EFFECT OF NON-CENTRALITY ON THE BARTLETT DECOMPOSITION OF A WISHART MATRIX

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1. Introduction

The central Wishart distribution is the distribution of the variance-covariance matrix of a sample drawn from a multivariate normal population assuming that the expected value of each variate is the same from observation to observation. The non-central distribution arises when the observations have different means. The necessity of this distribution is felt in obtaining the power function for many tests in multivariate analysis and also in problems such as testing collinearity, (Fisher 1938) comparing scales of measurements (Cochran 1943) and multiple regression in time-series analysis (Tintner, 1944). This distribution was first derived by Anderson and Girshick (1944) (see also Anderson 1946), for the linear and planer cases. Later, James (1955- a & b) gave an alternative and elegant derivation based on averaging over orthogonal groups.

In this paper, the method of random orthogonal transformations is employed to derive the distribution and to consider the effect of non-centrality on the distribution of the variates which form the 'Bartlett Decomposition' of a Wishart matrix in the central case. The method of random orthogonal transformations consists in using orthogonal matrices having elements dependent on certain random vectors. This is a useful tool in various distribution problems in multivariate analysis. This idea is not new but is often couched in geometrical language. This method usually leads to the results in a simple, direct and unified way. Wijsman (1957) and later the present author (1959) have demonstrated this in the case of the Wishart distribution in the central case. It is now extended to the non-central case. The motivation for presenting yet another derivation of the non-central distribution is that it is believed that the method presented here leads to the results faster than existing derivations and also enables us to study explicitly the complications introduced by non-centrality on the Bartlett-Decomposition of a Wishart matrix.

2. Canonical form of the problem

There is no loss of generality in assuming the distribution of the sample observations x_{it} $(i=1, \cdots, p; t=1, \cdots, n)$ on the $p<n$ variables

$x_1, \cdots, x_p$ to be

$$(2\pi)^{-np/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{p}\sum_{t=1}^{n}(x_{it}-\delta_{it}k_i)^2\right\}\prod_{i,t} dx_{it} , \tag{2.1}$$

where δ_{it} is the Kronecker delta. Anderson and Girshick (1944) have shown that this form can always be obtained by a series of linear transformations. The matrix $(p\times p)$

$$S=[s_{ij}]=\left[\sum_{t=1}^{n} x_{it}x_{jt}\right] \tag{2.2}$$

is the Wishart matrix of the sums of squares and products of the observations x_{it}, which, on account of (2.1) are all normal independent variables with unit variances. The expected value of each x_{it} is zero except for p of the variables, viz

$$E(x_{ii})=k_i , \qquad (i=1, \cdots, \mathrm{p}) . \tag{2.3}$$

Let X_i' denote the row vector

$$[x_{i1}, \cdots, x_{in}] , \qquad (i=1, \cdots, p) . \tag{2.4}$$

By the well known process of orthogonalization of vectors, we obtain new unit and orthogonal vectors

$$Y_i'=[y_{i1}, \cdots, y_{in}] , \qquad (i=1, \cdots, p) . \tag{2.5}$$

The relation between the X_i's and the Y_i's is

$$X_i=b_{i1}Y_1+b_{i2}Y_2+ \cdots +b_{ii}Y_i , \qquad (i=1, 2, \cdots, p) \tag{2.6}$$

so that each X_i depends on $Y_1, \cdots, Y_i$ only and vice-versa. This can also be expressed as

$$X'=BY' \tag{2.7}$$

where

$$\underset{n\times p}{X}=[X_1 \vdots X_2 \vdots \cdots \vdots X_p] , \tag{2.8}$$

$$\underset{n\times p}{Y}=[Y_1 \vdots Y_2 \vdots \cdots \vdots Y_p] \tag{2.9}$$

and

$$B=\begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_{p1} & b_{p2} & \cdots & b_{pp} \end{bmatrix} . \tag{2.10}$$

