ON TESTING THE HYPOTHESIS THAT SUBMATRICES OF THE MULTIVARIATE REGRESSION MATRICES OF k POPULATIONS ARE EQUAL

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1. Introduction and summary

Let $\prod_1, \prod_2, \cdots, \prod_k$ be k normal p variate populations which are independently distributed. Let

$$\begin{bmatrix} X_1^{(g)} \\ Z_1^{(g)} \end{bmatrix}, \begin{bmatrix} X_2^{(g)} \\ Z_2^{(g)} \end{bmatrix}, \cdots, \begin{bmatrix} X_{N_g}^{(g)} \\ Z_{N_g}^{(g)} \end{bmatrix}$$

denote N_g independent samples from the gth population $\prod_g$, where $X_\alpha^{(g)}$ and $Z_\alpha^{(g)}$ are vectors of observed and fixed variates, respectively. The relation between $E(X_\alpha^{(g)})$ and $Z_\alpha^{(g)}$ for the αth sample of the gth population is given by

$$E(X_\alpha^{(g)})=B_g Z_\alpha^{(g)} \qquad (\alpha=1, 2, \cdots, N_g,\ g=1, 2, \cdots, k),$$

where B_g is a $p\times q$ matrix. Partitioning B_g into two parts

$$B_g=(B_{g1}B_{g2})\,, \qquad g=1, 2, \cdots, k,$$

where B_{g1} is a $p\times q_1$ matrix and B_{g2} is a $p\times q_2$ matrix ($q=q_1+q_2$), we consider the hypothesis

$$H: \qquad B_{11}=B_{21}=\cdots=B_{k1}\ (=B_1^o)$$

and derive the likelihood ratio criterion and its distribution.

It is known that the likelihood ratio criterion is distributed as a ratio of central Wishart distributions with $N-kq$ and $N-kq_2$ degrees of freedom in the case where B_1^o is a known matrix. In this paper we consider the case where B_1^o is an unknown matrix. We assume that all the k populations have the same covariance matrix Σ. First we introduce the following matrix notations:

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$$X^{(g)}=[X_1^{(g)}, X_2^{(g)}, \cdots, X_{N_g}^{(g)}] \qquad (p\times N_g), \qquad (g=1, 2, \cdots, k)$$

$$X=[X^{(1)}, X^{(2)}, \cdots, X^{(k)}] \qquad (p\times N), \qquad N=\sum_{g=1}^{k} N_g,$$

$$Z^{(g)}=[Z_1^{(g)}, Z_2^{(g)}, \cdots, Z_{N_g}^{(g)}] \qquad (q\times N_g),$$

$$=\begin{bmatrix} Z_1^{(g)} \\ Z_2^{(g)} \end{bmatrix}=\begin{bmatrix} Z_{11}^{(g)}, Z_{21}^{(g)}, \cdots, Z_{N_g1}^{(g)} \\ Z_{12}^{(g)}, Z_{22}^{(g)}, \cdots, Z_{N_g2}^{(g)} \end{bmatrix} \qquad \begin{matrix} (q_1\times N_g) \\ (q_2\times N_g), \ q=q_1+q_2, \end{matrix}$$

where $Z_1^{(g)}$, $Z_2^{(g)}$, and $Z^{(g)}$ have ranks q_1, q_2 and q, respectively,

$$Z=[Z^{(1)}, Z^{(2)}, \cdots, Z^{(k)}] \qquad (q\times N)$$

$$=\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}=\begin{bmatrix} Z_1^{(1)}, Z_1^{(2)}, \cdots, Z_1^{(k)} \\ Z_2^{(1)}, Z_2^{(2)}, \cdots, Z_2^{(k)} \end{bmatrix} \qquad \begin{matrix} (q_1\times N) \\ (q_2\times N), \quad q=q_1+q_2 \end{matrix}$$

where Z_1, Z_2 and Z have ranks q_1, q_2 and q, respectively. We shall assume that $N_g\geqq p+q$ for all g.

