ON A FUNCTIONAL TRANSFORM

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1. Introduction

In the study of mathematical economics and operations research, we encounter the problem of determining the function of the type

$$(1.1) \qquad f(x) = \max_{\substack{x_1+x_2=x \\ x_1,\, x_2 \geqq 0}} [f_1(x_1) + f_2(x_2)]\,,$$

for given $f_1(x)$ and $f_2(x)$. We denote such f by $f = f_1 \oplus f_2$. Though this problem can be treated by means of the theory of dynamic programming ([3]), we can find the function (1.1) more easily by introducing the functional transform

$$(1.2) \qquad \mathscr{M}(f) = F$$

with the property that

$$(1.3) \qquad \mathscr{M}(f_1 \oplus f_2) = \mathscr{M}(f_1) + \mathscr{M}(f_2)\,,$$

([1], [2]).

In the present paper, we consider the transform, which is a special form of such transform (1.2) and is defined by

$$(1.4) \qquad F(y) = \max_{x \geqq 0} [f(x) - xy] \qquad (y \geqq 0)\,,$$

and give both the largest class of f's for which (1.4) can be defined and its subclass for which the inverse transform of $F(y)$ exists.

Bellman and Karush [1] have considered the transform

$$(1.5) \qquad M(f) = F$$

with the property that

$$(1.6) \qquad M(f_1 \otimes f_2) = M(f_1) M(f_2)$$

where the symbol $f_1 \otimes f_2$ designates a " convolution " of two functions f_1 and f_2, defined by

$$f_1 \otimes f_2(x) = \max_{\substack{x_1+x_2=x \\ x_1,\, x_2 \geqq 0}} [f_1(x_1) f_2(x_2)]\,.$$

They give an example of transform (1.5), namely

$$F(y)=\max_{x\geqq 0}\,[e^{-xy}f(x)] \qquad (y>0) \tag{1.7}$$

and remarked that, to ensure the existence of (1.7), it is sufficient to assume that $f(x)=0(x^c)$ for $x\geqq 0$ where $c\geqq 0$. When the equation

$$f'(x)-yf(x)=0$$

has a unique solution for each $y>0$, they have shown that there exists the inverse transform of $F(y)$ which is uniquely defined by

$$f(x)=\min_{y>0}\,[e^{xy}F(y)] \qquad (x\geqq 0)\,.$$

Considering the inverse transform of (1.4), they also gave a very special class for which the function (1.1) can be readily calculated ([2]).

We shall note that two transforms (1.2) and (1.5) are essentially the same in the following sense:

$$\log M(f_1\otimes f_2)=\log M(f_1)+\log M(f_2)$$

and since

$$\log f_1\oplus\log f_2=\log f_1\otimes f_2\,,$$

we have

$$\mathscr{M}\log(f_1\otimes f_2)=\mathscr{M}(\log f_1)+\mathscr{M}(\log f_2)\,,$$

which shows that both $\log\cdot M$ and $\mathscr{M}\cdot\log$ belong to the same class of functional transforms such that

$$\mathscr{N}(f_1\otimes f_2)=\mathscr{N}(f_1)+\mathscr{N}(f_2)\,.$$

From this it follows that our result is more general than theirs.

2. Theorem

Let $f(x)$ be a non-decreasing function defined for $x\geqq 0$ which has a continuous derivative $f'(x)$. If there exists a non-negative number m such that $f'(x)$ is monotone decreasing for $x\geqq m$ and tends to zero as $x\to\infty$, then a function (1.4) *can be defined, and is a continuous, strictly decreasing function of y.*

Furthermore, the function

$$f^*(x)=\min_{y\geqq 0}\,[F(y)+xy] \qquad (x\geqq 0) \tag{2.1}$$

can also be defined, and for $x\geqq 0$ we have

$$f(x)=f^*(x) \tag{2.2}$$

if and only if $m=0$.

