but the result was far from the satisfactory.

Yu V. Linnik discussed, in his elaborate paper [2], the more general problem. He treated the ch. f. $\varphi(t)$ satisfying the equation

$$\varphi(a_1 t)\cdots\varphi(a_n t)=\varphi(b_1 t)\cdots\varphi(b_n t)\,, \tag{3}$$

where a's and b's are given real constants. A necessary condition is given in the theorem 5 of the paper. We shall give in the present paper a necessary and sufficient condition under which (2) holds ($a\geqq b>0$), which contains the following proposition: if $\log a/\log b$ is an irrational number, the ch. f. $\varphi(t)$ satisfying the relation (2) is stable. We shall also show some properties of the corresponding probability distributions.

2. Some lemmas

Throughout this section, we assume $|\varphi(t)|\not\equiv 1$. From equation (2) we find for all positive integer n

$$\varphi(t)=\prod_{0}^{n}\varphi^{\binom{n}{k}}(a^k b^{n-k}t)\,. \tag{4}$$

LEMMA 1. *$1>|a|,\ |b|>0$.*

PROOF. Since $|\varphi(t)|$ is a symmetric ch. f. and is bounded by 1, we obtain, using (4),

$$\begin{aligned}|\varphi(t)| &= |\varphi(a^n t)|\cdot\prod_{0}^{n-1}|\varphi^{\binom{n}{k}}(a^k b^{n-k}t)|\\ &\leqq |\varphi(a^n t)| = |\varphi(|a|^n t)|\,.\end{aligned}$$

Hence if $|a|>1$,

$$|\varphi(t)|\geqq|\varphi(|a|^{-n}t)|\geqq|\varphi(0)|=1\,,\quad\text{or}\quad|\varphi(t)|\equiv 1\,.$$

If $|a|=1$, we obtain, from (2),

$$|\varphi(t)|=|\varphi(at)|\cdot|\varphi(bt)|=|\varphi(t)|\cdot|\varphi(bt)|\,.$$

Since $|\varphi(t)|>0$ in the suitable neighbourhood N of $t=0$, we have $|\varphi(bt)|=1$ on N. This implies $|\varphi(t)|\equiv 1$. q. e. d.

By the above lemma there exists a unique positive number α such that

$$|a|^\alpha+|b|^\alpha=1\,. \tag{5}$$

We shall denote the set of all ch. f.'s satisfying (1) and (5) by $T_\alpha(a, b)$.

LEMMA 2. *Every ch. f. $\varphi(t)$ in $T_\alpha(a, b)$ is infinitely divisible.*

PROOF[1]. It is enough to show that there exists a system $\{\varphi_{n,j}(t)\}$ of ch. f.'s such that

(i) $\varphi(t)=\prod_{1}^{j_n}\varphi_{n,j}(t)$ for all n

and

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1) See [3], p. 57.

(ii) $\lim_{n\to\infty}[\sup_{1\leq j\leq j_n} |\varphi_{n,j}(t)-1|]=0$ for all t.

For $j=1, 2, \cdots, 2^n$, let $\varphi_{n,j}(t)=\varphi(a^k\cdot b^{n-k}t)$, where k is a positive integer such that

$$\binom{n}{0}+\cdots+\binom{n}{k}\leqq j<\binom{n}{0}+\cdots+\binom{n}{k+1}.$$

Then from (4) we obtain

$$\varphi(t)=\prod_0^n \varphi^{\binom{n}{k}}(a^kb^{n-k}t)$$
$$=\prod_0^{2^n} \varphi_{n,j}(t).$$

On the other hand,

$$\lim_{n\to\infty}[\sup_{1\leq j\leq 2^n} |\varphi_{n,j}(t)-1|]=\lim_{n\to\infty}[\sup_{0\leqq k\leqq n}|\varphi(a^kb^{n-k}t)-1|]=0$$

q. e. d.

LEMMA 3. *$|\varphi(t)|<1$ if $t\neq 0$.*

PROOF. If we have $|\varphi(t_0)|=1$ for some $t_0\neq 0$, then $|\varphi(t)|$ is a periodic function with a period ρ, say. Using (4) with $t=m\rho$, where m is an arbitrary integer, we obtain

$$|\varphi(ma^n\rho)|=1.$$

Since the set $\{ma^n\rho;\ m, n \text{ integer}, n\geqq 0\}$ is dense in $(-\infty, \infty)$ and since $|\varphi(t)|$ is continuous, we have $|\varphi(t)|\equiv 1$. q. e. d.

In what follows we assume unless otherwise stated

$$1>a\geqq b>0$$

and

(6) $$a^\alpha+b^\alpha=1.$$

LEMMA 4. *Let $g(x)$ be a real valued function defined on $(0, \infty)$ such that*

(7) $$g(x)=a^\alpha g(ax)+b^\alpha g(bx)\geqq 0.$$

Then

(8) $$\sup_{x\geqq u} g(x)=\sup_{u/b\geqq x\geqq u} g(x),$$

(9) $$\inf_{x\geqq u} g(x)=\inf_{u/b\geqq x\geqq u} g(x).$$

PROOF. From (7) we obtain for all n

$$g(x)=\sum_0^n \binom{n}{k}(a^\alpha)^k(b^\alpha)^{n-k}g(a^kb^{n-k}x). \tag{10}$$