Hence

$$S=X'X=BY'YB'=BB' \tag{2.11}$$

as

$$Y'Y=I\,, \tag{2.12}$$

the Y_i's being unit and orthogonal vectors. From (2.11) we have

$$s_{ii}=\sum_{t=1}^{n} x_{it}^2=X_i'X_i=b_{i1}^2+\cdots+b_{ii}^2 \tag{2.13}$$

or

$$b_{ii}^2=X_i'X_i-b_{i1}^2-\cdots-b_{ii-1}^2 \qquad (i=1,2,\cdots,p)\,. \tag{2.14}$$

Also from (2.6)

$$b_{ij}=Y_j'X_i\,; \qquad j=1,\cdots,i\,; \qquad i=1,\cdots,p\,. \tag{2.15}$$

The elements of the lower-triangular matrix $\boldsymbol{B}$ are called the rectangular co-ordinates and these constitute the Bartlett Decomposition of the Wishart matrix. (See Bartlett, 1933). In the null-case i.e. when $k_1, \cdots, k_p$ are all zero, it has been proved either implicitly or explicitly by several authors [for example, Mahalanobis, Bose & Roy (1937), Elfving (1947), Mauldon (1955), Wijsman (1957 and 1959) and Kshirsagar (1959)] that the off-diagonal elements of $\boldsymbol{B}$ are standard normal variables and the diagonal elements b_{ii} are χ-variables with degrees of freedom $n-i+1$, all these being independent. We are now concerned with the non-null distribution of these statistics. We shall consider the following three cases

(i) $k_1\neq 0\,,\quad k_2=\cdots=k_p=0$

(ii) $k_1\neq 0\,,\quad k_2\neq 0\,,\quad k_3=\cdots=k_p=0$

and (iii) $k_1\neq 0\,,\quad k_2\neq 0\,,\quad k_3\neq 0\,,\quad k_4=\cdots=k_p=0\,.$

3. The linear case

Here we assume k_1 to be the only non-zero mean ; all the other k's are zero. This is called the linear case because in terms of geometry, the expected values of the observations lie on a line in the p-dimensional space, in this case.

Keep $X_1, \cdots, X_{i-1}$ fixed (hence $Y_1, \cdots, Y_{i-1}$ are also fixed due to (2.6)) and transform from X_i ($i=2, 3, \cdots, p$ but $i\neq 1$) to $b_{i1}, \cdots, b_{ii-1}, \eta_{ii}, \cdots, \eta_{in}$ by an orthogonal transformation in which (see 2.15)

$$\begin{aligned} b_{i1}&=Y_1'X_i\,,\\ &\cdots\cdots\\ b_{ii-1}&=Y_{i-1}'X_i \end{aligned} \tag{3.1}$$

and $\eta_{ii}, \cdots, \eta_{in}$ are to be chosen suitably to complete the orthogonal transformation. This is a random orthogonal transformation because $Y_1, \cdots, Y_{i-1}$ are orthogonal vectors depending on $X_1, \cdots, X_{i-1}$ which we are keeping fixed for the time being. Since for any $i\neq 1$,

$$E(X_i)=0$$

and as the transformation is orthogonal, $b_{i1}, \cdots, b_{ii-1}, \eta_{ii}, \cdots, \eta_{in}$ are all standard normal independent variables. But this is their conditional distribution when $X_1, \cdots, X_{i-1}$ are fixed. However, since these conditioning variates do not occur in that distribution, it is also their absolute distribution and further they $(b_{i1}, \cdots, b_{i-1}, \eta_{ii}, \cdots, \eta_{in})$ are independently distributed of $X_1, \cdots, X_{i-1}$. On account of the orthogonality of the transformation,

$$\begin{aligned}\eta_{ii}^2+\cdots+\eta_{in}^2 &= X_i'X_i-b_{i1}^2-\cdots-b_{ii-1}^2\\ &= b_{ii}^2\end{aligned} \tag{3.2}$$

by virtue of (2.14). Thus b_{ii}^2 is a χ^2 with $n-(i-1)$ degrees of freedom. Combining all these results we have (i) $b_{i1}, \cdots, b_{ii-1}$ are $N(0, 1)$; (ii) b_{ii}^2 is a χ^2 with $n-(i-1)$ d.f. and (iii) all these are independently distributed and are independent of $X_1, \cdots, X_{i-1}$ too. (3.4)

