ON A SIMPLIFIED METHOD OF THE ESTIMATION OF THE CORRELOGRAM FOR A STATIONARY GAUSSIAN PROCESS

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1. Introduction

Let $x(t)$ be a real valued weakly stationary process with discrete time parameter t, such that $Ex(t)=0$, $Ex(t)^2=\sigma^2$, $Ex(t)x(t+h)=\sigma^2\rho_h$. The problem considered in this paper is concerned with the estimation of the correlogram ρ_h. For this, we assume the variance σ^2 to be known. Further, since the correlogram ρ_h is symmetric about $h=0$, we take h to be nonnegative. For the correlogram ρ_h, the unbiased estimate

$$\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{t=1}^{N} x(t)x(t+h)$$

is usually considered. However, we do not take this estimate, as it is, but make modification. The essential part of our modification is to replace $x(t)x(t+h)$ by $x(t)sgn(x(t+h))$ in the estimate $\tilde{\gamma}_h$, where $sgn(y)$ means 1, 0, -1 correspondingly as $y>0$, $y=0$, $y<0$. The statistic thus obtained originates in Takahasi and Husimi's method of determining the period and decrement of a vibrating system exposed to irregular statistical forces [4]. This statistic has been used as a simplified estimate of the correlogram in many practical fields. However, its validity has not been assertained. This problem was first presented to the author by Prof. R. Kawashima, the Faculty of Fisheries, Hokkaido University. He used $(1/N)\sum_{t=1}^{N} x(t)sgn(x(t+h))$ to simplify calculations of the covariance in analysis of ship's rolling records [3].

In the present paper, we shall investigate mathematical and statistical properties of the statistic obtained by that replacement from $\tilde{\gamma}_h$. In section 2, it will be shown that, when $x(t)$ is a stationary Gaussian process, we can give an unbiased estimate of the correlogram, in terms of $x(t)sgn(x(t+h))$. The estimate is

$$\gamma_h=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\frac{1}{N}\sum_{t=1}^{N} x(t)sgn(x(t+h)).$$

The variance of this estimate will also be evaluated. In general, it is not easy to evaluate the variance, and, our discussion is mainly restricted to the case where $x(t)$ is a Markov process. Further, in section 3, we shall give numerical comparisons of the variances of our estimate γ_h with those of the ordinary estimate $\tilde{\gamma}_h$. From this comparison, it will be seen, at least in that Markov case, that our estimate has smaller variance than the ordinary one for small lag h.

2. Mean and variance of estimate γ_h

First, we shall show that, if $x(t)$ is a stationary Gaussian process, γ_h is an unbiased estimate of ρ_h.

In fact, for $h\neq 0$, putting simply $x(t)=x$ and $x(t+h)=y$, we have

$$\begin{aligned}
&Ex(t)sgn(x(t+h))\\
&=\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} x\,sgn(y)\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy\\
&=\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=0}^{\infty}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy\\
&-\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=-\infty}^{0}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy
\end{aligned}$$

and

$$\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=0}^{\infty}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy=\frac{\sigma\rho_h}{\sqrt{2\pi}}$$

$$\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=-\infty}^{0}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy=-\frac{\sigma\rho_h}{\sqrt{2\pi}}\,.$$

Consequently,

$$Ex(t)sgn(x(t+h))=\sqrt{\frac{2}{\pi}}\,\sigma\rho_h\,,$$

and finally

$$E(\gamma_h)=\rho_h\,.$$

For $h=0$, we can also show

$$E(\gamma_0)=1.$$

In the next place, we consider the variance $V(\gamma_h)$ of estimate γ_h. Evaluation of the variance of γ_0 proceeds as follows. We have

$$V(\gamma_0)=E(\gamma_0-1)^2=E\gamma_0^2-1\,,$$

$$E\gamma_0^2 = E\left(\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\frac{1}{N}\sum_{t=1}^{N} x(t)sgn(x(t))\right)^2$$
$$= \pi\frac{1}{\sigma^2}\frac{1}{N^2}\sum_{t=1}^{N}\sum_{\substack{s=1\\ s>t}}^{N} Ex(s)sgn(x(s))x(t)sgn(x(t))$$
$$+\frac{\pi}{2}\frac{1}{\sigma^2}\frac{1}{N^2}\sum_{t=1}^{N} Ex(t)^2 sgn^2(x(t)),$$

