ON THE NONCENTRAL DISTRIBUTION OF RAO'S U STATISTIC

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1. Introduction and summary

Let the joint density of a $(p+q)\times(p+q)$ positive definite symmetric matrix B, and a $p+q$ component column vector y be given by

$$(1.1)\qquad g(B, y)=C_1 \exp\left\{-\frac{1}{2} \operatorname{tr} \Sigma^{-1}[B+h^2(y-\mu)(y-\mu)']\right\},$$

where

$$(1.2)\qquad C_1=(h/2)^{(p+q)}(2\pi)^{-(p+q)(N+1)/2}|\Sigma|^{-(N+1)/2}\prod_{i=1}^{p+q} C(N-p-q+i).$$

Here h is a constant and $C(N)$ denotes the surface area of a unit N dimensional sphere. Let y be partitioned into two parts y_1 and y_2, y_1 being the column vector of first p components of y, and y_2 the column vector of next q components, and let the corresponding partitions of B, Σ, Σ^{-1}, and μ be

$$(1.3)\qquad B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}, \quad \Sigma=\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}, \quad \Sigma^{-1}=\begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22}\end{pmatrix}, \quad \mu=\begin{pmatrix} \mu_1 \\ \mu_2\end{pmatrix}.$$

Then Rao's U Statistic [5] may be defined by the relation

$$(1.4)\qquad 1+U=(1+T_{p+q}^2)/(1+T_p^2),$$

where

$$(1.5)\qquad T_{p+q}^2=h^2y'B^{-1}y, \qquad T_p^2=h^2y_1'B_{11}^{-1}y_1.$$

The U statistic is used for testing the hypothesis that $\mu_2=0$, against the alternative that $\mu_2\neq 0$. The noncentral joint density of the variates U and T_p^2 has been given by Narain [4]. Recently, Giri [2] has given an alternative derivation of the joint noncentral distribution of the variates R_2 and R_1 defined by

$$(1.6)\qquad R_2=(1-R_1)U/(1+U), \qquad R_1=T_p^2/(1+T_p^2).$$

Giri uses a different approach than that used by Narain for the derivation of the joint density of U and T_p^2. Giri also considers some properties of the statistic $Z=1/(1+U)$.

Since the distribution of U is of importance in discriminatory analysis, it might perhaps be useful to give a more concise, elegant, and straightforward derivation of the noncentral joint distribution of the variates U and T_p^2. We follow the same approach, for the derivation of the joint distribution, as that used by Narain, however, our method of derivation is different. We hope that the paper is at least of pedagogical interest.

Some results which are found useful in the sequel are stated in the next section.

2. Some useful results

Kabe [3] gives a generalization of Sverdrup's lemma [6]. The generalized lemma[1)] may be stated as follows. Let Y be a $p\times N$ matrix, D a given $q\times N$ matrix of rank q $(<N)$, $N\geqq p+q$. Then

$$(2.1)\quad \int_{YY'=G,\ DY'=V'} f(YY',\ DY')dY$$
$$=2^{-p}\prod_{i=1}^{p} C(N-p-q+i)|DD'|^{-p/2}f(G,\ V')|G-V(DD')^{-1}V'|^{(N-p-q-1)/2},$$

where V is $p\times q$, G is $p\times p$ positive definite symmetric, dY, as usual, denotes the product of the differentials of the elements of Y, and the integral is considered as a part of the volume integral over the appropriate range, i.e., $-\infty<Y<\infty$, $G\leqq YY'\leqq G+dG$, $V\leqq YD'\leqq V+dV$. In case f is a suitable density function, then the right hand side of (2.1) obviously represents the joint density of the matrices G and V.

In case D is a row vector d', then it immediately follows from the lemma that

$$(2.2)\quad \int_{(Y-\Omega)A(Y-\Omega)'=G,\ d'Y'=v'} f((Y-\Omega)A(Y-\Omega)',\ d'Y')dY$$
$$=2^{-p}\prod_{i=1}^{p} C(N-p-1+i)(d'A^{-1}d)^{-p/2}|A|^{-p/2}f(G,\ v')$$
$$\cdot\ |G-(v-\Omega d)(v-\Omega d)'/(d'A^{-1}d)|^{(N-p-2)/2},$$

where Ω is $p\times N$, A is positive definite symmetric $N\times N$, and v is a column vector of p components.

It is easily proved that

1) The lemma has been recommended for publication in the Annals of Mathematical Statistics.

(2.3) $$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(x-v)'M^{-1}(x-v)-\frac{1}{2}v'H^{-1}v\right\}dv$$
$$=(2\pi)^{p/2}|M^{-1}+H^{-1}|^{-1/2}\exp\left\{-\frac{1}{2}x'(M^{-1}-M^{-1}(M^{-1}+H^{-1})^{-1}M^{-1})x\right\}.$$

Here x is $p\times 1$, M is $p\times p$ positive definite symmetric, and H is also $p\times p$ positive definite symmetric.

Now we proceed to obtain the noncentral joint distribution of the variates U and T_p^2. We assume that all the integrals occurring in this paper are evaluated over the appropriate ranges of the variables of integration.