This is true for $i=2, 3, \cdots, p$. But when we come to $i=1$, since

$$E(X_1')=[k_1, 0, \cdots, 0]\neq 0$$

we find that

$$b_{11}^2=X_1'X_1$$

is a non-central χ^2 with n degrees of freedom and non-centrality parameter k_1^2. Its distribution is

$$\frac{(b_{11}^2)^{(n/2-1)}\exp(-\frac{1}{2}b_{11}^2)\,db_{11}^2}{2^{n/2}\Gamma(n/2)}\cdot\exp(-\tfrac{1}{2}k_1^2)\sum_{\alpha=0}^{\infty}\frac{(k_1^2/2)^{\alpha}}{\alpha!}\cdot\frac{\Gamma(n/2)}{2^{\alpha}}\cdot\frac{b_{11}^{2\alpha}}{\Gamma((n/2)+\alpha)} \tag{3.5}$$

Using (3.4) and (3.5) we can write down the joint distribution of all the b_{ij} and then transform to the Wishart matrix S by (2.11). The Jacobian of transformation from B to S is (see Deemer and Olkin 1951)

$$2^{-p}\prod_{i=1}^{p} b_{ii}^{-(p+1-i)}\,. \tag{3.6}$$

Consequently the distribution of S comes out as

$$\frac{|S|^{(n-p-1)/2}\exp(-\frac{1}{2}t_rS)\,dS}{2^{np/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma((n+1-i)/2)}\cdot\exp(-\tfrac{1}{2}k_1^2)\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}\left(\frac{k_1^2}{2}\right)^{\alpha}\frac{\Gamma(n/2)s_{11}^{\alpha}}{2^{\alpha}\Gamma((n/2)+\alpha)} \tag{3.7}$$

where dS stands for the volume element. This is the non-central Wishart distribution of the linear case. The effect of non-centrality on the Bartlett discomposition is evident from (3.5).

4. The planar case

Here we assume $k_1 \neq 0$, $k_2 \neq 0$, but the remaining k's are all zero. In geometrical language this means that the means of the observations lie in a plane in the p dimensional space.

Proceeding exactly as in the previous section, we can prove that the result (3.4) holds for $i=3, \cdots, p$ but not for $i=1$ and 2. We, therefore, keep X_1 fixed and transform from X_2 to $b_{21}, \eta_{22}, \cdots, \eta_{2n}$ by an orthogonal transformation in which

$$b_{21} = Y_1' X_2 , \tag{4.1}$$

$$\eta_{22} = \frac{x_{22} - b_{21} y_{12}}{(1-y_{12}^2)^{1/2}} \tag{4.2}$$

and $\eta_{23}, \cdots, \eta_{2n}$ are chosen suitably to complete the orthogonal transformation. From (2.6) and (2.15)

$$b_{21} = Y_1' X_2 = \frac{1}{b_{11}} X_1' X_2$$

and

$$y_{12} = \frac{1}{b_{11}} x_{12} \tag{4.3}$$

and therefore, the coefficients of $x_{21}, x_{22}, \cdots, x_{2n}$ in η_{22} given by (4.2) are, apart from a constant multiplier $(1-y_{12}^2)^{-1/2}$,

$$-\frac{x_{11}x_{12}}{b_{11}^2}, \quad 1-\frac{x_{12}^2}{b_{11}^2}, \quad -\frac{x_{13}x_{12}}{b_{11}^2}, \quad \cdots, \quad -\frac{x_{1n}x_{12}}{b_{11}^2} . \tag{4.4}$$

The sum of squares of these coefficients is unity and the sum of products of these coefficients with the corresponding coefficients in (4.1) is zero. The requirements of an orthogonal transformation are thus fulfilled. It should be observed that (4.1) and (4.2) is a random orthogonal transformation. Consequently, $b_{21}, \eta_{22}, \cdots, \eta_{2n}$ are independent normal variables with unit variances.