and, for simplicity, putting $x(s)=x$ and $x(t)=y$,

$$Ex(s)sgn(x(s))x(t)sgn(x(t)) = E|x(s)||x(t)|$$
$$= \frac{1}{2\pi\sigma^2\sqrt{1-\rho_{s-t}^2}}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}|x||y|\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(x^2-2\rho_{s-t}xy+y^2)\right)dxdy.$$

Using the expansion (S. O. Rice [2], section 3.5)

$$\int_0^\infty\int_0^\infty u^l v^m \exp(-u^2-v^2-2auv)dudv$$
$$= \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-2a)^k}{k!}\Gamma\left(\frac{l+k+1}{2}\right)\Gamma\left(\frac{m+k+1}{2}\right),$$

we get, for example,

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty} xy\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(x^2-2\rho_{s-t}xy+y^2)\right)dxdy$$
$$= \sigma^4(1-\rho_{s-t}^2)^2\left(\sum_{k=0}^{\infty}\frac{(2\rho_{s-t})^k}{k!}\Gamma\left(\frac{k+2}{2}\right)^2\right).$$

Thus, we have

$$Ex(s)sgn(x(s))x(t)sgn(x(t))$$
$$= \frac{2\sigma^2(1-\rho_{s-t}^2)^{3/2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2m)!}\Gamma(m+1)^2\right),$$

and finally,

$$V(\gamma_0) = \frac{2}{N^2}\sum_{t=1}^{N}\sum_{\substack{s=1\\ s>t}}^{N}(1-\rho_{s-t}^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2m)!}\Gamma(m+1)^2\right)+\frac{\pi}{2}\frac{1}{N}-1$$
$$= \frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)+\frac{\pi}{2}\frac{1}{N}-1.$$

For $h>0$, the variance of γ_h is given as follows. As was stated in section 1, we hereafter restrict our attention to the case where $x(t)$ is a stationary Gaussian Markov process. A process $x(t)$ is called a Markov process in the sense of J. L. Doob [1] when $x(t)$ satisfies the following condition:

for any integer $n \geqq 1$ *and any parameter values* $t_1 < t_2 < \cdots < t_n$, *the conditional probabilities of* $x(t_n)$, *relative to* $x(t_1), x(t_2), \cdots, x(t_{n-1})$, *are the same as those relative to* $x(t_{n-1})$ *in the sense that for each* λ

$$P\{x(t_n) \leqq \lambda \mid x(t_1), x(t_2), \cdots, x(t_{n-1})\} = P\{x(t_n) \leqq \lambda \mid x(t_{n-1})\}$$

with probability 1.

In this case, the correlogram is expressed as

$$\rho_h = a^{|h|} \qquad (|a| \leqq 1)$$

(see J. L. Doob [1]). Let

$$f_n(x_1, x_2, \cdots, x_n)$$

be the probability density function of the n-dimensional Gaussian distribution. Then, for a Gaussian Markov process $x(t)$, and for any $t_1 < t_2 < \cdots < t_{n-2} < t_{n-1} < t_n$, we have

$$\begin{aligned} & f_n(x(t_1), x(t_2), \cdots, x(t_{n-2}), x(t_{n-1}), x(t_n)) \\ & = \frac{f_2(x(t_1), x(t_2))}{f_1(x(t_2))} \cdots \frac{f_1(x(t_{n-2}), x(t_{n-1}))}{f_1(x(t_{n-1}))} f_2(x(t_{n-1}), x(t_n)). \end{aligned}$$