3. Proof of the theorem

Since a function

$$g_y(x)=f(x)-xy \tag{3.1}$$

is differentiable with respect to x, for each $y>0$, among the solutions of the equation

$$g_y'(x)=f'(x)-y=0 \tag{3.2}$$

we can find a value $u=u(y)$ of x at which the function (3.1) takes its maximum value. If there is no solution of equation (3.2), then $g_y'(x)<0$ for $x\geqq 0$ so that $g_y(x)$ takes its maximum value at $x=0$. Therefore, putting $u(y)=0$ in such cases and $u(0)=\infty$, we can define a function $u=u(y)$ for $y\geqq 0$, which is not necessarily unique. We shall denote by U the set of y for which $u(y)$ is unique. When $y\in U'\equiv[0, \infty]-U$, there exist $u_*=u_*(y)<u^*=u^*(y)$ such that

$$g_y(x)<g_y(u_*)=g_y(u^*)=g_y(u(y)) \tag{3.3}$$

for $0\leqq x<x_*$ or $x>x^*$.

LEMMA. *The function* $u=u(y)$ *is a monotone decreasing function of* y. *If* $y\in U$, *then* $u(y)$ *is continuous at* y. *If* $y\in U'$, *then*

$$\lim_{y'\downarrow y} u(y')=u(y+0)=u_*(y)\,, \tag{3.4}$$

$$\lim_{y'\uparrow y} u(y')=u(y-0)=u^*(y)\,. \tag{3.5}$$

PROOF. Assume that $u(y)\leqq u(y')$ for $y<y'$. Then

$$\begin{aligned} g_y(u(y)) &= g_{y'}(u(y))+u(y)(y'-y) \\ &< g_{y'}(u(y'))+u(y)(y'-y) \\ &= g_y(u(y'))\,. \end{aligned}$$

This contradicts the definition of $u(y)$.

Let $y\in U$. Since $g_y(x)$ is continuous and takes its maximum value at only a single point u, there exist a positive number c and positive numbers $\varepsilon_1>\varepsilon_2$ depending on c such that

$$\begin{gathered} g_y'(x) \text{ is monotone decreasing for } |u-x|<\varepsilon_1\,, \\ g_y(u)-g_y(x)>c \qquad \text{for } |u-x|\geqq\varepsilon_1\,, \end{gathered}$$

$$g_y(u)-g_y(x)<c/2 \qquad \text{for } |u-x|<\varepsilon_2 .$$

Let $\{l, l'\}$ be the largest solutions of equations $g_y'(x)=0$, $g_{y'}'(x)=0$, respectively. By the assumption that $f'(x)$ is monotone decreasing for $x \geqq m$ and that $f'(x)$ tends to zero as $x \to \infty$, there exist positive numbers $\delta_0=\delta_0(y)$ and $\varepsilon_0=\varepsilon_0(\delta_0)=\varepsilon_0(y)$ such that $l' \leqq l+\varepsilon_0$ for $y'>y-\delta_0$. Put

$$\delta^*=\delta^*(y)=\min\left(\delta_0, \frac{c}{2(l+\varepsilon_0)}\right) .$$

Then, by the continuity of $f'(x)$ there exists a positive number ε_3 such that $|f'(u)-f'(x)|<\delta^*$ for $|u-x|<\varepsilon_3$. Since $f'(x)$ is monotone decreasing for $|u-x|<\varepsilon_1$, for any positive number ε less than $\varepsilon^*=\min(\varepsilon_0, \varepsilon_2, \varepsilon_3)$ there exists a positive number δ(less than δ^*) such that

"for every y' with $|y-y'|<\delta$ there exists x_0 such that $|u-x_0|<\varepsilon$ and that $f'(x_0)=y'$."

Now we can show that $x_0=u'=u(y')$, which implies that $u(y)$ is continuous at $y \in U$. Assume that $x_0 \neq u'$. Since $f'(x_0)=f'(u')=y'$ and $f'(x)$ is monotone decreasing for $|u-x|<\varepsilon_1$ and $|u-x_0|<\varepsilon \leqq \varepsilon_2$, we have $|u-u'|>\varepsilon_1$ so that $g_y(u)-g_y(u')>c$, $g_y(u)-g_y(x_0)<c/2$. Since $y'>y-\delta_0$, $0<u' \leqq l' \leqq l+\varepsilon_0$ and since $0 \leqq x_0<u+\varepsilon<l+\varepsilon_0$, we have $|u'-x_0|<l+\varepsilon_0$. Therefore, we have

$$\begin{aligned} g_{y'}(x_0)-g_{y'}(u') &= g_y(x_0)-g_y(u')-(y'-y)(x_0-u') \\ &\geqq [g_y(u)-g_y(u')]-[g_y(u)-g_y(x_0)]-|y'-y||x_0-u'| \\ &> c-c/2-\delta(l+\varepsilon_0)>0 . \end{aligned}$$

This contradicts the definition of $u'=u(y')$.