$$\begin{aligned} E(b_{21}) = Y_1' E(X_2) &= [y_{11}, \cdots, y_{1n}][0, k_2, \cdots, 0]' \\ &= k_2 y_{12} \end{aligned} \tag{4.5}$$

and

$$E(\eta_{22}) = \frac{E(x_{22}) - y_{12}E(b_{21})}{(1-y_{12}^2)^{1/2}}$$
$$= k_2(1-y_{12}^2)^{1/2} . \tag{4.6}$$

Thus

$$\{E(b_{21})\}^2 + \{E(\eta_{22})\}^2 = k_2^2 ,$$

and as the transformation is orthogonal, it is obvious that the means of $\eta_{23}, \cdots, \eta_{2n}$ are all zero. Hence,

$$\eta_{23}^2 + \cdots + \eta_{2n}^2 = X_2'X_2 - b_{21}^2 - \eta_{22}^2$$
$$= b_{22}^2 - \eta_{22}^2 \quad \text{(by 2.14)}$$
$$= v , \quad \text{say} \tag{4.7}$$

is a χ^2 with $n-2$ degrees of freedom. The joint distribution of b_{21}, η_{22} and v when X_1 is fixed, is therefore,

$$\frac{1}{\sqrt{2\pi}} \exp[-\tfrac{1}{2}(b_{21} - k_2 y_{12})^2]\, db_{21}$$
$$\times \frac{1}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2}[\eta_{22} - k_2(1-y_{12}^2)^{1/2}]^2\}\, d\eta_{22}$$
$$\times \frac{1}{2^{(n-2)/2}\Gamma((n-2)/2)} v^{(n-2)/2-1} \exp(-\tfrac{1}{2}v)\, dv . \tag{4.8}$$

From (2.6)

$$X_2 = b_{21} Y_1 + b_{22} Y_2 ,$$

and, therefore,

$$y_{22} = (x_{22} - b_{21} y_{12})/b_{22}$$
$$= \eta_{22}(1-y_{12}^2)^{1/2}/b_{22} \tag{4.9}$$

by virtue of (4.2). Consequently from (4.7),

$$v = b_{22}^2(1 - y_{12}^2 - y_{22}^2)/(1-y_{12}^2) . \tag{4.10}$$

In (4.8) transform from η_{22} and v to b_{22} and y_{22}. The Jacobian of transformation is

$$2b_{22}^2/(1-y_{12}^2)^{1/2}$$

and the conditional distribution of b_{21}, b_{22} and y_{22} when X_1 is fixed, comes out as

$$\exp[-\tfrac{1}{2}k_2^2 - \tfrac{1}{2}(b_{21}^2 + b_{22}^2) + k_2(b_{21}y_{12} + b_{22}y_{22})]$$
$$\times \frac{1}{\pi 2^{(n-2)/2}\Gamma((n-2)/2)} \cdot \frac{b_{22}^{n-2}(1-y_{12}^2-y_{22}^2)^{(n-4)/2}}{(1-y_{12}^2)^{(n-3)/2}} db_{21}\, db_{22}\, dy_{22} . \tag{4.11}$$

This conditional distribution involves the conditioning variates X_1 only in the form

$$y_{12}=\frac{1}{b_{11}}x_{12}=x_{12}/(x_{11}^2+x_{12}^2+\cdots+x_{1n}^2)^{1/2} \tag{4.12}$$

but $x_{11},\cdots,x_{1n}$ are all normal independent variables with means $k_1, 0, \cdots, 0$ respectively and unit variances. From this the joint distribution of b_{11} and y_{12} comes out as

$$\sqrt{\frac{2}{\pi}}\exp\left(-\tfrac{1}{2}k_1^2-\tfrac{1}{2}b_{11}^2\right)\sum_{\alpha=0}^{\infty}\frac{(k_1^2/2)^\alpha}{\alpha!}\cdot\frac{(b_{11}^2)^{((n-1)/2)+\alpha)}(1-y_{12}^2)^{((n-3)/2)+\alpha)}}{\Gamma[((n-1)/2)+\alpha]2^{((n-1)/2)+\alpha}}db_{11}\,dy_{12}\,. \tag{4.13}$$