We use this fact for calculation of the variance of γ_h. For simplicity, we assume that N is sufficiently large and $N > h$. Then we obviously, have

$$\text{variance of } \gamma_h = V(\gamma_h) = E(\gamma_h - \rho_h)^2 = E\gamma_h^2 - \rho_h^2,$$

and

$$\begin{aligned} E\gamma_h^2 &= E\left(\sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \frac{1}{N} \sum_{t=1}^{N} x(t) sgn(x(t+h))\right)^2 \\ &= \frac{\pi}{2} \frac{1}{\sigma^2} \frac{1}{N^2} E\left(\sum_{t=1}^{N} \sum_{s=1}^{N} x(t) sgn(x(t+h)) x(s) sgn(x(s+h))\right) \\ &= \frac{\pi}{2} \frac{1}{\sigma^2} \frac{1}{N^2} \Bigg[2 \sum_{t=1}^{N} \sum_{\substack{s=1 \\ s>t \\ t+h>s}}^{N} Ex(t) sgn(x(t+h)) x(s) sgn(x(s+h)) \\ &\qquad + 2 \sum_{t=1}^{N} \sum_{\substack{s=1 \\ s>t+h}}^{N} Ex(t) sgn(x(t+h)) x(s) sgn(x(s+h)) \\ &\qquad + 2 \sum_{t=1}^{N-h} Ex(t) sgn(x(t+h)) x(t+h) sgn(x(t+2h)) \\ &\qquad + \sum_{t=1}^{N} Ex(t)^2 sgn^2(x(t+h)) \Bigg]. \end{aligned}$$

In the following we evaluate each part of summation.

i) When $s > t$ and $t+h > s$, we have

$$f_4(x(t),\, x(s),\, x(t+h),\, x(s+h))$$
$$=\frac{f_2(x(t),\, x(s))}{f_1(x(s))}\,\frac{f_2(x(s),\, x(t+h))}{f_1(x(t+h))}\,f_2(x(t+h),\, x(s+h)).$$

For simplicity, we put

$$x(t)=x,\ x(s)=y,\ x(t+h)=\tilde{x},\ x(s+h)=\tilde{y}.$$

Then we have

$$f_4(x,\, y,\, \tilde{x},\, \tilde{y})$$
$$=\frac{1}{(2\pi)^2\sigma^4(1-\rho_{s-t}^2)\sqrt{1-\rho_{t+h-s}^2}}\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(x-\rho_{s-t}y)^2\right)$$
$$\times\exp\left(-\frac{1}{2\sigma^2(1-\rho_{t+h-s}^2)}(y-\rho_{t+h-s}\tilde{x})^2-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(\tilde{x}^2-2\rho_{s-t}\tilde{x}\tilde{y}+\tilde{y}^2)\right).$$

Now

$$Ex(t)x(s)sgn(x(t+h))sgn(x(s+h))$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xysgn(\tilde{x})sgn(\tilde{y})f_4(x,\, y,\, \tilde{x},\, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$=\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x,\, y,\, \tilde{x},\, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$-\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x,\, y,\, \tilde{x},\, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$-\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x,\, y,\, \tilde{x},\, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$+\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x,\, y,\, \tilde{x},\, \tilde{y})dxdyd\tilde{x}d\tilde{y},$$

and

$$\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x,\, y,\, \tilde{x},\, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$=\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x,\, y,\, \tilde{x},\, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$=\frac{\rho_{s-t}(1-\rho_{t+h-s}^2)}{2\pi\sqrt{1-\rho_{s-t}^2}}\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(\tilde{x}^2-2\rho_{s-t}\tilde{x}\tilde{y}+\tilde{y}^2)\right)d\tilde{x}d\tilde{y}$$
$$+\frac{\rho_{s-t}\rho_{t+h-s}^2}{2\pi\sigma^2\sqrt{1-\rho_{s-t}^2}}\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\tilde{x}^2\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(\tilde{x}^2-2\rho_{s-t}\tilde{x}\tilde{y}+\tilde{y}^2)\right)d\tilde{x}d\tilde{y}.$$

Using the expansion formula used in the evaluation of $V(\gamma_0)$, we can get

$$\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\frac{\sigma^2\rho_{s-t}(1-\rho_{t+h-s}^2)\sqrt{1-\rho_{s-t}^2}}{4\pi}\left(\sum_{k=0}^{\infty}\frac{(2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+1}{2}\right)^2\right)$$

$$+\frac{\sigma^2\rho_{s-t}\rho_{t+h-s}^2(1-\rho_{s-t}^2)^{3/2}}{2\pi}\left(\sum_{k=0}^{\infty}\frac{(2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{k+1}{2}\right)\right).$$