For $x\geqq u$, let

$$K_{n,x}=\{k\,;\ g(a^kb^{n-k}x)\geqq g(x)\},$$

which is non-void. Since

$$g(a^kb^{n-k}x)=a^\alpha g(a^{k+1}b^{n-k}x)+b^\alpha g(a^kb^{n+1-k}x),$$

$K_{n,x}\ni k$ implies either $K_{n+1,x}\ni k$ or $K_{n+1,x}\ni k+1$, and $a^kb^{n-k}x\geqq u/b$ implies $a^{k+1}b^{n-k}x\geqq a^kb^{n-k+1}x\geqq u$. But, since $\max_{0\leqq k\leqq n}|a^kb^{n-k}x|=|a^nx|$ tends to 0 as n tends to infinity, we can choose a positive integer n and k in $K_{n,x}$ such that

$$u\leqq y\equiv a^kb^{n-k}x\leqq u/b.$$

By the definition of $K_{n,x}$, we see that

$$g(y)\geqq g(x).$$

Hence we have

$$\sup_{x\geqq u} g(x)\geqq \sup_{u/b\geqq x\geqq u} g(x).$$

The reverse inequality is trivial and we obtain (8). The relation (9) is obtained similarly.

LEMMA 5. *If a real valued function $M(x)\not\equiv 0$ defined on $(0,\infty)$ is monotone non-decreasing and satisfies*

$$M(x)=M\left(\frac{1}{a}x\right)+M\left(\frac{1}{b}x\right), \tag{11}$$

then $M(x)<0$ for all x and $\lim_{x\to\infty}M(x)=0$. A necessary and sufficient condition that $\int_0^u x^\gamma dM(x)<\infty$ for all $u>0$ is that $\gamma>\alpha$. Moreover, if $0<\beta<\alpha$, then,

$$\int_1^\infty x^\beta dM(x)<\infty. \tag{12}$$

PROOF. From (11) we have as usual that for all n

$$M(x)=\sum_0^n \binom{n}{k} M\left(\frac{x}{a^kb^{n-k}}\right). \tag{13}$$

Then $M(x)\leqq 0$ and $\lim_{x\to\infty}M(x)=0$ readily follow from the monotonicity and the relation (13). If $M(x_0)=0$ for some $x_0>0$, then $M(x)=0$ for all $x\geqq x_0$.

For a given $y>0$, choose n so large that $y/a^k b^{n-k} \geqq x_0$, $k=1, 2, \cdots, n$. Then

$$M(y)=\sum_0^n \binom{n}{k} M\left(\frac{y}{a^k b^{n-k}}\right)=0 .$$

Thus $M(x)\equiv 0$ contrary to the hypothesis of the lemma.

To prove the second assertion, let

$$g(x)=-\frac{1}{x^\alpha} M\left(\frac{1}{x}\right) .$$

The function $g(x)$ satisfies the relation (7) of lemma 4.

Since both x^α and $M(1/x)$ are monotone, we have

$$0<C\equiv -M\left(\frac{u}{b^2}\right) \cdot \left(\frac{b}{u}\right)^\alpha \leqq g(x) \leqq D \equiv -M\left(\frac{u}{b}\right) \cdot \left(\frac{u}{b^2}\right)^\alpha < \infty$$

$$\text{for} \quad \frac{b^2}{u} \leqq x \leqq \frac{b}{u}, \quad u>0 .$$

Hence, using lemma 4 we obtain for all $x \geqq b/u$

$$C \leqq g(x) \leqq D$$

or equivalently for all $x \leqq b/u$

$$0<C\leqq -M(x) \cdot x^\alpha \leqq D < \infty . \tag{14}$$

Now we have, on the one hand,

$$\begin{aligned} \int_{a^n u}^{u} x^r dM(x) &\leqq \sum_0^{n-1} [M(a^k u)-M(a^{k+1}u)](a^k u)^r \\ &= \sum_0^{n-1}\left[-M\left(\frac{a^{k+1}}{b}u\right) \cdot (a^k u)^r\right] \\ &= \sum_0^{n-1}\left[-M\left(\frac{a^{k+1}}{b}u\right) \cdot \left(\frac{a^{k+1}}{b}u\right)^\alpha\right] \cdot \left(\frac{b}{a}\right)^\alpha \cdot (a^k u)^{r-\alpha} \\ &\leqq D \cdot \left(\frac{b}{a}\right)^\alpha u^{r-\alpha}(1+a^{r-\alpha}+ \cdots +(a^{r-\alpha})^{n-1}) , \end{aligned} \tag{15}$$

and on the other hand,

$$\begin{aligned} \int_{b^n u}^{u} x^r dM(x) &\geqq \sum_0^{n-1} [M(b^k u)-M(b^{k+1}u)](b^{k+1}u)^r \\ &\geqq C \cdot \left(\frac{a}{b}\right)^\alpha (bu)^{r-\alpha}(1+b^{r-\alpha}+ \cdots +(b^{r-\alpha})^{n-1}) . \end{aligned} \tag{16}$$

The second assertion is an immediate consequence of the inequalities (15) and (16).