2. The maximum likelihood estimates

Suppose that $\{X^{(g)}\}$ are k independent observation matrices and $\{Z^{(g)}\}$ are k known matrices. Since $\{X_\alpha^{(g)}\}$ are independently distributed according to $N[B_g Z_\alpha^{(g)}, \Sigma]$, the likelihood function of $B_1, B_2, \cdots, B_k$ and Σ is

$$L(X)=\prod_{g=1}^{k} L(X^{(g)})$$

$$=\frac{1}{(2\pi)^{(1/2)pN}|\Sigma|^{(1/2)N}}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}[X-(B_1Z^{(1)}, B_2Z^{(2)}, \cdots, B_kZ^{(k)})]\right.$$

$$\left.\cdot[X-(B_1Z^{(1)}, B_2Z^{(2)}, \cdots, B_kZ^{(k)})]'\right\}.$$

Let $\hat{B}_g^{\Omega}$ $(g=1, \cdots, k)$ and $\hat{\Sigma}_\Omega$ be the maximum likelihood estimates of B_g $(g=1, 2, \cdots, k)$ and Σ over the entire parameter space. Then we have

$$\hat{B}_g^{\Omega}=X^{(g)}Z^{(g)\prime}[Z^{(g)}Z^{(g)\prime}]^{-1} \qquad (g=1, 2, \cdots, k),$$

$$N\hat{\Sigma}_\Omega=\sum_{g=1}^{k} X^{(g)}[I_{N_g}-Z^{(g)\prime}(Z^{(g)}Z^{(g)\prime})^{-1}Z^{(g)}]X^{(g)\prime},$$

where I_{N_g} is the $N_g\times N_g$ identity matrix. Let

$$A=\begin{bmatrix} Z^{(1)\prime}(Z^{(1)}Z^{(1)\prime})^{-1}Z^{(1)} & & & 0 \\ & Z^{(2)\prime}(Z^{(2)}Z^{(2)\prime})^{-1}Z^{(2)} & & \\ 0 & & \ddots & \\ & & & Z^{(k)\prime}(Z^{(k)}Z^{(k)\prime})^{-1}Z^{(k)} \end{bmatrix}.$$

Then, we can represent $N\hat{\Sigma}_\Omega$ as

$$N\hat{\Sigma}_\Omega=X(I_N-A)X',$$

where I_N is the $N\times N$ identity matrix. Concerning this A, we have $A^2=A$, that is, A is an idempotent matrix, and rank $A=\mathrm{tr}A=kq$. Further, it is known that $X(I-A)X'$ is distributed according to the central Wishart distribution with $N-kq$ degrees of freedom, [1]. Now we estimate B_g and Σ under the hypothesis H. The likelihood function is written as

$$L(X)=\frac{1}{(2\pi)^{(1/2)PN}|\Sigma|^{(1/2)N}}\cdot\exp\Big\{-\frac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}[X-\overset{o}{B}_1Z_1-(B_{12}Z_2^{(1)},B_{22}Z_2^{(2)},\cdots,B_{k2}Z_2^{(k)})]\cdot[X-\overset{o}{B}_1Z_1-(B_{12}Z_2^{(1)},B_{22}Z_2^{(2)},\cdots,B_{k2}Z_2^{(k)})]'\Big\}.$$

Let $\hat{B}_1^{o\omega}$, $\hat{B}_{g2}^{\omega}$ $(g=1,2,\cdots,k)$ and $\hat{\Sigma}_\omega$ be the maximum likelihood estimates of $\overset{o}{B}_1$, B_{g2} $(g=1,2,\cdots,k)$ and Σ in the parameter space restricted by the null hypothesis H, respectively. Then we have

$$\hat{B}_1^{o\omega}=\sum_{g=1}^{k}X^{(g)}(Z_1^{(g)}P^{(g)})'A_{11.2}^{-1},$$

where

$$P^{(g)}=I_{N_g}-Z_2^{(g)\prime}(Z_2^{(g)}Z_2^{(g)\prime})^{-1}Z_2^{(g)},\qquad (g=1,2,\cdots,k),$$

$$A_{11.2}=\sum_{g=1}^{k}[Z_1^{(g)}Z_1^{(g)\prime}-(Z_1^{(g)}Z_2^{(g)\prime})(Z_2^{(g)}Z_2^{(g)\prime})^{-1}(Z_2^{(g)}Z_1^{(g)\prime})],$$

and

$$\hat{B}_{g2}^{\omega}=X^{(g)}Z_2^{(g)\prime}(Z_2^{(g)}Z_2^{(g)\prime})^{-1}-\hat{B}_1^{o\omega}(Z_1^{(g)}Z_2^{(g)\prime})(Z_2^{(g)}Z_2^{(g)\prime})^{-1},\qquad (g=1,2,\cdots,k),$$

and

$$N\hat{\Sigma}_\omega=XX'-[\sum_{g=1}^{k}X^{(g)}(Z_1^{(g)}P^{(g)})']A_{11.2}^{-1}[\sum_{g=1}^{k}X^{(g)}(Z_1^{(g)}P^{(g)})']'-\sum_{g=1}^{k}X^{(g)}Z_2^{(g)\prime}(Z_2^{(g)}Z_2^{(g)\prime})^{-1}Z_2^{(g)}X^{(g)\prime}.$$

