A CHARACTERIZATION OF A BIVARIATE DISTRIBUTION BY THE MARGINAL AND THE CONDITIONAL DISTRIBUTIONS OF THE SAME COMPONENT*

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1. Introduction

The problem of determining a bivariate distribution of x and y with the knowledge of the marginal distribution of a component, say x, together with the conditional distributions of the same component x is a problem of the mixtures of distributions and their identifiability. Some interesting results on mixtures and their identifiability have been obtained by Teicher ([2], [3]). The problem on mixtures can be also viewed as a problem on compound distributions. In this paper, some results on mixtures are obtained essentially by the use of the Laplace transform. At the end, three plausible definitions of a general character are suggested for the bivariate generalizations of a univariate distribution of the given form.

2. Results

Let x and y be two random variables which may be continuous, discrete or mixed. Let $f_{x|y}(x\mid y)$, $f_y(y)$, $f_x(x)$ be the conditional *p.d.f.* of x given y, the marginal *p.d.f.* of y and that of x respectively. Given $f_x(x)$ and $f_{x|y}(x\mid y)$, the bivariate *p.d.f.* given by $f_{x,y}(x, y)=f_y(y)\cdot f_{x|y}(x\mid y)$ can be found once one obtains $f_y(y)$ and it is obvious that $f_{x,y}(x, y)$ is unique if $f_y(y)$ is unique. We give a theorem here for the continuous case. Other cases can be treated similarly.

THEOREM 1. *Given $f_x(x)$ and $f_{x|y}(x|y)$, a sufficient condition for $f_y(y)$ (and hence for $f_{x,y}(x, y)$) to be unique is that the conditional p.d.f. of x given y is of the exponential form*

$$f_{x|y}(x|y)= \exp\,[y\,A(x)+B(x)+C(y)] \tag{1}$$

where an interval is contained in the range of $A(x)$.

PROOF. It is well-known that

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$$\int f_{x|y}(x \mid y) f_y(y) dy = f_x(x) \tag{2}$$

which because of (1), reduces to

$$\int \exp [y\, A(x)] \phi(y) dy = \psi(x) \tag{3}$$

where

$$\phi(y) = f_y(y) \exp [C(y)] \tag{4}$$

and

$$\psi(x) = f_x(x) \exp [-B(x)]\ . \tag{5}$$

Now, for given $f_x(x)$ and $f_{x|y}(x \mid y)$, $\psi(x)$ is known and by the uniqueness of the inverse Laplace transform, $\phi(y)$ and therefore $f_y(y)$ can be determined uniquely.

THEOREM 2. *For a bivariate distribution of x and y, where x and y are not independent, the following hold:*

(*i*) *x Poisson* $\Rightarrow$ *$x|y$ is not Poisson*

(*ii*) *$x|y$ Poisson* $\Rightarrow$ *x is not Poisson*

PROOF. That the Poisson distribution cannot be obtained by compounding two Poisson distributions has been obtained by Teicher [2]. We obtain the result of this theorem as follows:

Let

$$f_x(x) = \frac{e^{-\mu}\mu^x}{x\,!}\ , \quad x = 0, 1, 2, \cdots$$

and

$$f_{x|y}(x \mid y) = e^{-\lambda(y)}[\lambda(y)]^x / x\,!\ .$$

Denoting the *m.g.f.*'s of x and $x \mid y$ by $M_x(t)$ and $M_{x|y}(t)$, respectively, we have

$$\sum M_{x|y}(t) f_y(y) = M_x(t)$$

which, upon writing $e^t - 1 = s$ where $-1 < s < \infty$, gives

$$\sum e^{s\lambda(y)} f_y(y) = e^{\mu s}\ . \tag{6}$$

(6) implies that, if x and y are not independent (in which case $\lambda(y)$ depends on y), no solution exists for $f_y(y)$. The theorem now follows immediately.

