ON HIGH ORDER MOMENTS OF THE NUMBER OF RENEWALS

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Summary

Renewal equations and their solutions are obtained for high order moments of the number of renewals in any finite interval of time. Moreover, as to two examples, the computational method for coefficients of their series expansions in time is given.

1. Introduction

Concerning the renewal function associated with the Weibull distribution, its evaluation in finite intervals of time by the method of series expansion was treated in [1]. In treating the number of renewals in finite interval of time, however, it should be noticed to consider its high order moments with its expectation—the renewal function. Moreover, such moments might be useful when considering the degree of approximation of the asymptotic normality for the distribution function of the number of renewals in a sufficiently large interval of time.

2. Renewal equations of high order moments and their solutions

At first, we introduce some notations which are necessary later.

$\{X_n\}$: a sequence of random variables, each being independently and identically distributed according to a common distribution function $F(x)$ with a finite and positive mean.

$F^{(n)}(x)$: nth convolution of $F(x)$, especialy $F^{(0)}(x)=1$ and $F^{(1)}(x)=F(x)$

$N(x)$: the number of renewals in time interval $(0, x]$,

$$\text{i.e.}\qquad N(x)=\max\,\{n;\ \sum_{i=1}^{n} X_i \leqq x\}.$$

$H_r(x)$: rth moment of $N(x)$, i.e. $H_r(x)=E\{(N(x))^r\}$, $r\geqq 1$, especially $H_1(x)=E\{N(x)\}$ is called a renewal function.

$\Gamma(p)$: Gamma function, i.e. $\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx$.

$B(p, q)$: Beta function, i.e. $B(p, q)=\int_0^1 (1-x)^{p-1}x^{q-1}dx$

It is well known, that the renewal function $H_1(x)$ satisfies the renewal equation, (2.1) $$H_1(x)=F(x)+\int_0^x H_1(x-y)dF(y)\,,$$

and has the unique solution

(2.2) $$H_1(x)=\sum_{n=1}^{\infty} F^{(n)}(x)\,.$$

In [1] a computational method to obtain coefficients of a series expansion of $H_1(x)$ was given when $F(x)$ is the Weibull distribution.

In the following we shall give renewal equations of $H_r(x)$ recursively defined in r, and show the existence and expressions of their solutions. According to the above definition, $H_r(x)$ is given by

(2.3) $$H_r(x)=\sum_{n=1}^{\infty} n^r\{F^{(n)}(x)-F^{(n+1)}(x)\}\,.$$

To obtain the renewal equation of $H_r(x)$, let us make the convolution of $H_r(x)$ and $F(x)$. Then we have,

(2.4) $$\int_0^x H_r(x-y)dF(y)=\sum_{n=1}^{\infty}(n-1)^r\{F^{(n)}(x)-F^{(n+1)}(x)\}\,.$$

and this is rewritten as

(2.5) $$\int_0^x H_r(x-y)dF(y)=H_r(x)+\sum_{s=1}^{r}\binom{r}{s}(-1)^s H_{r-s}(x)\,,$$

where $H_0(x)=F(x)$. Especially, in case when $r=1$, it reduces to the well known renewal equation (2.1). Suppose that the functions $\{H_k(x);\ 1\leqq k\leqq r-1\}$ are already known, and put $\Psi_r(x)=-\sum_{s=1}^{r}\binom{r}{s}(-1)^s H_{r-s}(x)$. Then the equation (2.5) is rewritten as

(2.6) $$H_r(x)=\Psi_r(x)+\int_0^x H_r(x-y)dF(y),\quad r=1,\,2,\,3,\,\cdots$$

These are the renewal equations to be satisfied by $H_r(x)$'s.

Now we shall show that each equation of (2.6) has the unique solution $H_r(x)$ with the following properties:

1° $H_r(x)$ is non-negative and bounded.

2° $H_r(x)$ is monotone increasing in r and x.