Here we assume that rank $A_{11.2}=q_1$. It can easily be seen that $\hat{B}_1^{o\omega}$ and $\hat{B}_{g2}^{\omega}$ are unbiased estimates of B_1^o and B_{g2}, respectively. Let

$$B=[Z_1^{(1)}P^{(1)}, Z_1^{(2)}P^{(2)}, \cdots, Z_1^{(k)}P^{(k)}]'A_{11.2}^{-1}[Z_1^{(1)}P^{(1)}, Z_1^{(2)}P^{(2)}, \cdots, Z_1^{(k)}P^{(k)}] \quad (N\times N),$$

and

$$C=\begin{bmatrix} Z_2^{(1)\prime}(Z_2^{(1)}Z_2^{(1)\prime})^{-1}Z_2^{(1)} & & & 0 \\ & Z_2^{(2)\prime}(Z_2^{(2)}Z_2^{(2)\prime})^{-1}Z_2^{(2)}, & & \\ & & \ddots & \\ 0 & & & Z_2^{(k)\prime}(Z_2^{(k)}Z_2^{(k)\prime})^{-1}Z_2^{(k)} \end{bmatrix} \quad (N\times N).$$

Then we see that $B^2=B$, $C^2=C$ and rank $B=\operatorname{tr} B=q_1$, rank $C=\operatorname{tr} C=kq_2$. We can write $N\hat{\Sigma}_\omega$ as

$$N\hat{\Sigma}_\omega=X(I_N-B-C)X'.$$

It is easily seen that $N\hat{\Sigma}_\omega$ is invariant for change of location

$$Y=X-E(X)=X-B_1^oZ_1-(B_{12}Z_2^{(1)}, B_{22}Z_2^{(2)}, \cdots, B_{k2}Z_2^{(k)}).$$

Y $(p\times N)$ is distributed according to

$$\frac{1}{(2\pi)^{(1/2)PN}|\Sigma|^{(1/2)N}}\cdot\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}YY'\right\},$$

and we have

$$N\hat{\Sigma}_\omega=Y(I_N-B-C)Y'.$$

3. The distribution of $N\hat{\Sigma}_\omega$

We now prove that $N\hat{\Sigma}_\omega$ is distributed according to the central Wishart distribution with $N-kq_2-q_1$ degrees of freedom.

1°. $YCY'=\sum_{g=1}^{k} Y^{(g)}Z_2^{(g)\prime}(Z_2^{(g)}Z_2^{(g)\prime})^{-1}Z_2^{(g)}Y^{(g)\prime}.$

Since $Z_2^{(g)}Z_2^{(g)\prime}(q_2\times q_2)$ has rank q_2, there exists a non-singular matrix $F^{(g)}$ such that

$$F^{(g)}(Z_2^{(g)}Z_2^{(g)\prime})F^{(g)\prime}=I_{q_2}.$$

Let

$$E_3^{(g)}=F^{(g)}Z_2^{(g)}\ (q_2\times N_g),\qquad (g=1,2,\cdots,k).$$

Then

$$E_3^{(g)}E_3^{(g)\prime}=F^{(g)}Z_2^{(g)}Z_2^{(g)\prime}F^{(g)\prime}=I_{q_2}.$$

Let

$$E_{3(kq_2\times N)}=\begin{bmatrix} E_3^{(1)} & & & & 0 \\ & E_3^{(2)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & E_3^{(k)} \end{bmatrix}$$

$$=\begin{bmatrix} F^{(1)} & & & & 0 \\ & F^{(2)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & F^{(k)} \end{bmatrix}\begin{bmatrix} Z_2^{(1)} & & & & 0 \\ & Z_2^{(2)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & Z_2^{(k)} \end{bmatrix}.$$

Then

$$E_3E_3'=I_{kq_2}.$$

By using $E_3^{(g)}$'s, we can represent C as

$$C=\begin{bmatrix} E_3^{(1)\prime}E_3^{(1)} & & & & 0 \\ & E_3^{(2)\prime}E_3^{(2)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & E_3^{(k)\prime}E_3^{(k)} \end{bmatrix}=E_3'E_3.$$

2°. $YBY'=Y[Z_1^{(1)}P^{(1)}, Z_1^{(2)}P^{(2)}, \cdots, Z_1^{(k)}P^{(k)}]'A_{11.2}^{-1}[Z_1^{(1)}P^{(1)}, Z_1^{(2)}P^{(2)}, \cdots, Z_1^{(k)}P^{(k)}].$