Finally we prove (12). To this end we define c_n, $n=0, 1, \cdots$, as

$$c_0=1.$$

$$c_n=\begin{cases} c_{n-1}/a & \text{if } M\left(\frac{c_{n-1}}{a}\right)\geqq a^\alpha M(c_{n-1}), \\ c_{n-1}/b & \text{if } M\left(\frac{c_{n-1}}{a}\right)<a^\alpha M(c_{n-1}). \end{cases}$$

The sequence $\{c_n\}$ is strictly monotone increasing and tends to infinity as $n\to\infty$. For a given positive integer k, let l be the number of indices i's such that $1\leqq i\leqq k$, $c_i=c_{i-1}/a$. Then since, as easily seen, $M(c_{i-1}/a)<a^\alpha M(c_{i-1})$ is equivalent to $M(c_{i-1}/b)>b^\alpha M(c_{i-1})$, we obtain

$$\begin{aligned}[M(c_k)-M(c_{k-1})]c_k^\beta &\leqq -M(c_{k-1})\cdot c_k^\beta\leqq -b^{-\beta}M(c_{k-1})\cdot c_{k-1}^\beta \\ &\leqq -(a^{\alpha-\beta})^l(b^{\alpha-\beta})^{k-l-1}\cdot M(1)\cdot b^{-\alpha}\leqq -(a^{\alpha-\beta})^{k-1}\cdot M(1).\end{aligned}$$

Hence

$$\begin{aligned}\int_1^{C_n} x^\beta dM(x) &\leqq \sum_1^n [M(c_k)-M(c_{k-1})]c_k^\beta \\ &\leqq -M(1)\frac{1}{b^\alpha(1-a^{\alpha-\beta})}<\infty\end{aligned}$$

as was to be proved. q. e. d.

3. Representation of $\varphi(t)$

Let $\boldsymbol{M}$ be the set of all monotone non-decreasing functions $M(x)$ defined on $(0, \infty)$ such that $\lim_{x\to\infty} M(x)=0$ and $\int_0^u x^2dM(x)<\infty$ for all $u>0$. Then the complex valued function $\varphi(t)$ defined on the real axis is infinitely divisible ch. f. if and only if it admits the P. Lévy representation,

$$\text{(17)}\qquad \log\varphi(t)=i\gamma t-\frac{\sigma^2}{2}t^2+\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x) + \int_{-\infty}^0\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x)$$

where $$M(x),\ -N(-x)\in \boldsymbol{M}.$$

The expression (17) is unique.

Let $\boldsymbol{M}_\alpha(a, b)$ the set of all monotone non-decreasing functions $M(x)$ defined on $(0, \infty)$ such that

$$M(x)=M\left(\frac{x}{a}\right)+M\left(\frac{x}{b}\right) \tag{18}$$

holds. It follows from lemma 5 that if $\alpha<2$ every function of $\boldsymbol{M}_\alpha(a, b)$ also belongs to $\boldsymbol{M}$, while if $\alpha\geqq 2$ no function except $M(x)\equiv 0$ does.

We shall prove the following theorem:

THEOREM 1. *In order that $\boldsymbol{T}_\alpha(a, b)$ contains a non-degenerate ch. f., it is necessary and sufficient that $\alpha\leqq 2$. If $\alpha\leqq 2$, ch. f. $\varphi(t)\in T_\alpha(a, b)$ admits the following representation, with $M(x)$, $-N(-x)\in \boldsymbol{M}_\alpha(a, b)$.*

(*i*) *case $\alpha=2$*

$$\log\varphi(t)=-\frac{\sigma^2}{2}t^2 \qquad \text{(the normal distribution)} \tag{19}$$

(*ii*) *case $1<\alpha<2$*

$$\log\varphi(t)=\int_0^\infty (e^{itx}-1-itx)dM(x)+\int_{-\infty}^0 (e^{itx}-1-itx)dN(x) \tag{20}$$

(*iii*) *case $\alpha=1$*

$$\log\varphi(t)=i\gamma t+\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x)+\int_{-\infty}^0\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x) \tag{21}$$

with

$$\lim_{\epsilon\to 0}\left[a\int_\epsilon^{\epsilon/a} xdH(x)+b\int_\epsilon^{\epsilon/b} xdH(x)\right]=0 \tag{22}$$

$$H(x)\equiv M(x)+N(-x)$$

(*iv*) *case $0<\alpha<1$*

$$\log\varphi(t)=\int_0^\infty (e^{itx}-1)dM(x)+\int_{-\infty}^0 (e^{itx}-1)dN(x)\,. \tag{23}$$

Conversely, the complex valued function $\varphi(t)$ determined by either (20) or (21) or (23) is a ch. f. in $\boldsymbol{T}_\alpha(a, b)$.