Similarly, we get

$$\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\frac{\sigma^2\rho_{s-t}(1-\rho_{t+h-s}^2)\sqrt{1-\rho_{s-t}^2}}{4\pi}\left(\sum_{k=0}^{\infty}\frac{(-2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+1}{2}\right)^2\right)$$

$$+\frac{\sigma^2\rho_{s-t}\rho_{t+h-s}^2(1-\rho_{s-t}^2)^{3/2}}{2\pi}\left(\sum_{k=0}^{\infty}\frac{(-2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{k+1}{2}\right)\right).$$

Therefore we obtain

$$Ex(t)x(s)sgn(x(t+h)sgn(x(s+h))$$

$$=\frac{\sigma^2\rho_{s-t}(1-\rho_{t+h-s}^2)\sqrt{1-\rho_{s-t}^2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\Gamma(m+1)^2\right)$$

$$+\frac{2\sigma^2\rho_{s-t}\rho_{t+h-s}^2(1-\rho_{s-t}^2)^{3/2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\Gamma(m+2)\Gamma(m+1)\right).$$

ii) When $s>t+h$, we have, using the same notation as in i),

$$f_4(x(t), x(t+h), x(s), x(s+h))$$

$$=\frac{1}{(2\pi)^2\sigma^4(1-\rho_h^2)\sqrt{1-\rho_{s-t-h}^2}}\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x-\rho_h\tilde{x})^2\right)$$

$$\times\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t-h}^2)}(\tilde{x}-\rho_{s-t-h}y)^2-\frac{1}{2\sigma^2(1-\rho_h^2)}(y^2-2\rho_h y\tilde{y}+\tilde{y}^2)\right).$$

In this case, we have

$$E(x/\tilde{x}, y, \tilde{y})=\rho_h\tilde{x}$$

$$E(xy/\tilde{x}, \tilde{y})=E(yE(x/\tilde{x}, y, \tilde{y})/\tilde{x}, \tilde{y})$$

$$=\frac{\rho_{s-t}(1-\rho_h^2)}{1-\rho_{s-t}^2}\tilde{x}^2+\frac{\rho_h^2(1-\rho_{s-t-h}^2)}{1-\rho_{s-t}^2}\tilde{x}\tilde{y}.$$

Consequently, we get

$$Ex(t)sgn(x(t+h))x(s)sgn(x(s+h))$$
$$=\frac{2\sigma^2\rho_{s-t}(1-\rho_h^2)\sqrt{1-\rho_{s-t}^2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)$$
$$+\frac{2\sigma^2\rho_h^2(1-\rho_{s-t-h}^2)\sqrt{1-\rho_{s-t}^2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2m)!}\Gamma(m+1)^2\right).$$

iii) When $s=t+h$, the necessary joint probability density function is $f_3(x(t), x(t+h), x(t+2h))$. With the same notation as in i), we have

$$f_3(x(t), x(t+h), x(t+2h))$$
$$=\frac{f_2(x(t), x(t+h))}{f_1(x(t+h))}f_2(x(t+h), x(t+2h))$$
$$=\frac{1}{(2\pi)^{3/2}\sigma^3(1-\rho_h^2)}\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x-\rho_h y)^2\right)$$
$$\times\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(y^2-2\rho_h y\tilde{y}+\tilde{y}^2)\right).$$

Using this expression, we get

$$Ex(t)sgn(x(t+h))x(t+h)sgn(x(t+2h))$$
$$=\frac{2\sigma^2\rho_h(1-\rho_h^2)^{3/2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_h)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right).$$

iv) When $t=s$, we have

$$Ex(t)^2(sgn(x(t+h)))^2=\sigma^2.$$

Using these results, we finally get

$$V(\gamma_h)=E\gamma_h^2-\rho_h^2$$
$$=\frac{1}{N^2}\Bigg[\sum_{k=1}^{h-1}(N-k)\Bigg\{\rho_k(1-\rho_{h-k}^2)\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+1)^2\right)$$
$$+2\rho_k\rho_{h-k}^2(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)\Bigg\}$$
$$+2\sum_{k=h+1}^{N-1}(N-k)\Bigg\{\rho_k(1-\rho_h^2)\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)$$
$$+\rho_h^2(1-\rho_{k-h}^2)\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)\Bigg\}$$
$$+2(N-h)\rho_h(1-\rho_h^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_h)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)+\frac{\pi}{2}N\Bigg]$$
$$-\rho_h^2.$$