First, we prove that the equation (2.6) has the unique solution in the class of bounded functions. If two bounded functions $H_r'(x)$ and $H_r''(x)$ satisfy the equation (2.6), then the difference of these two functions, $G_r(x)=H_r'(x)-H_r''(x)$, has to satisfy the following homogeneous equation:

(2.7) $$G_r(x)=\int_0^x G_r(x-y)dF(y)\,.$$

Therefore, it can be shown recursively that $G_r(x)$ must satisfy the equations

$$G_r(x)=\int_0^x G_r(x-y)dF^{(n)}(y), \text{ for all } n\geqq 1 . \tag{2.8}$$

Since $F^{(n)}(x)$ is the nth convolution of the distribution function $F(x)$ with a finite and positive mean, $F^{(n)}(x)$ tends to zero as n increases for any finite (fixed) value of x (see remarks in section 3). Therefore, from (2.8), we can conclude that

$$G_r(x)=\lim_{n\to\infty}\int_0^x G_r(x-y)dF^{(n)}(y)=0, \text{ for all finite } x\geqq 0 , \tag{2.9}$$

since $G_r(x)$ is bounded under the assumption that both $H'_r(x)$ and $H''_r(x)$ are bounded. This shows the uniqueness of the solution $H_r(x)$ in the class of bounded functions.

Next we shall show recursively that the solutions $H_r(x)$'s exist and have the properties 1° and 2° stated above. By the same method as uses the so-called Liouville-Neumann series, it may be ascertained that there exists at least one solution of (2.6) such that

$$\begin{aligned} H_r(x)&=\Psi_r(x)+\sum_{n=1}^{\infty}\int_0^x \Psi_r(x-y)dF^{(n)}(y) \\ &=\Psi_r(x)+\int_0^x \Psi_r(x-y)dH_1(y) . \end{aligned} \tag{2.10}$$

From the above stated, this solution must be unique in the class of bounded functions. In fact, this is true for $r=1$, since $\Psi_1(x)=F(x)$ and $H_1(x)=\sum_{n=1}^{\infty} F^{(n)}(x)$. Here, we suppose that the solutions $\{H_s(x);\ 1\leqq s\leqq r-1\}$ have the properties 1° and 2°, and are expressed in the form

$$H_s(x)=\sum_{n-1}^{\infty} n^s\{F^{(n)}(x)-F^{(n+1)}(x)\}, \quad 1\leqq s\leqq r-1 . \tag{2.11}$$

Then, it is easily seen that the function $\Psi_{r-1}(x)$ is expressed as

$$\Psi_r(x)=\sum_{n=1}^{\infty}\{n^r-(n-1)^r\}\{F^{(n)}(x)-F^{(n+1)}(x)\} , \tag{2.12}$$

which is clearly non-negative, bounded in x, and monotone increasing in r. Hence, the solution $H_r(x)$, given in (2.10), is non-negative, bounded in x, and monotone increasing in r and x. Moreover, it is ascertained that the function $H_r(x)$ is given by

$$H_r(x)=\sum_{n=1}^{\infty} n^r\{F^{(n)}(x)-F^{(n+1)}(x)\} , \tag{2.13}$$

and satisfies the equation (2.6) by substituting (2.12) and (2.13) into (2.6). Thus the existence of the unique solution of each in (2.6), with the properties 1° and 2°, is ascertained.

3. Series expansion

In practical applications of series expansion, we have to pay much attentions to the speed of convergence and easiness of computation of the coefficients in a series expansion rather than its formal general theory (see for example [3], [4] and [5]). In this section, we shall give two examples in which adequate series expansions can be found such that their coefficients of series expansion of $H_r(x)$ are computed easily and recursively.