PROOF. By lemma 2, ch. f. $\varphi(t)$ in $\boldsymbol{T}_\alpha(a, b)$ can be written in the canonical form (17). But we have for any $c>0$

$$\int_0^\infty\left(e^{ictx}-1-\frac{ictx}{1+x^2}\right)dM(x)=i\gamma^+(c)t+\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM\left(\frac{x}{c}\right) \tag{24}$$

$$\int_{-\infty}^{0}\left(e^{ictx}-1-\frac{ictx}{1+x^2}\right)dN(x)=i\gamma^-(c)t+\int_{-\infty}^{0}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN\left(\frac{x}{c}\right)$$

where

$$\gamma^+(c)=\int_0^\infty\left(\frac{x}{1+x^2}-\frac{x}{1+(x/c)^2}\right)dM\left(\frac{x}{c}\right)$$

$$\gamma^-(c)=\int_{-\infty}^0\left(\frac{x}{1+x^2}-\frac{x}{1+(x/c)^2}\right)dN\left(\frac{x}{c}\right).$$

Since the expression (17) is unique, we obtain from (2) and (17), using the relations (24),

$$\begin{aligned}&\sigma^2(a^2+b^2)=\sigma^2\\ &M(x)=M(x/a)+M(x/b) \qquad x>0\\ &N(x)=N(x/a)+N(x/b) \qquad x<0.\end{aligned} \tag{25}$$

Since both $M(x)$ and $-N(-x)$ are monotone non-decreasing, we see that they belong to $\boldsymbol{M}_\alpha(a, b)$. The first assertion follows from (25) and from the fact that $\alpha\geqq 2$ implies $M(x)\equiv 0$, $N(x)\equiv 0$.

(i) From (2) and (17) with $M(x)\equiv 0$, $N(x)\equiv 0$, we further deduce that $\gamma=0$.

(ii) By lemma 5, we have

$$0\leqq\gamma_1\equiv\int_0^\infty\left(x-\frac{x}{1+x^2}\right)dM(x)\leqq\int_0^1 x^3dM(x)+\int_1^\infty xdM(x)<\infty, \tag{26}$$

and

$$0\geqq\gamma_2\equiv\int_{-\infty}^0\left(x-\frac{x}{1+x^2}\right)dN(x)=-\int_0^\infty\left(x-\frac{x}{1+x^2}\right)d(-N(-x))>-\infty. \tag{27}$$

Hence (17) reduces to

$$\begin{aligned}\log\varphi(t)=i(\gamma+\gamma_1+\gamma_2)t&+\int_0^\infty(e^{itx}-1-itx)dM(x)\\ &+\int_{-\infty}^0(e^{itx}-1-itx)dN(x).\end{aligned} \tag{28}$$

Using the condition (2), we easily obtain $\gamma+\gamma_1+\gamma_2=0$ as was to be proved. Suppose conversely $\varphi(t)$ is expressed like (20) with $M(x)$, $-N(-x)$ $\in\boldsymbol{M}_\alpha(a, b)(\subseteqq\boldsymbol{M})$. Then, owing to (26) and (27), it can be rewritten as (17). $\varphi(t)$ clearly satisfies the relation (2).

(iii) We have

$$\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x)$$

$$=\int_0^\infty\left(e^{iatx}-1-\frac{iatx}{1+x^2}\right)dM(x)+\int_0^\infty\left(e^{ibtx}-1-\frac{ibtx}{1+x^2}\right)dM(x)$$

$$+i\int_0^\infty(\sin tx-\sin atx-\sin btx)dM(x)\,.$$

Similar equation holds also for the negative part. Hence, in order that (2) holds it is necessary and sufficient that $\sigma^2=0$ and that

$$\int_0^\infty(\sin tx-\sin atx-\sin btx)dH(x)=0 \tag{29}$$

where
$$H(x)=M(x)+N(-x)\,.$$

An easy calculation shows that (29) is equivalent to (22).

(iv) By lemma 5, we have

$$0\leqq\gamma_3\equiv\int_0^\infty\frac{x}{1+x^2}dM(x)\leqq\int_0^1 xdM(x)+\int_1^\infty\frac{1}{2}dM(x)<\infty\,, \tag{30}$$

$$0\geqq\gamma_4\equiv\int_{-\infty}^0\frac{x}{1+x^2}dN(x)=-\int_0^\infty\frac{x}{1+x^2}d(-N(-x))>-\infty\,. \tag{31}$$

Hence (17) becomes

$$\log\varphi(t)=i(\gamma+\gamma_3+\gamma_4)t+\int_0^\infty(e^{itx}-1)dM(x)+\int_{-\infty}^0(e^{itx}-1)dN(x)\,.$$

We obtain using (2) that $\gamma+\gamma_3+\gamma_4=0$. The converse statement follows from the same argument as in the case (ii). q. e. d.

COROLLARY[1]. *Let a and b be non-zero constants such that $a^2+b^2=1$. Let X and Y be independent and identically distributed random variables. If $aX+bY$ is distributed as X, then the distribution is normal with mean zero.*

PROOF. Let $\psi(t)$ be the ch. f. of the distribution of X. Then $\varphi(t)\equiv\psi(t)\cdot\psi(-t)$ as well as $\psi(t)$ belongs to $\boldsymbol{T}_2(a, b)$. But since $\varphi(t)$ is symmetric, $\varphi(t)$ also belongs to $\boldsymbol{T}_2(|a|, |b|)$. By the theorem we have $\log\varphi(t)=e^{-(\sigma^2/2)t^2}$ and the desired result follows from the well-known H. Cramér theorem and the fact that $a+b\neq1$. q. e. d.

[1] See theorems I and II of [2], theorem I of [4].