THEOREM 3. *For a bivariate distribution of x and y, if the marginal distribution of x is binomial with parameters n and p_1 and the conditional distribution of x given y is binomial with parameters $n-y$ and p_2, then the marginal distribution of y is binomial with parameters n and (p_1/p_2) provided $p_1 \leqq p_2 \neq 0$.*

PROOF. We note that theorem 1 does not apply here. We offer the following proof. We have

$$f_x(x) = \binom{n}{x} p_1^x (1-p_1)^{n-x}, \quad x=0, 1, 2 \cdots n ; \quad 0<p_1<1$$

and

$$f_{x|y}(x \mid y) = \binom{n-y}{x} p_2^x (1-p_2)^{n-y-x}, \quad x=0, 1, 2 \cdots n-y ; \quad 0<p_2<1 .$$

Now,

$$\sum_{y=0}^{n} f_{x|y}(x \mid y) f_y(y) = f_x(x) \tag{7}$$

Multiplying (7) by t^x and summing over x, gives the identity in factorial $m.g.f.$'s as

$$\sum_{y=0}^{n} (p_2 t + q_2)^{n-y} f_y(y) = (p_1^t + q_1)^n , \tag{8}$$

where $q_i = 1 - p_i$, $i=1, 2$. To establish the uniqueness of the solution for $f_y(y)$, assume that $g_y(y)$ satisfies (8). Letting $d(y) = g_y(y) - f_y(y)$, we have

$$\sum_{y=0}^{n} \left(\sum_{k=0}^{n-y} a(k, y) t^k \right) d(y) = 0 , \tag{9}$$

where

$$a(k, y) = \binom{n-y}{k} p_2^k q_2^{n-y-k} .$$

It follows from (9) that for every k, one has

$$\sum_{y=0}^{n} a(k, y) d(y) = 0 , \qquad k=0, 1, 2, \cdots n$$

which yields a system of $(n+1)$ homogeneous equations linear in $d(y)$. The coefficient matrix is triangular and it can be easily verified that its determinant is $\pm p_2^{\frac{n(n+1)}{2}} \neq 0$. Hence $d(y)=0$ and the solution of $f_y(y)$ is unique. It can be verified that the solution is given by

$$f_y(y)=\binom{n}{y}\left(1-\frac{p_1}{p_2}\right)^y\left(\frac{p_1}{p_2}\right)^{n-y}, \quad \text{where } \frac{p_1}{p_2}\leqq 1 .$$

THEOREM 4. *For a bivariate distribution of x and y, if the marginal distribution of x is the Beta distribution of the second kind with parameters a and b and the conditional distribution of x given y is a Pearson Type III distribution with parameters y and a (y being the scale parameter), then the marginal distribution of y is a Pearson Type III distribution with parameters 1 and b.*

PROOF.

Let
$$f_x(x)=\frac{1}{B(a,\, b)}\,\frac{x^{a-1}}{(1+x)^{a+b}}\,, \quad 0<x<\infty$$

and

$$f_{x|y}(x\mid y)=\frac{e^{-yx}y^a x^{a-1}}{\Gamma(a)}\,, \quad y>0 .$$

Using (2), we obtain

$$\int_0^\infty e^{-yx}\phi(y)dy=\frac{\Gamma(a+b)}{\Gamma(b)}\cdot\frac{1}{(1+x)^{a+b}}$$

and by the uniqueness theorem of the Laplace transform

$$\phi(y)=\frac{e^{-y}y^{a+b-1}}{\Gamma(b)}\,,$$

and therefore

$$f_y(y)=\frac{e^{-y}y^{b-1}}{\Gamma(b)}\,.$$

COROLLARY. *If we replace the word " Beta distribution " in theorem 4 by " Pareto distribution " the results of the theorem hold.*

In particular if we let $f_x(x)=\dfrac{\alpha^\lambda e^{-\alpha x}x^{\lambda-1}}{\Gamma(\lambda)}\,,\ x>0$

and
$$f_{x|y}(x\mid y)=\frac{(y+\alpha)^{\lambda+1}e^{-x(y+\alpha)}x^\lambda}{\Gamma(\lambda+1)}\,, \quad \lambda>0\,,\ x,\, y>0\,,$$

using the same arguments it can be shown that

$$f_y(y)=\frac{\lambda}{\alpha}\left(\frac{\alpha}{y+\alpha}\right)^{\lambda+1}: \quad \text{the Pareto distribution.}$$

Finally we consider two exponential distributions discussed by Gumbel [1].