3. Distribution of the U statistic

Using (1.5), we find that

(3.1) $$T_{p+q}^2=h^2[y_1'B_{11}^{-1}y_1+z_2'D_{22}^{-1}z_2],$$

and using (1.1) we have that

(3.2) $$g(B_{11}, D_{22}, B_{21}, y_1, z_2)=C_1\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_{11}^{-1}B_{11}-\frac{1}{2}\operatorname{tr}\Sigma^{22}D_{22}\right.$$
$$-\frac{1}{2}\operatorname{tr}\Sigma^{22}(B_{21}-\Sigma_{21}\Sigma_{11}^{-1}B_{11})B_{11}^{-1}(B_{21}-\Sigma_{21}\Sigma_{11}^{-1}B_{11})'$$
$$-\frac{1}{2}h^2(y_1-\mu_1)'\Sigma_{11}^{-1}(y_1-\mu_1)$$
$$\left.-\frac{1}{2}h^2(z_2-\eta_2+B_{21}B_{11}^{-1}y_1-\Sigma_{21}\Sigma_{11}^{-1}y_1)'\Sigma^{22}(z_2-\eta_2+B_{21}B_{11}^{-1}y_1-\Sigma_{21}\Sigma_{11}^{-1}y_1)\right\},$$

where

(3.3) $$D_{22}=B_{22}-B_{21}B_{11}^{-1}B_{12}, \quad \eta_2=\mu_2-\Sigma_{21}\Sigma_{11}^{-1}\mu_1, \quad z_2=y_2-B_{21}B_{11}^{-1}y_1.$$

We now consider the integral

(3.4) $$I=\int_R \exp\left\{-\frac{1}{2}h^2(z_2-\eta_2+B_{21}B_{11}^{-1}y_1-\Sigma_{21}\Sigma_{11}^{-1}y_1)'\Sigma^{22}(z_2-\eta_2+B_{21}B_{11}^{-1}y_1\right.$$
$$\left.-\Sigma_{21}\Sigma_{11}^{-1}y_1)-\frac{1}{2}\operatorname{tr}\Sigma^{22}(B_{21}-\Sigma_{21}\Sigma_{11}^{-1}B_{11})B_{11}^{-1}(B_{21}-\Sigma_{21}\Sigma_{11}^{-1}B_{11})'\right\}dB_{21},$$

where the region R of integration is determined by the conditions

(3.5) $$(B_{21}-\Sigma_{21}\Sigma_{11}^{-1}B_{11})B_{11}^{-1}(B_{21}-\Sigma_{21}\Sigma_{11}^{-1}B_{11})'=G, \quad y_1'B_{11}^{-1}B_{21}'=v'.$$

The integral (3.4), assuming $p\geqq q$, is evaluated by using the result (2.2);

and we find that

$$(3.6)\qquad I=2^{-q}\prod_{i=1}^{q}C(p-q-1+i)(y_1'B_{11}^{-1}y_1)^{-q/2}|B_{11}|^{q/2}$$
$$\cdot\ \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{22}G-\frac{1}{2}h^2(z_2-\eta_2+v-\Sigma_{21}\Sigma_{11}^{-1}y_1)'\Sigma^{22}(z_2-\eta_2+v-\Sigma_{21}\Sigma_{11}^{-1}y_1)\right\}$$
$$\cdot\ |G-(v-\Sigma_{21}\Sigma_{11}^{-1}y_1)(v-\Sigma_{21}\Sigma_{11}^{-1}y_1)'/(y_1'B_{11}^{-1}y_1)|^{(p-q-2)/2}.$$

Again we consider the integral

$$(3.7)\qquad I_1=\int I\,dG\,dv.$$

Setting

$$(3.8)\qquad G_1=G-(v-\Sigma_{21}\Sigma_{11}^{-1}y_1)(v-\Sigma_{21}\Sigma_{11}^{-1}y_1)'/(y_1'B_{11}^{-1}y_1),$$

and integrating (3.7) with respect to G_1, by using the result ([1], p. 176, example 6), we find that

$$(3.9)\qquad I_1=(y_1'B_{11}^{-1}y_1)^{-q/2}|B_{11}|^{q/2}(2\pi)^{q(p-1)/2}|\Sigma^{22}|^{-(p-1)/2}$$
$$\cdot\ \int\exp\left\{-\frac{1}{2}h^2(z_2-\eta_2+v-\Sigma_{21}\Sigma_{11}^{-1}y_1)'\Sigma^{22}(z_2-\eta_2+v-\Sigma_{21}\Sigma_{11}^{-1}y_1)\right.$$
$$\left.-\frac{1}{2}(v-\Sigma_{21}\Sigma_{11}^{-1}y_1)'\Sigma^{22}(v-\Sigma_{21}\Sigma_{11}^{-1}y_1)/(y_1'B_{11}^{-1}y_1)\right\}dv.$$