3. Comparison of γ_h with $\tilde{\gamma}_h$

The variance of $\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{t=1}^{N}x(t)x(t+h)$ for a stationary Gaussian process is as follows:

i) When $h=0$, we have

$$\text{variance of } \tilde{\gamma}_0=V(\tilde{\gamma}_0)=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1+2\rho_k^2)+\frac{3}{N}-1\,.$$

ii) When $h\neq 0$, assuming Markov property, we have

$$\begin{aligned}\text{variance of } \tilde{\gamma}_h= V(\tilde{\gamma}_h)&=E\tilde{\gamma}_h^2-\rho_h^2\\ &=\frac{1}{N^2}\Big[2\sum_{k=1}^{h-1}(N-k)\rho_k^2(1+2\rho_{h-k}^2)\\ &\quad+2\sum_{k=h+1}^{N-1}(N-k)\rho_h^2(1+2\rho_{k-h}^2)\\ &\quad+6(N-h)\rho_h^2+N(1+2\rho_h^2)\Big]\\ &\quad-\rho_h^2\,.\end{aligned}$$

In this section, we compare, numerically, the variance of γ_h with that of $\tilde{\gamma}_h$.

We are considering a Markov process, so we have

$$\rho_k=a^{|k|} \qquad (|a|\leqq 1).$$

Numerical comparisons are made for the following cases:

$$a=(0.8)^5, \qquad 0.8,$$
$$N=50, \qquad 500.$$

The results are shown in Table 1 and Figure 1.

Taking into account the present numerical results and the ease of computation of γ_h, we can say that the estimate γ_h is a fairly good estimate of the correlogram for a stationary Gaussian Markov process. This will also be referred to in future by M. Sibuya from the point of view of estimation theory.

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TABLE 1*)

$a=0.8$

h	ρ_h	$N=500$ $V(\gamma_h)$	$N=500$ $V(\tilde{\gamma}_h)$	$N=50$ $V(\gamma_h)$	$N=50$ $V(\tilde{\gamma}_h)$
0	1.0000	0.0048	0.0181	0.0464	0.1743
1	0.8000	0.0070	0.0174	0.0630	0.1671
2	0.6400	0.0084	0.0160	0.0808	0.1535
3	0.5120	0.0093	0.0146	0.0894	0.1391
4	0.4096	0.0098	0.0133	0.0949	0.1263
5	0.3277	0.0102	0.0122	0.0984	0.1159
6	0.2621	0.0104	0.0113	0.1007	0.1079
7	0.2097	0.0106	0.0107	0.1021	0.1019
8	0.1678	0.0107	0.0102	0.1031	0.0975
9	0.1342	0.0107	0.0099	0.1037	0.0943
10	0.1074	0.0108	0.0096	0.1040	0.0921
11	0.0859	0.0108	0.0095	0.1043	0.0916
12	0.0687	0.0108	0.0093	0.1044	0.0901
13	0.0550	0.0108	0.0093	0.1045	0.0892
14	0.0440	0.0108	0.0092	0.1046	0.0885
15	0.0352	0.0108	0.0092	0.1046	0.0879
20	0.0115	0.0108	0.0091	0.1047	0.0873
25	0.0038	0.0108	0.0091	0.1047	0.0872
30	0.0012	0.0108	0.0091	0.1047	0.0872