4. Determination of $M(x)$

A well-known example of the function $M(x)$ in $\boldsymbol{M}_\alpha(a, b)$ is

$$M(x)=-\lambda/x^\alpha\,, \qquad \lambda>0\,,$$

which corresponds to a stable ch. f., i.e., if further $-N(-x)=-\mu/x^\alpha$, ch. f. $\varphi(t)$ determined by either (20) or (21) with $\lambda=\mu$ or (23) is the non-normal stable ch. f. But, in fact, $\boldsymbol{M}_\alpha(a, b)$ can not be exhausted by such functions. In this section we shall derive all possible functions.

For $M(x)\not\equiv 0$ in $\boldsymbol{M}_\alpha(a, b)$, let

$$f(t)=M(e^{-t}) \tag{32}$$

and let $A=\log a$, $B=\log b$. Then $f(t)$ is a monotone non-increasing function defined on the real axis and satisfies the relation

$$f(t)=f(t+A)+f(t+B)<0\,. \tag{33}$$

Thus the problem is reduced to solve the functional equation (33). Though the function $f(t)$ need not be continuous, the argument of Yu. V. Linnik [2] well applies.

Let $\boldsymbol{Z}$ be the set of all zeros of the entire function

$$\sigma(z)=1-a^z-b^z \tag{34}$$

of a complex variable $z=x+iy$. The only real zero of $\sigma(z)$ is clearly $z=\alpha$.

LEMMA 6. *All the zeros of $\sigma(z)$ are simple and are located in some strip $x_0\leqq x\leqq\alpha$, and the number of zeros in every circle of radius 1 does not exceed a finite number k.*

PROOF. It suffices to consider the case $a>b$. If $z=x+iy\in\boldsymbol{Z}$, we have

$$a^x(\cos yA+i\sin yA)+b^x(\cos yB+i\sin yB)=1$$

from which we get

$$(a^x-b^x)^2\leqq 1\leqq(a^x+b^x)^2. \tag{35}$$

The second assertion follows from (35) and the fact that α is the unique real zero of $\sigma(z)$.

Set $\eta=A/B$. Since $0>A=\log a>\log b=B$, we have $0<\eta<1$.

From the relations

$$\sigma(z)=1-a^z-b^z=0 \tag{36}$$

$$\sigma'(z) = -a^z \cdot A - b^z \cdot B = 0\,, \tag{37}$$

we deduce easily that

$$a^x = 1/(1-\eta)\,, \quad b^x = \eta/(1-\eta)$$

whence

$$0 = x \cdot A - x \cdot A = \eta \cdot xB - xA = \eta \log \eta + (1-\eta)\log(1-\eta) < 0\,.$$

This is a contradiction. Thus (36) and (37) can not be satisfied simultaneously, which proves the first assertion.

The third assertion follows from the Jensen theorem [5], which is applied to circles with centers at $\alpha+1+iy$ and radii equal to $2(\alpha-x_0+1)$. q. e. d.

The following lemma is the same as lemma 4 of [2].

LEMMA 7. *For any $\varepsilon>0$, there exists a positive constants $C(\varepsilon)$ such that $|z_0-\rho|>\varepsilon$ for all $\rho\in \boldsymbol{Z}$ implies $|\sigma(z_0)|>C(\varepsilon)$.*

From the inequality (14) and the definition (32) of $f(t)$, we find that

$$\begin{aligned} &0 \geqq f(t) > -C\cdot e^{\alpha t} && \text{if } t\geqq 0 \\ &0 \geqq f(t) > -C && \text{if } t<0 \end{aligned} \tag{38}$$

and that

$$\lim_{t\to-\infty} f(t) = 0\,. \tag{39}$$

Hence if $x = \operatorname{Re} z > \alpha$, the integral

$$\chi(z) = \int_0^\infty e^{-zt} f(t)dt \tag{40}$$

converges and is regular. Since for any real constant c,

$$\int_0^\infty e^{-zt} f(t+c)dt = e^{cz}\chi(z) - e^{cz}\int_0^c e^{-zt} f(t)dt\,,$$

we obtain, using the relation (33),

$$\sigma(z)\chi(z) = -a^z\cdot\int_0^A e^{-zt} f(t)dt - b^z\int_0^B e^{-zt} f(t)dt \equiv E(z)\,,$$

whence

$$\chi(z) = \frac{E(z)}{\sigma(z)} \qquad \text{if } \operatorname{Re} z > \alpha\,. \tag{41}$$

LEMMA 8. *$|E(z)/\sigma(z)|$ is bounded on the horizontal contours $z=x\pm$*

iy_n, $x \leqq x_1$, $n=1, 2, \cdots$, where $x_1 > \alpha$.

PROOF. First of all we note that $0 > A = \log a > B = \log b$. When $x = \operatorname{Re} z \leqq -x_1 < 0$, we have

$$|E(z)| \leqq C\left(e^{Ax}\int_A^0 e^{-xt}dt + e^{Bx}\cdot\int_B^0 e^{-xt}dt\right) < \frac{2C}{x_1}e^{Bx}$$

(cf. (38)).

On the other hand, if $x_2(>0)$ is sufficiently large we can choose $C_1 > 0$ so that

$$|\sigma(z)|^2 = 1 + e^{2Ax} + e^{2Bx} + 2e^{Ax}\cdot e^{Bx}\cos y(A-B) - 2(e^{Ax}\cos yA + e^{Bx}\cos yB) \geqq C_1^2 \cdot e^{2Bx}$$

holds when $x \leqq -x_2$. Therefore we obtain

$$\left|\frac{E(z)}{\sigma(z)}\right| \leqq \frac{2C}{C_1 x_1}$$

whenever $\operatorname{Re} z \leqq \min(-x_1, -x_2)$. The desired result follows from lemma 7.

q. e. d.