Example 1. $F(x)=1-e^{-x^m}$, $m>0$, (Weibull distribution). By the Taylor expansion of e^x, we get

$$F(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}(x^m)^n\ . \tag{3.1}$$

Then, it may be supposed that $H_1(x)$ is expanded in such a form that

$$H_1(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}A_{1k}}{\Gamma(km+1)}x^{km}\ , \tag{3.2}$$

as is shown in [1]. Moreover, the coefficients $\{A_{1k}\}$ can be obtained recursively by the relations

$$\begin{aligned} A_{11}&=\gamma_1 \\ A_{1k}&=\gamma_k-\sum_{j=1}^{n-1}A_{1,\,k-j} \end{aligned} \tag{3.3}$$

where
$$\gamma_k=\frac{\Gamma(km+1)}{k\,!}\ .$$

Now, we try to obtain the coefficients $\{A_{rk}\}$ recursively in $r(\geqq 2)$. By the same reason as stated above, it may be supposed that $H_r(x)$ is expanded in the form

$$H_r(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}A_{rk}}{\Gamma(km+1)}x^{km}\ . \tag{3.4}$$

Then we can obtain, after some computations

$$\int_0^x H_r(x-y)dF(y)=\sum_{h=1}^{\infty}\frac{(-1)^h x^{(h+1)m}}{\Gamma((h+1)m+1)}\Bigg\{\sum_{\substack{k+l=h\\ k\geqq 1\\ l\geqq 0}}A_{rk}\gamma_{l+1}\Bigg\}\ . \tag{3.5}$$

On the other hand, we obtain from the right hand side of (2.5) another expansion

$$\int_0^x H_r(x-y)dF(y)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\Gamma(km+1)}x^{km} \tag{3.6}$$
$$\times\left\{A_{rk}+\sum_{s=1}^{r-1}\binom{r}{s}(-1)^s A_{r-s,\,k}+(-1)^r\gamma_k\right\}.$$

From (3.5) and (3.6) we obtain the recurrence relations for the coefficients $\{A_{rk}\}$:

$$\begin{cases} A_{r1}+\sum_{s=1}^{r-1}\binom{r}{s}(-1)^s A_{r-s,\,1}+(-1)^r\gamma_1=0 \\ A_{r2}+\sum_{s=1}^{r-1}\binom{r}{s}(-1)^s A_{r-s,\,2}+(-1)^r\gamma_2=-A_{r1}\gamma_1 \\ A_{r3}+\sum_{s=1}^{r-1}\binom{r}{s}(-1)^s A_{r-s,\,3}+(-1)^r\gamma_3=-(A_{r2}\gamma_1+A_{r1}\gamma_2) \\ \cdots\cdots\cdots \\ A_{rk}+\sum_{s=1}^{r-1}\binom{r}{s}(-1)^s A_{r-s,\,k}+(-1)^r\gamma_k=-\sum_{j=1}^{k-1}\gamma_j A_{r,\,k-j}. \end{cases} \tag{3.7}$$

For example, we obtain

$A_{r1}=\gamma_1\ (r\geqq 1)$,
$A_{r2}=\gamma_2-(2^r-1)\gamma_1^2\ (r\geqq 1)$,
$A_{13}=\gamma_3-2\gamma_1\gamma_2+\gamma_1^2$, $A_{23}=\gamma_3-6\gamma_2\gamma_1+5\gamma_1^3$, $A_{33}=\gamma_3-14\gamma_2\gamma_1+19\gamma_1^3$,
$A_{43}=\gamma_3-30\gamma_2\gamma_1+65\gamma_1^3$, $A_{53}=\gamma_3-62\gamma_2\gamma_1+211\gamma_1^3$, etc.

Example 2. $F(x)=\int_0^x\frac{1}{\Gamma(\alpha)}y^{\alpha-1}e^{-y}dy,\ \alpha>0$, (Gamma distribution).

It is easily seen that the probability density function $f(x)$ and its nth convolution $f^{(n)}(x)$ are represented as

$$f(x)=\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}$$

$$f^{(n)}(x)=\frac{1}{\Gamma(n\alpha)}x^{n\alpha-1}e^{-x}.$$

Then we can obtain the expansion of the nth convolution $F^{(n)}(x)$

$$F^{(n)}(x)=\int_0^x\frac{1}{\Gamma(n\alpha)}y^{n\alpha-1}e^{-y}dy=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(n\alpha)k!}\frac{x^{n\alpha+k}}{n\alpha+k}. \tag{3.8}$$

Hence we get the series expansion of $H_1(x)$ such that

$$H_1(x)=\sum_{n=1}^{\infty}F^{(n)}(x)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(n\alpha)k\,!\,(n\alpha+k)}x^{n\alpha+k}\,, \tag{3.9}$$

which is the solution of the equation (2.1).