Since rank $A_{11.2}=q_1$, there exists also non-singular matrix R such that

$$RA_{11.2}R'=I_{q_1}.$$

Let

$$E_2=R[Z_1^{(1)}P^{(1)}, Z_1^{(2)}P^{(2)}, \cdots, Z_1^{(k)}P^{(k)}]\ (q_1\times N).$$

Then

$$E_2E_2'=RA_{11.2}R'=I_{q_1}.$$

We also have

$$B=E_2'E_2$$

and

$$E_2E_3'=R[Z_1^{(1)}P^{(1)}, Z_1^{(2)}P^{(2)}, \cdots, Z_1^{(k)}P^{(k)}]$$

$$\cdot\begin{bmatrix} Z_2^{(1)\prime} & & & 0 \\ & Z_2^{(2)\prime} & & \\ & & \ddots & \\ 0 & & & Z_2^{(k)\prime} \end{bmatrix}\begin{bmatrix} F^{(1)\prime} & & & 0 \\ & F^{(2)\prime} & & \\ & & \ddots & \\ 0 & & & F^{(k)\prime} \end{bmatrix}=0.$$

3°. Thus, $E_{23}=\begin{bmatrix} E_2 \\ E_3 \end{bmatrix}$ $((kq_2+q_1)\times N)$ has (kq_2+q_1) orthogonal rows such that $E_{23}E_{23}'=I_{kq_2+q_1}$. It is possible to find a $(N-kq_2-q_1)\times N$ matrix E_1 such that $E=\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$ is orthogonal ([1], p. 225, equation (12)).

Now, let

$$Y=[U_1, U_2, \cdots, U_N]E=UE.$$

Then

$$\begin{aligned} N\hat{\Sigma}_\omega &= UU'-U\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} E_2'E_2[E_1'E_2'E_3']U' \\ &\quad -U\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} E_3'E_3[E_1'E_2'E_3']U' \\ &= UU'-U\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{q_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}U'-U\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{kq_2} \end{bmatrix}U' \\ &= \sum_{\alpha=1}^{N} U_\alpha U_\alpha' - \sum_{\alpha=N-kq_2-q_1+1}^{N-kq_2} U_\alpha U_\alpha' - \sum_{\alpha=N-kq_2+1}^{N} U_\alpha U_\alpha' \\ &= \sum_{\alpha=1}^{N-kq_2-q_1} U_\alpha U_\alpha'. \end{aligned}$$

Here U_α is distributed according to $N[0, \Sigma]$ and independently of U_β $(\beta\neq\alpha)$. Therefore $N\hat{\Sigma}_\omega$ is distributed according to the central Wishart distribution with $N-kq_2-q_1$ degrees of freedom.

4. The likelihood ratio criterion

By the preceding consideration, under the null hypothesis, the likelihood ratio criterion λ is given by

$$V=\lambda^{2/N}=\frac{|N\hat{\Sigma}_\Omega|}{|N\hat{\Sigma}_\omega|}=\frac{|Y(I-A)Y'|}{|Y(I-B-C)Y'|}$$
$$=\frac{|Y(I-A)Y'|}{|Y(I-A)Y'+Y(A-B-C)Y'|}.$$

It is easily seen that $A-B-C$ is an idempotent matrix. Now, we apply theorem 1 of Hogg's [2] to

$$Y(I-B-C)Y'=Y(I-A)Y'+Y(A-B-C)Y',$$

where $Y(I-B-C)Y'$ and $Y(I-A)Y'$ are distributed according to the central Wishart distribution with $N-kq_2-q_1$ and $N-kq$ degrees of freedom, respectively, and $A-B-C$ is positive semi-definite. Then we see that $Y(A-B-C)Y'$ is distributed according to the central Wishart distribution with $(k-1)q_1$ degrees of freedom, independently of $Y(I-A)Y'$. Thus $\lambda^{2/N}$ is a U-statistic such that $U_{p\cdot(k-1)q_1\cdot N-kq}$ is distributed as $\lambda^{2/N}=\prod_{i=1}^{p} X_i$, where X_i has the beta density $\beta(x;\frac{1}{2}(N-kq+1-i),\frac{1}{2}(k-1)q_1)$, and $X_1, X_2, \cdots, X_p$ are independent with each other, [1].

Although we have treated the aspect of testing submatrices, the testing of matrices follows on similar lines, and to test the hypothesis $B_1=B_2=B_3=\cdots=B_k$, we have the criterion

$$\lambda_1^{2/N}=\frac{|YY'-YAY'|}{|YY'-YAY'+YAY'-YZ'(ZZ')^{-1}ZY'|},$$

whose distribution is that of $U_{p\cdot(k-1)q\cdot N-kq}$.

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