$a=(0.8)^5=0.32768$

h	ρ_h	$N=500$ $V(\gamma_h)$	$N=500$ $V(\tilde{\gamma}_h)$	$N=50$ $V(\gamma_h)$	$N=50$ $V(\tilde{\gamma}_h)$
0	1.00000	0.0014	0.0050	0.0138	0.0494
1	0.32768	0.0034	0.0032	0.0338	0.0316
2	0.10737	0.0036	0.0026	0.0359	0.0259
3	0.03518	0.0036	0.0025	0.0362	0.0249
4	0.01153	0.0036	0.0025	0.0362	0.0247
5	0.00378	0.0036	0.0025	0.0362	0.0247
10	0.00001	0.0036	0.0025	0.0362	0.0247

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*) As was stated in H. Akaike [5], the variance of $\tilde{\gamma}_h$ is asymptotically of order $1/N$. The present results are in accordance with this fact.

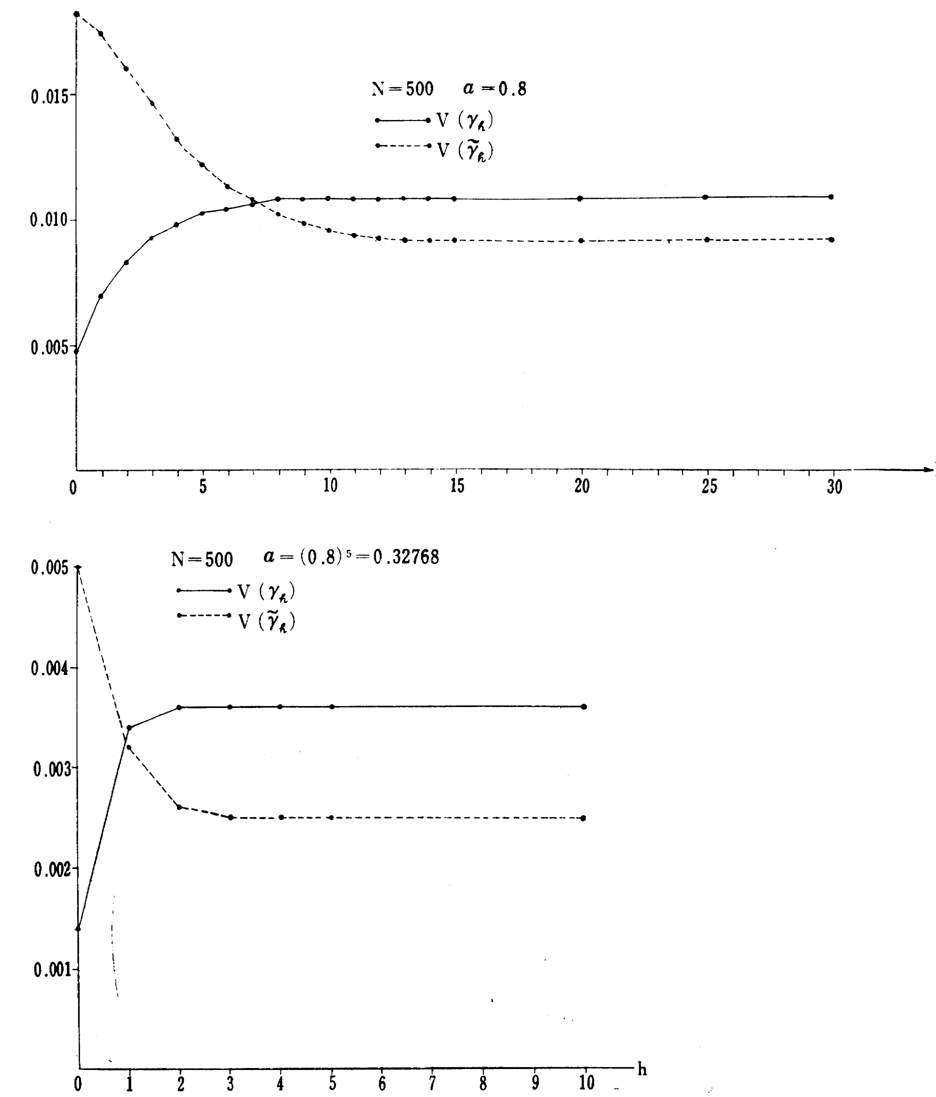

Fig. 1.

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