ON CHARACTERIZING THE EXPONENTIAL DISTRIBUTION BY ORDER STATISTICS

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(Received July 22, 1963; revised August 1, 1964)

1. Introduction

It is well known that if X_1 and X_2 ($X_1<X_2$) are two ordered observations from an exponential distribution having the probability density function (p.d.f.)

$$(1)\qquad f(x)=\frac{1}{\theta}e^{-x/\theta}, \qquad X,\ \theta>0 ,$$

then the random variables X_1 and $\omega=X_2-X_1$ are independently distributed. In this note we shall prove the converse result and thus the following theorem will be established:

THEOREM 1. *Let $F(x)$ be an absolutely continuous distribution function with $F(0)=0$. A necessary and sufficient condition that the p.d.f. $f(x)$ given in (1) is the density function of $F(x)$ is that the random variables X_1 and $\omega=X_2-X_1$ are independently distributed.*

Another characterization theorem is also given.

2. Exponential and power distributions

To establish the above theorem we first of all note the following relationship between the exponential and the power distribution.

LEMMA. *If the variable X has the p.d.f. given in (1), then the variable Y defined by*

$$(2)\qquad Y^{m+1}=b^{m+1}e^{-X/\theta}, \qquad m\geqq 0,\ b>0$$

follows the power distribution having the density function $\varphi(y)$ defined by

$$(3)\qquad \varphi(y)=\frac{m+1}{b^{m+1}}y^m, \qquad 0\leqq \underset{m\geqq 0}{Y}\leqq b$$

1) This research has been supported in part by the National Science Foundation under Grant No. G-19126 at the University of Minnesota.

and conversely.

The proof of the lemma is obvious.

It is to be noted that if X_r be the rth ordered observation in a sample of size n where X follows (1), then

$$Y_r=(b^{m+1}e^{-X_r/\theta})^{1/(m+1)}, \qquad r=1, 2, \cdots, n$$

will be the $(n-r+1)$th ordered observation in the corresponding sample of size n from the corresponding power distribution. Now to prove theorem 1 we refer to the following theorem proved by Fisz [4].

THEOREM 2. *Let $F(y)$ be an absolutely continuous distribution function with $F(0)=0$. The function $\varphi(y)$ given by (3) is the density function of $F(y)$ if and only if the random variables Y_1 and $Y_2/Y_1\,(Y_2<Y_1)$ are independently distributed (where Y_1 and Y_2 are two ordered observations from a population having the distribution function $F(y)$).*

3. Proof of the sufficiency condition of theorem 1

Let Y_1 and Y_2 be the values corresponding to X_1 and X_2 obtained from (2). That is

$$Y_1=(b^{m+1}e^{-X_1/\theta})^{1/(m+1)}, \qquad (X_2>X_1)$$

and

$$Y_2=(b^{m+1}e^{-X_2/\theta})^{1/(m+1)}, \qquad (Y_1>Y_2).$$

Now

$$Y_2/Y_1=\left\{\exp\left(-\frac{1}{\theta}(X_2-X_1)\right)\right\}^{1/(m+1)}. \tag{4}$$

Clearly, independence of Y_1 and Y_2/Y_1 implies independence of X_1 and (X_2-X_1) and vice versa.

Hence, if X_1 and X_2-X_1 are independently distributed Y_1 and Y_2/Y_1 are also independently distributed and by theorem 2, Y has the p.d.f. (3). Finally, from the lemma of section 2, X follows the exponential distribution.

Thus the sufficiency condition of theorem 1 is proved and hence theorem 1 is established.

An elementary direct proof of theorem 1, obtained through the courtesy of Professor E. S. Pearson [5], runs as follows. The joint density of X_1 and X_2 is

$$2f(x_1)f(x_2), \qquad X_1<X_2$$

and hence of X_1 and ω is

$$2f(x_1)f(x_1+\omega), \qquad \omega>0 .$$

But this must be, for independence, $g(x_1)h(\omega)$ for some $g(\ )$ and $h(\ )$. Putting $X_1=0$, $\omega=0$ in turn we have

$$f(x_1+\omega)=f(x_1)f(\omega)/f(0)$$

whence

$$f(x_1) \text{ is of the form } Ae^{-\lambda x_1} .$$

The case of n observations follows similarly. And hence independence of X_1, the smallest observation and $\omega=X_n-X_1$, the sample range in a sample of size n, characterizes the exponential distribution. Ferguson [3] has also characterized the exponential distribution using the above result. But his proof is different. Tanis [7] has used a different property to characterize the exponential distribution.

4. An alternative way of characterization

From theorems 1 and 2 the following theorems easily follow.

THEOREM 3. *Let $F(y)$ be an absolutely continuous distribution function with $F(0)=0$. Let $Y_1\geqq Y_2\geqq \cdots \geqq Y_n$ be an ordered set of n observations from a population having the distribution function $F(y)$. A necessary and sufficient condition that the function $\varphi(y)$ given by (3) is the density function of $F(y)$ is that the random variables*

$$(5) \qquad Z_1=\frac{Y_n}{Y_{n-1}}, \ Z_2=\frac{Y_{n-1}}{Y_{n-2}}, \ Z_3=\frac{Y_{n-2}}{Y_{n-3}}, \ \cdots, \ Z_{n-1}=\frac{Y_2}{Y_1}, \ Z_n=Y_1$$

are mutually independently distributed.

THEOREM 4. *Let $F(x)$ be an absolutely continuous distribution function with $F(0)=0$. Let $X_1\leqq X_2\leqq \cdots \leqq X_n$ be an ordered set of n observations from a population having the distribution function $F(x)$. A necessary and sufficient condition that the p.d.f. $f(x)$ given in (1) is the density function of $F(x)$ is that the random variables $Z_1=X_1$, $Z_2=X_2-X_1$, $Z_3=X_3-X_2$, $\cdots$, $Z_n=X_n-X_{n-1}$ are independently distributed.*

It follows that for the exponential distribution X_r and $X_l-X_m (l>m\geqq r)$ are independently distributed. In particular, the rth ordered statistic X_r and rth quasi-range $\omega_r=X_{n-r}-X_{r+1}$ are independently distributed.

Incidentally, above theorems furnish alternative proofs to many of the well known theorems and lemmas proved directly by Epstein and Sobel in [2].

Acknowledgments

The author is grateful to the referee for his valuable suggestions which considerably improved the presentation of the material.

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