Let $y \in U'$. Since $g_y(x)<g_y(u_*)$ for $0 \leqq x<u_*$, there exist a positive number c' and positive numbers $\varepsilon_1'>\varepsilon_2'$ depending on c' such that

$$\begin{aligned} &g_y'(x) \text{ is monotone decreasing} && \text{for } u_*-\varepsilon_1'<x<u_* , \\ &g_y(u_*)-g_y(x)>c' && \text{for } 0 \leqq x \leqq u_*-\varepsilon_1' , \\ &0<g_y(u_*)-g_y(x)<c'/2 && \text{for } u_*-\varepsilon_2''<x<u_* . \end{aligned}$$

Since $f'(x)$ is monotone decreasing for $u_*-\varepsilon_1'<x<u_*$, for any positive number ε less than $\min(\varepsilon_0, \varepsilon_2', \varepsilon_3)$ these exists a positive number δ (less than δ^*) such that

"for every y' for which $0<y'-y<\delta$ there exists x_0 such that $u_*-\varepsilon<x_0<u_*$ and that $f'(x_0)=y'$."

As before, we can show that $x_0=u'$, which shows that (3.4) holds. By using the fact that $g_y(x)<g_y(u^*)$ for $x \geqq u^*$, we can also show that the relation (3.5) holds. This completes the proof of the lemma.

We shall now proceed to the proof of the theorem. It readily follows that the function $F(y)$ is strictly decreasing. For, if $y<y'$, then

$$F(y)-F(y')=g_y(u)-g_y(u')+u'(y'-y)>0\,.$$

To prove that $F(y)$ is continuous at $y\geqq 0$, we select a positive number ε arbitrarily. Then, by the continuity of $g_y(u)=f(u)-uy$ there exists a positive number δ_1 depending on both ε and u, consequently on both ε and y, such that

$$|g_y(u)-g_y(u')|<\varepsilon/2$$

for any u' with $|u-u'|<\delta_1$. It follows from the lemma that there exists a positive number δ_2 depending on δ_1, y such that, when $y\in U$,

$$|u(y)-u(y')|<\delta_1$$

and, when $y\in U'$,

$$\max\left([u_*(y)-u(y')],\ [u(y')-u^*(y)]\right)<\delta_1$$

for any y' with $|y-y'|<\delta_2$. We put

$$\delta=\min\left(\delta_2,\ \delta_0,\ \frac{\varepsilon}{2(l+\varepsilon_0)}\right),$$

which depends on ε and y. Then, for every y' with $|y-y'|<\delta$, we have

$$\begin{aligned}|F(y)-F(y')| &\leqq |g_y(u)-g_y(u')|+|u'(y'-y)|\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\,.\end{aligned}$$

Finally, we prove that the function (2.1) can be defined for $x\geqq 0$ and that (2.2) holds if and only if $m=0$. We put

$$G_x(y)=F(y)+xy=f(u(y))-(u(y)-x)y\,.$$

Since

$$\frac{G_x(y')-G_x(y)}{y'-y}=\left\{\frac{f(u(y'))-f(u(y))}{u(y')-u(y)}-y\right\}\frac{u(y')-u(y)}{y'-y}-u(y')+x\,,$$

when $y\in U$, we have

$$G'_x(y)=-u(y)+x\,,$$

and when $y\in U'$, we have

$$\begin{aligned}G_x^+(y)&=-u_*(y)+x\,,\\ G_x^-(y)&=-u^*(y)+x\,.\end{aligned}$$

Then, to every $x\geqq 0$, there corresponds either $v=v(x)\in U$ such that $G'_x(v)=0$ i.e. $u(v)=x$, or $v=v(x)\in U'$ such that $G_x^-(v)\leqq 0\leqq G_x^+(v)$ i.e. $u_*(v)\leqq$