Now, it may be supposed that $H_r(x)$ can be expanded in such a series as

$$H_r(x)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k B_{nk}^{(r)}}{\Gamma(n\alpha+k+1)k\,!}x^{n\alpha+k}\,. \tag{3.10}$$

Comparing (3.9) with (3.10), we obtain the relation

$$B_{nk}^{(1)}=\frac{\Gamma(n\alpha+k+1)}{\Gamma(n\alpha)\cdot(n\alpha+k)}=\frac{\Gamma(n\alpha+k)}{\Gamma(n\alpha)}=\prod_{j=1}^{k}(n\alpha+k-j)\equiv\beta_{nk} \tag{3.11}$$

which is easily computed. Therefore, we may suppose that all coefficients of $F(x)$ and $H_1(x)$ are known. Moreover, it is convenient to represent the expansion of $F(x)$ as follows:

$$F(x)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k B_{nk}^{(0)}}{\Gamma(n\alpha+k+1)k\,!}x^{n\alpha+k}\,, \tag{3.12}$$

where $$B_{1k}^{(0)}=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\equiv\beta_{1k} \text{ and } B_{nk}^{(0)}=0 \text{ for } n\geqq 2\,.$$

Then

$$\int_0^x H_r(x-y)dF(y)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k B_{nk}^{(r)}}{\Gamma(n\alpha+k+1)k\,!}\int_0^x(x-y)^{n\alpha+k}f(y)dy\,,$$

where
$$\begin{aligned}\int_0^x(x-y)^{n\alpha+k}f(y)dy&=\sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(\alpha)l\,!}\int_0^x(x-y)^{n\alpha+k}y^{\alpha+l-1}dy\\&=\sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(\alpha)l\,!}x^{(n+1)\alpha+(k+l)}\int_0^x\left(1-\frac{y}{x}\right)^{n\alpha+k}\left(\frac{y}{x}\right)^{\alpha+l-1}d\left(\frac{y}{x}\right)\\&=\sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(\alpha)l\,!}\frac{\Gamma(n\alpha+k+1)\Gamma(\alpha+l)}{\Gamma((n+1)\alpha+(k+l)+1)}x^{(n+1)\alpha+(k+l)}\,.\end{aligned}$$

Therefore, we obtain the expansion

$$\begin{aligned}&\int_0^x H_r(x-y)dF(y)\\&=\sum_{n-1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}\Gamma(n\alpha+k+1)\Gamma(\alpha+l)B_{nk}^{(r)}x^{(n+1)\alpha+(k+l)}}{\Gamma(n\alpha+k+1)\Gamma(\alpha)\Gamma((n+1)\alpha+(k+l)+1)k\,!\,l\,!} \qquad (3.13)\\&=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m x^{(n+1)\alpha+m}}{\Gamma((n+1)\alpha+m+1)m\,!}\left\{\sum_{k+l=m}\frac{(k+l)\,!\,\Gamma(\alpha+l)}{k\,!\,l\,!\,\Gamma(\alpha)}B_{nk}^{(r)}\right\}\end{aligned}$$

$$=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m x^{(n+1)\alpha+m}}{\Gamma((n+1)\alpha+m+1)m!}\left\{\sum_{k=0}^{m}\binom{m}{k}\beta_{1,m-k}B_{nk}^{(r)}\right\}.$$

On the other hand, from (2.5), we obtain

$$\int_0^x H_r(x-y)dF(y)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k x^{n\alpha+k}}{\Gamma(n\alpha+k+1)k!}\left\{\sum_{s=0}^{r}\binom{r}{s}(-1)^s B_{nk}^{(r-s)}\right\}. \tag{3.14}$$

Comparing (3.13) with (3.14), we can obtain the recursive relations between $B_{nk}^{(r)}$'s

$$\begin{aligned}&\sum_{s=0}^{r}\binom{r}{s}(-1)^s B_{1m}^{(r-s)}=0,\ r\geqq 1,\ m\geqq 0,\ \text{for } n=1, \quad\text{and}\\ &\sum_{s=0}^{r}\binom{r}{s}(-1)^s B_{nm}^{(r-s)}=\sum_{k=0}^{\infty}\binom{m}{k}\beta_{1,m-k}B_{n-1,k}^{(r)},\ r\geqq 1,\ m\geqq 0,\ \text{for } n\geqq 2.\end{aligned} \tag{3.15}$$

The coefficients $B_{nk}^{(r)}$'s are obtained recursively as follows.