Between every two lines $y=n$ and $y=n+1 (n>0)$, draw vertical straight lines $y=y_n$ at a distance greater than $\varepsilon>0$ from all zeros of $\sigma(z)$. (cf. lemma 6).

We have, for $t>0$,

$$(42) \qquad \int_0^t f(\tau)d\tau = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{x_1-iy_n}^{x_1+iy_n} e^{tz}\frac{E(z)}{z\sigma(z)}dz\,.$$

Using lemmas 7 and 8, we see that

$$\frac{1}{2\pi i}\int_{x_1-iy_n}^{x_1+iy_n} e^{tz}\frac{E(z)}{z\cdot\sigma(z)}dz = \sum_0^{n-1} S_m(t) + O\left(\frac{1}{n}\right),$$

where $S_m(t)$ denotes the sum of residues of the integrand between two pairs of straight lines: $y=y_m$, $y=y_{m+1}$; $y=-y_m$, $y=-y_{m+1}$, and where $S_0(t)$ denotes the sum of residues between $y=y_1$ and $y=-y_1$.

Therefore (42) becomes

$$(43) \qquad \int_0^t f(\tau)d\tau = \sum_0^\infty S_n(t)\,.$$

We shall say that a zero $\rho \in \boldsymbol{Z}$ is active if the residue of the integrand $e^{tz}\dfrac{E(z)}{z\cdot\sigma(z)}$ at ρ does not vanish.

LEMMA 9. *If ρ is an active zero, then $Re\,\rho=\alpha$.*

PROOF. The lemma is an immediate consequence of the lemma 6 of [2]. In fact, if σ_1 is the greatest lower bound of the real part of active zeros, then from that lemma, σ_1 itself is an active zero, whence $\sigma_1=\alpha$. q. e. d.

Let

$$\xi_0=\begin{cases} 0 & \text{if } \log a/\log b \text{ is irrational} \\ -2\pi k/\log a & \text{if } \log a/\log b=k/l \end{cases} \tag{44}$$

where k and l are relatively prime positive integers. Then, it is easily seen that all the active zeros of $\sigma(z)$ are exhausted by

$$\rho_n=\alpha+in\xi_0\,, \qquad n=0,\ \pm1,\ \pm2\,,\cdots$$

Since $\sigma(0)=-1\neq0$, the poles of $e^{tz}\dfrac{E(z)}{z\sigma(z)}$ are all simple and we obtain

$$\text{Res}\,\rho_n\left(e^{tz}\,\frac{E(z)}{z\sigma(z)}\right)=\eta_n\,e^{t\rho_n}=\eta_n e^{\alpha t+in\xi_0 t} \tag{45}$$

where
$$\eta_n=\frac{E(\rho_n)}{\rho_n}\cdot\lim_{z\to\rho_n}\frac{z-\rho_n}{\sigma(z)}\,,$$

and

$$\text{Res}_0\left(e^{tz}\frac{E(z)}{z\sigma(z)}\right)=\frac{E(0)}{\sigma(0)}\equiv\gamma\,. \tag{46}$$

Using (43), (45) and (46), and writing

$$g(t)=\sum_{-\infty}^{\infty}\eta_n e^{in\xi_0 t} \qquad -\infty<t<\infty\,, \tag{47}$$

we find that for all $t>0$

$$\int_0^t f(\tau)d\tau=\gamma+e^{\alpha t}\cdot g(t)\,. \tag{48}$$

Suppose first that $\xi_0\neq0$. Differentiating (48) we obtain

$$\begin{aligned} f(t)&=e^{\alpha t}\cdot(g(t)+g'(t)) \qquad t>0\,,\\ &=e^{\alpha t}\cdot h(t)\,, \end{aligned} \tag{49}$$

where $h(t)\equiv g(t)+g'(t)(-\infty<t<\infty)$ is a periodic function with the period $C\equiv2\pi/\xi_0$ as well as $g(t)$ and $g'(t)$.

Rewrite (33) as

(50) $$f(t)=f(t-A)+f(t+B-A)\,.$$

Since $0>A>B$, the value of $f(t)$ at $t\leqq 0$ can be computed from the values at $t>0$ through (50). But the function $e^t\cdot h(t)$ satisfies the relation (33) for all t, whence the equation (49) holds for all t. Thus we obtain that

(51) $$h(t)=e^{-\alpha t}\cdot f(t)=e^{-\alpha t}\cdot M(e^{-t})$$

is a periodic function with the period $C=2\pi/\xi_0$.