$x \leqq u^*(v)$. Therefore, $G_x(y)$ takes its minimum value at $v=v(x)$ so that the function $f^*(x)$ can be defined by (2.1) for $x \geqq 0$. Concerning this $f^*(x)$, we have

$$f^*(x)=\min_{y\leq 0} G_x(y)=G_x(v)=F(v)+xv \\ =f(u(v))-u(v)\cdot v+xv=f(x) \tag{3.6}$$

when $v=v(x) \in U$, and when $v=v(x) \in U'$, we have

$$f^*(x)=F(v)+xv=f(x)+F(v)-g_v(x) \geqq f(x)\,, \tag{3.7}$$

where equality sign holds if and only if

$$F(v)-g_v(x)=0\,, \tag{3.8}$$

namely $f'(x)=v$ for all x such that $u_*(v) \leqq x \leqq u^*(v)$.

When noticing that $f'(x)$ is strictly decreasing at x to which there corresponds $v=v(x) \in U$, we can conclude that, in order that (2.2) holds for $x \geqq 0$, it is sufficient that $f'(x)$ is monotone non-increasing, and tends to zero for $x \geqq 0$ i.e. $m=0$. Conversely, assume that $m=0$. If there exists such a interval $[a, b]$ that $f'(x)=v$ (constant) for all $a \leqq x \leqq b$, then we have $u_*(v)=a$, $u^*(v)=b$ so that (3.8) holds and consequently (2.2) holds for $a \leqq x \leqq b$. To the point x at which $f'(x)$ is strictly decreasing, there corresponds $v=v(x) \in U$ so that (3.6) holds, and consequently (2.2) holds. Therefore (2.2) holds for $x \geqq 0$. This completes the proof of the theorem.

4. Remarks

(i) A more general transform than (1.4) is given by

$$F(y)=\max_{x\geq 0}\,[f(x)-xY(y)] \qquad (y \geqq 0)$$

where $Y(y)$ is some function of y. Therefore, when we consider the transform defined by

$$F(y)=\max_{x\geq 0}\,[f(x)-x(y+c)] \qquad (y \geqq 0) \tag{4.1}$$

where c is any non-negative number, we can weaken the assumption concerning the function $f(x)$ of the theorem.

COROLLARY. *If we replace the part "tends to zero" in the theorem by "tends to c", then the function* (4.1) *can be defined and is also a continuous strictly decreasing function of y. Furthermore, the function*

$$f^*(x)=\min_{y\geq 0}\,[F(y)+x(y+c)] \qquad (x \geqq 0)$$

can also be defined, and $f(x)=f^*(x)$ *for* $x\geqq 0$ *if and only if* $m=0$.

(ii) When both f_1' and f_2' are non-increasing functions, we can also find the function (1.1) in the following way.

First, we can find a point $x_0\geqq 0$ such that

$$\max(f_1'(x_0),\ f_2'(x_0))=\min(f_1'(0),\ f_2'(0))\ .$$

Next, we seek the solution $w=w(x)$ of equation

$$f_1'(w)=f_2'(x-w)$$

for each $x\geqq x_0$. For $0\leqq x<x_0$, we define

$$w=w(x)=\begin{cases} 0 & \text{when } f_1'(0)<f_2'(0) \\ x & \text{when } f_1'(0)>f_2'(0)\ . \end{cases}$$

Then the function (1.1) is given by

$$f(x)=f_1(w)+f_2(x-w)\ .$$

But this method is more complicated when the number of functions is more than two.

(iii) Since the transform (1.5) is not linear, namely

$$M(f-a)\geqq M(f)-M(a)\ ,$$

the relation (4.5) in [1] does not hold. But only the relation

$$f(x)\leqq \min_{z\geqq 0}\frac{e^{xz}A(z)}{1-G(z)}$$

holds for such a function g that $G(z)<1$ for $z\geqq 0$, where A, G are the transforms (1.7) of functions a and g.

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