Now we use the result (2.3) to integrate the right hand side expression of (3.9) with respect to v and we obtain that

$$(3.10)\qquad I_1=(2\pi)^{qp/2}|\Sigma^{22}|^{-p/2}(1+h^2y_1'B_{11}^{-1}y_1)^{-q/2}|B_{11}|^{q/2}$$
$$\cdot\ \exp\left\{-\frac{1}{2}h^2(z_2-\eta_2)'\Sigma^{22}(z_2-\eta_2)/(1+h^2y_1'B_{11}^{-1}y_1)\right\}.$$

Using the results (3.2) and (3.10) it follows that

$$(3.11)\qquad g(B_{11},D_{22},y_1,z_2)$$
$$=C_2\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_{11}^{-1}B_{11}-\frac{1}{2}\operatorname{tr}\Sigma^{22}D_{22}-\frac{1}{2}h^2(y_1-\mu_1)'\Sigma_{11}^{-1}(y_1-\mu_1)\right.$$
$$\left.-\frac{1}{2}h^2(z_2-\eta_2)'\Sigma^{22}(z_2-\eta_2)/(1+h^2y_1'B_{11}^{-1}y_1)\right\}$$
$$\cdot\ (1+h^2y_1'B_{11}^{-1}y_1)^{-q/2}|D_{22}|^{(N-p-q-1)/2}|B_{11}|^{(N-p-1)/2},$$

where

$$(3.12)\qquad C_2=C_1(2\pi)^{qp/2}|\Sigma^{22}|^{-p/2}.$$

Noting that $h^2 y_1' B_{11}^{-1} y_1 = T_p^2$ has Hotelling's T^2 distribution, and using (3.11) we find the joint density of D_{22}, z_2, and T_p^2 to be

$$(3.13) \quad g(D_{22}, z_2\, T_p^2) = C_3 T_p^{(p-2)} (1+T_p^2)^{-(N+1)/2} {}_1F_1\left[\frac{N+1}{2}, \frac{p}{2}, \frac{(h^2\mu_1'\Sigma_{11}^{-1}\mu_1)T_p^2}{2(1+T_p^2)}\right] \cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{22}D_{22} - \frac{1}{2}h^2(z_2-\eta_2)'\Sigma^{22}(z_2-\eta_2)/(1+T_p^2)\right\} \cdot (1+T_p^2)^{-q/2}|D_{22}|^{(N-p-q-1)/2},$$

where

$$(3.14) \quad C_3 = C_2 h^{-p}(2\pi)^{p(N+1)/2}|\Sigma_{11}|^{(N+1)/2} \cdot \exp\left\{-\frac{1}{2}h^2\mu_1'\Sigma_{11}^{-1}\mu_1\right\}\left[B\left(\frac{1}{2}p, \frac{1}{2}(N+1-p)\right)\right]^{-1}.$$

However, for fixed T_p^2, we have $h^2 z_2' D_{22}^{-1} z_2/(1+T_p^2) = U$, and obviously for fixed T_p^2, U has Hotelling T^2 distribution, hence using (3.13) we find the joint density of U and T_p^2 to be

$$(3.15) \quad g(U, T_p^2) = C_4 \exp\left\{-\frac{1}{2}h^2\eta_2'\Sigma^{22}\eta_2/(1+T_p^2)\right\} \cdot T_p^{(p-2)}(1+T_p^2)^{-(N+1)/2}U^{(q-2)/2}(1+U)^{-(N-p+1)/2} \cdot \sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}[N+1]+r)}{\Gamma(\frac{1}{2}p+r)\ r!}\left(\frac{1}{2}h^2\mu_1'\Sigma_{11}^{-1}\mu_1\right)^r T_p^{2r}(1+T_p^2)^{-r} \cdot \sum_{s=0}^{\infty}\frac{\Gamma(\frac{1}{2}[N-p+1]+s)}{\Gamma(\frac{1}{2}q+s)\ s!}\left(\frac{1}{2}h^2\eta_2'\Sigma^{22}\eta_2/(1+T_p^2)\right)^s\left(U/(1+U)\right)^s,$$

where

$$(3.16) \quad C_4 = \exp\left\{-\frac{1}{2}h^2\mu_1'\Sigma_{11}^{-1}\mu_1\right\}\left[\Gamma\left(\frac{1}{2}[N-p+1]\right)\Gamma\left(\frac{1}{2}[N-p-q+1]\right)\right]^{-1}.$$

In the equation (3.15) first set $N+1=N_1$ and then set $h^2=N_1$, $p+q=p_1$, $p=q_1$ and further setting

$$(3.17) \quad R_2 = (1-R_1)U/(1+U), \qquad R_1 = T_p^2/(1+T_p^2)$$

write down the expression for the joint density of R_2 and R_1, in this joint density now substitute $p_1=p$, $q_1=q$, and $N_1=N$, and thus get the expression for the joint density of R_2 and R_1 as given by Giri ([2], p. 187, equation (2.13)).

4. Concluding remarks and acknowledgment

Since the conditional distribution of U, given T_p^2, is that of Hotelling's

T^2 statistic, U is conditionally uniformly most powerful invariant similar test.

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