1) $n=1$

From the relation

$$\sum_{s=0}^{r}\binom{r}{s}(-1)^s B_{1m}^{(r-s)}=0,$$

we can obtain easily

$$B_{1m}^{(r)}=B_{1m}^{(0)}=\beta_{1m},\quad m\geqq 1.$$

2) $n=2$

From the relation

$$\sum_{s=0}^{r}\binom{r}{s}(-1)^s B_{2m}^{(r-s)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,m-k}B_{1k}^{(r)},$$

we obtain the relation for $r=1$,

$$B_{2m}^{(1)}-B_{2m}^{(0)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,m-k}B_{1k}^{(1)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,m-k}\beta_{1k}.$$

As $B_{2m}^{(1)}=\beta_{2m}$ and $B_{2m}^{(0)}=0$ for all $m\geqq 0$, we can obtain

$$\beta_{2m}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,m-k}\beta_{1k},\quad m\geqq 0. \tag{3.16}$$

In fact, we have:

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\beta_{1,m-k}\beta_{1k}&=\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(\alpha+m-k)}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\\ &=\sum_{k=0}^{m}\binom{m}{k}\frac{B(\alpha+m-k,\ \alpha+k)\Gamma(2\alpha+m)}{B(\alpha,\ \alpha)\Gamma(2\alpha)}\end{aligned}$$

$$=\frac{\Gamma(2\alpha+m)}{\Gamma(2\alpha)B(\alpha,\,\alpha)}\int_0^1\sum_{k=0}^{m}\binom{m}{k}(1-x)^{\alpha+m-k-1}x^{\alpha+k-1}dx$$

$$=\beta_{2m}\cdot\frac{1}{B(\alpha,\,\alpha)}\int_0^1(1-x)^{\alpha-1}x^{\alpha-1}dx=\beta_{2m}\,.$$

When $r=2$, from the relation

$$B_{2m}^{(2)}-2B_{2m}^{(1)}+B_{2m}^{(0)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,\,m-k}B_{1k}^{(2)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,\,m-k}\beta_{1k}=\beta_{2m}\,,$$

where $B_{2m}^{(1)}=\beta_{2m}$ and $B_{2m}^{(0)}=0$, we obtain $B_{2m}^{(2)}=3\beta_{2m}$. In the same way, we can generally obtain

$$B_{2m}^{(r)}=(2^r-1)\beta_{2m}\,,\qquad m\geqq 0\,.$$

3) $n=3$

When $r=1$, the relation

$$B_{3m}^{(1)}-B_{3m}^{(0)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,\,m-k}B_{2k}^{(1)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,\,m-k}\beta_{2k}=\beta_{3m}$$

holds, and we obtain $B_{3m}^{(1)}=\beta_{3m}$.
When $r=2$, the relation

$$B_{3m}^{(2)}-2B_{3m}^{(1)}+B_{3m}^{(0)}=\sum_{k=0}^{m}\binom{m}{k}\beta_{1,\,m-k}B_{2k}^{(2)}=(2^2-1)\sum_{k=0}^{m}\binom{m}{k}\beta_{1,\,m-k}\beta_{2k}=3\beta_{3m}\,,$$

holds, and we obtain $B_{3m}^{(2)}=5\beta_{3m}$. In the same way, we can obtain $B_{3m}^{(3)}=19\beta_{3m}$, $B_{3m}^{(4)}=65\beta_{3m}$, and so on.