THEOREM 5. *For a bivariate distribution of x and y, if the marginal distribution of x is exponential and the conditional distribution of x given y is $f_{x|y}(x\,|\,y)=e^{-x(1+\delta y)}[(1+\delta x)(1+\delta y)-\delta]$, $0\leqq\delta\leqq1$, $x\geqq0$, $y\geqq0$, then the marginal distribution of y is unique and is the exponential distribution.*

PROOF. Let $f_x(x)=e^{-x}$, $x\geqq0$

and
$$f_{x|y}(x\,|\,y)=e^{-x(1+\delta y)}[(1+\delta x)(1+\delta y)-\delta]\,,$$
$$x\geqq0,\ y\geqq0,\ 0\leqq\delta\leqq1\,.$$

Using (2), we have the following integral equation:

$$\int_0^\infty e^{-x(1+\delta y)}[(1+\delta x)(1+\delta y)-\delta]f_y(y)dy=e^{-x}\,.$$

Putting $\theta=\delta x$, we have

$$(1+\theta-\delta)\int_0^\infty e^{-\theta y}f_y(y)dy+(1+\theta)\delta\int_0^\infty ye^{-\theta y}f_y(y)dy=1\,.$$

Writing $h(\theta)=\int_0^\infty e^{-\theta y}f_y(y)dy$, we obtain a linear differential equation of the first order $(1+\theta)\delta\cdot h'(\theta)-(1+\theta-\delta)h(\theta)+1=0$. Since $h(0)=1$, the solution of the above equation is clearly $h(\theta)=1/(1+\theta)$ and hence, by the uniqueness theorem of the Laplace transform,

$$f_y(y)=e^{-y},\ y\geqq0\,.$$

THEOREM 6. *For a bivariate distribution of x and y if the marginal distribution of x is exponential and the conditional distribution of x given y is*

$$f_{x|y}(x\,|\,y)=e^{-x}(1+\alpha-2\alpha e^{-y})-2\alpha e^{-2x}(1-2e^{-y})\,,$$
$$-1\leqq\alpha\leqq1,\ x\geqq0,\ y\geqq0\,,$$

the marginal distribution of y is not unique.

PROOF. By theorem 1 we obtain the following integral equation for $f_y(y)$:

$$\int_0^\infty[e^{-x}(1+\alpha-2\alpha e^{-y})-2\alpha e^{-2x}(1-2e^{-y})]\,f_y(y)dy=e^{-x}\,.$$

Upon simplification we have

$$\int_0^\infty e^{-y} f_y(y) dy = \frac{1}{2} .$$

In this case $f_y(y)$ is not unique.

3. Bivariate generalization of a univariate distribution

There can be several ways of defining the bivariate generalization of a univariate distribution of the given form. Therefore setting up plausible definitions of a general character for a bivariate generalization can be a meaningful problem. In the light of the results of this paper we offer here three such possibilities. Let the random variable $x \in P$, the family of univariate distributions of a given form. Let (x, y) be a bivariate random variable under consideration.

DEFINITION 1. The bivariate random variable (x, y) is said to belong to BP, the family of distributions constituting the bivariate generalization of the given form if

(i) $x \in P$

(ii) $x|y \in P$ and

(iii) x and y enjoy symmetry with respect to (i) and (ii),

i.e., the marginals and the conditionals of (x, y) belong to P.

DEFINITION 2. The bivariate random variable (x, y) is said to belong to BP if

(i) $x \in P$

(ii) x and $x | y$ recover (x, y) uniquely, and

(iii) x and y enjoy symmetry with respect to (i) and (ii),

i.e., the marginals of (x, y) belong to P and the knowledge of the marginal and the conditional of each component determines the bivariate distribution of (x, y) uniquely.

DEFINITION 3. The bivariate random variable (x, y) is said to belong to BP if (i) $x \in P$ and (ii) $y \in P$, i.e., the marginals of (x, y) belong to P.

Remarks: (i) Consistent with all definitions 1, 2 and 3, the traditional forms of the bivariate normal and bivariate binomial distributions are, in view of theorems 1 and 3, bivariate generalizations of univariate normal and binomial distributions respectively.

(ii) In view of theorem 2 of this paper, there connot be a bivariate generalization of the Poisson distribution consistent with definition 1.

(iii) The bivariate distribution determined by theorem 5 of this paper can be labeled bivariate exponential distribution consistent with definition 2 and definition 3 and not 1.

(iv) The bivariate distribution determined by theorem 6 with $f_y(y) = e^{-y}$ could not be labeled bivariate exponential distribution according to definition 1 and definition 2.

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References

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