If, conversely, real valued function $M(x)$ defined on $(0, \infty)$ is such that $h(t)=e^{-\alpha t}\cdot M(e^{-t})$ is a periodic function with the period C, then $M(x)$ satisfies

$$M(x)=M(x/a)+M(x/b)\,.$$

Thus we have,

THEOREM 2. *Suppose that $\log a/\log b=k/l$ is a rational number with k and l being relatively prime positive integers. A necessary and sufficient condition that a real valued function $M(x)$ defined on $(0, \infty)$ belongs to $\boldsymbol{M}_\alpha(a, b)$ is that $M(x)$ be monotone non-decreasing and that $h(t)\equiv e^{-\alpha t}M(e^{-t})$ be a periodic function with the period*

$$C=2\pi/\xi_0=-\log a/k=-\log b/l=-A/k=-B/l\,.$$

Suppose next that $\xi_0=0$. Then writing $\lambda=-g(0)$, we find from (47) and (48),

(52) $$f(t)=-\lambda e^{\alpha t} \qquad t>0\,.$$

As before (52) holds in fact for all t, and hence

(53) $$M(x)=-\lambda/x^\alpha \qquad x>0\,.$$

Thus we have,

THEOREM 3. *If $\log a/\log b$ is irrational, every ch. f. of $\boldsymbol{T}_\alpha(a, b)$ is stable.*

5. Examples

When $\log a/\log b$ is rational, we can construct functions $M(x)$ of $\boldsymbol{M}_\alpha(a, b)$ other than $M(x)=-\lambda/x^\alpha$. Let $\xi_0(\neq 0)$ be defined as (44), and let λ_k, μ_k, $k=1, 2, \cdots, n$ be arbitrary real numbers.

Take λ sufficiently large so that

(54) $$\lambda>\sum_1^n(|\lambda_k|+|\mu_k|)(1+k\xi_0/\alpha)\,.$$

Then the function

(55) $$f(t)=e^{\alpha t}(-\lambda+\sum_{1}^{n}(\lambda_k \cos k\xi_0 t+\mu_k \sin k\xi_0 t))$$

is monotone decreasing. Moreover, $f(t)e^{-\alpha t}$ is a periodic function with the period $C=2\pi/\xi_0$.

Thus,

(56) $$M(x)\equiv f(-\log x) = \frac{-\lambda+\sum_{1}^{n}(\lambda_k \cos(k\xi_0 \log x)-\mu_k \sin(k\xi_0 \log x))}{x^{\alpha}}$$

belongs to $\boldsymbol{M}_\alpha(a, b)$.

A simple example of $\xi_0=0$ is given by $\alpha=1$, $a=2/3$ and $b=1/3$. Thus, if X and Y are independent and identically distributed random variables, and if $(2X+Y)/3$ is distributed as X, then the distribution is a Cauchy distribution.

6. Properties

THEOREM 4. *The distribution function $F(x)$ corresponding to non-degenerate ch. f. $\varphi(t)$ of $\boldsymbol{T}_\alpha(a, b)$ is absolutely continuous and it's probability density function $p(x)$ can be differentiated infinitely many times. It is an analytic function if $\alpha\geqq 1$, and is an entire function if $2\geqq\alpha>1$.* p(x)

PROOF. Let

(57) $$g(t)=\begin{cases} -\log|\varphi(t)|/|t|^{\alpha} & t\neq 0 \\ 0 & t=0 \end{cases}$$

(58) $$|\varphi(t)|=e^{-g(t)|t|^{\alpha}}.$$

Then $g(t)$ and $g(-t)$ satisfy the conditions of lemma 4. Moreover, there exists a positive number N such that $g(t)\geqq N>0$, $g(-t)\geqq N>0$ for $1\leqq t\leqq 1/b$, since otherwise $g(t_0)=0$ or $|\varphi(t_0)|=1$ for some $1\leqq t_0\leqq 1/b$, contrary to lemma 3.

Hence from lemma 4,

(59) $$\inf_{|x|\geqq 1} g(t)=\inf_{1/b\geqq|x|\geqq 1} g(t)\geqq N>0.$$

By conversion formula,

$$F(x)-F(x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx_0}-e^{-itx}}{it}\varphi(t)dt.$$

Differentiating both sides formally $n+1$ times, we get

$$p^{(n)}(x)\equiv F^{(n+1)}(x)=\frac{(-1)^n}{2\pi}\int_{-\infty}^{\infty} t^n e^{-itx}\varphi(t)dt\,. \tag{60}$$

Using (58) and (59), we obtain from (60)

$$|p^{(n)}(x)|\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty} t^n\cdot e^{-g(t)|t|^\alpha}dt\leqq\frac{1}{\pi}\Big[\int_0^1 t^n dt+\int_1^\infty t^n e^{-N\cdot t^\alpha}dt\Big]$$
$$<\frac{1}{\pi}\Big[\frac{1}{n+1}+\frac{1}{\alpha}\Gamma\Big(\frac{n+1}{\alpha}\Big)N^{-(n+1)/\alpha}\Big]\leqq\frac{1}{\pi\alpha}\Gamma\Big(\frac{n+1}{\alpha}\Big)N_0^{-(n+1)/\alpha}<\infty$$

where $N_0>0$ is taken sufficiently small. Using Stirling's formula we see easily that

$$\lim_{n\to\infty}\frac{1}{n!}\Big[\frac{1}{\pi\alpha}\Gamma\Big(\frac{n+1}{\alpha}\Big)N_0^{-(n+1)/\alpha}\Big]^{1/n}=\begin{cases}0 & \text{if } \alpha>1\\ N_0^{-1} & \text{if } \alpha=1\,.\end{cases}$$

q. e. d.