4) $n=4$

We obtain, in the same way as above,

$$B_{4m}^{(1)}=\beta_{4m},\ B_{4m}^{(2)}=7\beta_{4m},\ B_{4m}^{(3)}=39\beta_{4m},\ B_{4m}^{(4)}=180\beta_{4m},\ \text{and so on.}$$

Generally, we obtain the coefficient $B_{nm}^{(r)}$ in such a form as

$$B_{nm}^{(r)}=a_{nm}^{(r)}\beta_{nm}\,,$$

where $a_{nm}^{(r)}$'s are some positive integers.

Remark It is necessary to state that $F^{(n)}(x)$ tends to zero as n increases for any fixed value of $x(\geqq 0)$ if the distribution function $F(x)$ has a finite positive mean, and that the two series

$$H_r(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}A_{rk}x^{km}}{\Gamma(km+1)}\quad\text{and}\quad H_r(x)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^kB_{nk}^{(r)}}{\Gamma(n\alpha+k+1)k\,!}x^{n\alpha+k}$$

are absolutely convergent.

First, let us show that $F^{(n)}(x)$ tends to zero, under the condition stated above. As is well known, $F^{(n)}(x)$ is the distribution function associated with the random variable $\sum_{i=1}^{n} X_i$, where X_i's are independently and identically distributed random variables with the distribution function $F(x)$. Therefore, from the weak law of large number, it can be seen that, for any positive ε and δ, there exists a positive number $N(\varepsilon, \delta)$ such that

$$P_r\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i-\mu\right|>\delta\right\}<\varepsilon, \text{ for } n>N(\varepsilon, \delta)\,,$$

where μ designates the mean of $F(x)$ and is positive. By putting $\delta=\mu/2$, it is easily seen that

$$P_r\left\{\frac{1}{n}\sum_{i=1}^{n} X_i<\frac{\mu}{2}\right\}\leqq P_r\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i-\mu\right|>\frac{\mu}{2}\right\}<\varepsilon, \text{ for } n>N\left(\varepsilon, \frac{\mu}{2}\right).$$

Since $F^{(n)}(x)=P_r\left\{\sum_{i=1}^{n} X_i\leqq x\right\}=P_r\left\{\frac{1}{n}\sum_{i=1}^{n} X_i\leqq\frac{x}{n}\right\}$, we can choose n sufficiently large for a fixed value of $x(\geqq 0)$ so that

$$F^{(n)}(x)=P_r\left\{\frac{1}{n}\sum_{i=1}^{n} X_i\leqq\frac{x}{n}\right\}\leqq P_r\left\{\frac{1}{n}\sum_{i=1}^{n} X_i<\frac{\mu}{2}\right\}<\varepsilon\,.$$

This shows that $F^{(n)}(x)$ tends to zero as n increases infinitely. It is noticed that the above result does hold only on the assumption that the distribution function $F(x)$ have a finite and positive mean μ.*

Second, let us show that the two series for $H_r(x)$ are absolutely convergent. In fact, it is ascertained by induction that

$$|A_{rk}|\leqq 2^{(r+k-2)r}, \text{ if } m\leqq 1\,,$$

and

$$\left|\frac{A_{rk}}{\Gamma(km+1)}\right|\leqq\frac{2^{(r+k-2)r}}{k\,!}C(1+C)^{k-2},\ (C>1), \text{ if } m\geqq 2\,,$$

in the series expansion

$$H_r(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}A_{rk}}{\Gamma(km+1)}x^{km}\,.$$

Hence, we can see that it is absolutely convergent for all values of x.

In the same way, we can evaluate

$$|B_{nk}^{(r)}|\leqq 2^{(r+k-2)r}\beta_{nk}=2^{(r+k-2)r}\frac{\Gamma(n\alpha+k)}{\Gamma(n\alpha)}\,,$$

* The author is indebted to Mr. K. Isii, The Institute of Statistical Mathematics, for this proof.

concerning the series expansion

$$H_r(x)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k B_{nk}^{(r)}}{\Gamma(n\alpha+k+1)\cdot k!}x^{n\alpha+k}.$$

Hence, it is easily seen that the above series is also absolutely convergent for all values of x.

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