THEOREM 5. *Suppose that $0<\alpha<2$. Every distribution corresponding to $\varphi(t)\in \boldsymbol{T}_\alpha(a, b)$ has finite absolute moments of order β $(0<\beta<\alpha)$. On the other hand, all absolute moments of order $\geqq\alpha$ are infinite, provided $|\varphi(t)|\not\equiv 1$.*

PROOF. It suffices to consider the symmetric case. Then, by theorem 1,

$$-\log\varphi(t)=\int_0^\infty(1-\cos tx)dG(x)\,, \tag{61}$$

where $G(x)\equiv M(x)-N(-x)\in \boldsymbol{M}_\alpha(a, b)\,.$

Noting that

$$A_\gamma\equiv\sup_{x>0}\frac{1-\cos x}{x^\gamma}<\infty \qquad (\gamma\leqq 2)$$

and

$$\lim_{x\to 0}\frac{1-\cos x}{x^\gamma}=0 \qquad (\gamma<2)\,,$$

we obtain for $t>0$, $\gamma<\alpha<2$,

$$-\log\varphi(t)/t^\gamma=\int_0^\infty\frac{1-\cos tx}{t^\gamma}dG(x)$$

$$\leqq t^{2-\gamma}\cdot A_2\int_0^1 x^2 dG(x)+A_\gamma\cdot\int_1^\infty x^\gamma dG(x)<\infty\ .$$

Hence if $0<\beta<\gamma<\alpha$, we find that

$$-\log\varphi(t)/t^\beta\to 0 \qquad \text{as} \qquad t\to 0$$

and hence that

$$(1-\varphi(t))/t^\beta\to 0 \qquad \text{as} \qquad t\to 0\,. \tag{62}$$

Letting $0<u<v<\infty$, we get

$$\begin{aligned}(1-\varphi(t))/t^\beta &=\frac{1}{\pi}\int_0^\infty\frac{1-\cos tx}{t^\beta}p(x)dx\\ &=\frac{1}{\pi}\int_0^\infty\frac{1-\cos x}{x^{1+\beta}}\left(\frac{x}{t}\right)^{1+\beta}p\left(\frac{x}{t}\right)dx\\ &\geqq\frac{1}{\pi}\int_u^v\frac{1-\cos x}{x^{1+\beta}}\left(\frac{x}{t}\right)^{1+\beta}p\left(\frac{x}{t}\right)dx\\ &=\left(\frac{x_0}{t}\right)^{1+\beta}p\left(\frac{x_0}{t}\right)\frac{1}{\pi}\int_u^v\frac{1-\cos x}{x^{1+\beta}}dx\ ,\end{aligned}$$

where $u\leqq x_0\leqq v$.

Using (62) we deduce

$$x^{1+\beta}p(x)\to 0 \qquad \text{as } x\to\infty.$$

Thus,

$$\int_{-\infty}^\infty |x|^\beta p(x)dx=2\cdot\int_0^\infty x^\beta p(x)dx<\infty\ .$$

On the other hand, if

$$\int_0^\infty x^\alpha p(x)dx<\infty\ ,$$

then

$$\begin{aligned}\frac{1-\varphi(t)}{t^\alpha}&=\frac{1}{\pi}\int_0^\infty\frac{1-\cos tx}{(tx)^\alpha}x^\alpha p(x)dx\\ &\leqq\frac{1}{\pi}\int_0^u\frac{1-\cos tx}{(tx)^\alpha}x^\alpha p(x)+A_\alpha\frac{1}{\pi}\int_u^\infty x^\alpha\cdot p(x)dx\\ &=\frac{1}{\pi}\ \frac{1-\cos tx_0}{(tx_0)^\alpha}\cdot\int_0^u x^\alpha p(x)dx+A_\alpha\frac{1}{\pi}\int_u^\infty x^\alpha p(x)dx\ .\end{aligned}$$

Making u so large that

$$A_{\alpha}\frac{1}{\pi}\int_{u}^{\infty}x^{\alpha}\cdot p(x)dx<\varepsilon/2\ ,$$

and then making t so small that

$$\frac{1}{\pi}\ \frac{1-\cos tx_{0}}{(tx_{0})^{\alpha}}\int_{0}^{u}x^{\alpha}p(x)dx<\varepsilon/2\ ,$$

we get

$$0<\frac{1-\varphi(t)}{t^{\alpha}}<\varepsilon\ .$$

Hence

$$g(t)\equiv-\frac{\log\varphi(t)}{t^{\alpha}}\rightarrow 0 \text{ as } t\rightarrow 0\ .$$

Since $g(t)$ satisfies the relation (7), we obtain $g(t)\equiv 0$ or $\varphi(t)\equiv 1$. q. e. d.

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REFERENCES

[1] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sum of Independent Random Variable*, Moscow 1949; English translation, Addison-Wesley, Cambridge, Mass., 1954.

[2] Yu. V. Linnik, "Linear forms and statistical criteria," *Ukrain. Mat. Zurnal*, 5 (1953), English translation which appeared in *Selected Translations in Math. Stat. and Prob.*, 3 (1962). 1-90.

[3] E. Lukacs, *Characteristic Functions*, Griffin's Stat. Mons., London, 1960.

[4] R. Shimizu, "Characterization of the normal distribution II," *Ann. Inst. Stat. Math.*, 14 (1962), 173-178.

[5] E. C. Titchmarsh, *The Theory of Function*, Oxford, 1932.

[6] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, 1946.