From (4.11) and (4.13), the joint distribution of b_{11}, b_{21}, b_{22} and y_{12}, y_{22} is

$$\begin{aligned}&\exp\left[-\tfrac{1}{2}(k_1^2+k_2^2)-\tfrac{1}{2}(b_{11}^2+b_{21}^2+b_{22}^2)+k_2(b_{21}y_{12}+b_{22}y_{22})\right]\\&\quad\times\frac{b_{11}^{n-1}b_{22}^{n-2}(1-y_{12}^2-y_{22}^2)^{(n-4)/2}}{\pi^{3/2}2^{n-2}\Gamma((n-2)/2)}\sum_{\alpha=0}^{\infty}\frac{(k_1^2/2)^\alpha}{\alpha!}\cdot\frac{b_{11}^{2\alpha}(1-y_{12}^2)^\alpha}{2^\alpha\Gamma[(n-1)/2+\alpha]}\\&\quad\times db_{11}\,db_{21}\,db_{22}\,dy_{12}\,dy_{22}\,.\end{aligned} \tag{4.14}$$

By making the substitutions

$$y_{12}=\sin\theta\,,$$
$$y_{22}=\cos\theta\sin\phi$$

and using certain results in Bessal functions, the irrelevant variables y_{12} and y_{22} can be integrated out. The details of this are uninteresting and can be found in Anderson's paper (1944). The distribution of b_{11}, b_{21} and b_{22} comes out in the form

$$\begin{aligned}&\exp\left[-\tfrac{1}{2}(k_1^2+k_2^2)-\tfrac{1}{2}(b_{11}^2+b_{21}^2+b_{22}^2)\right]b_{11}^{n-1}b_{22}^{n-2}\\&\quad\times\frac{1}{2^{n-2}\sqrt{\pi}}\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\frac{(k_1^2k_2^2b_{11}^2b_{22}^2)^\alpha[k_1^2b_{11}^2+k_2^2(b_{21}^2+b_{22}^2)]^\beta}{2^{4\alpha+2\beta}\alpha!\beta!\,\Gamma[((n-1)/2)+\alpha]\Gamma[(n/2)+2\alpha+\beta]}\\&\quad\times db_{11}\,db_{21}\,db_{22}\,.\end{aligned} \tag{4.15}$$

Taking this distribution in conjunction with the distribution of the other b_{ij}'s, stated at the beginning of this section and then transforming from $\boldsymbol{B}$ to $\boldsymbol{S}$ as in section 3, the non-central Wishart distribution in the planar case is obtained as

$$\begin{aligned}&\frac{|S|^{(n-p-1)/2}\exp\left(-\tfrac{1}{2}t_rS\right)dS}{2^{np/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(n+1-i)/2)}\\&\quad\times\exp\left[-\tfrac{1}{2}(k_1^2+k_2^2)\right]\sum_{\alpha,\beta=0}^{\infty}\frac{\Gamma(n/2)\Gamma((n-1)/2)[k_1^2k_2^2(s_{11}s_{22}-s_{12}^2)]^\alpha(k_1^2s_{11}+k_2^2s_{22})^\beta}{2^{4\alpha+2\beta}\alpha!\beta!\,\Gamma[(n/2)+2\alpha+\beta]\Gamma[((n-1)/2)+\alpha)}\,.\end{aligned} \tag{4.16}$$

The complications introduced by non-centrality in the ditribution of the statistics b_{ij} forming the 'Bartlett Decomposition' of $\boldsymbol{S}$ are thus made clear by (4.15). In the linear case, the b_{ij}'s remained independent but the distribution of b_{11}^2 is a non-central χ^2 and not a central χ^2 as in the null case. But in the planar case, the b_{ij}'s are no longer independent. There is an additional complication worth noting. In the central case the elements of the matrices $\boldsymbol{B}$ and $\boldsymbol{Y}$ of (2.7) are independently distributed. This is, however, not so in the planar case. This is evident from (4.14) which shows that the b_{ij}'s and y_{12}, y_{22} are not independent. In fact, the main trouble in deriving the non-central Wishart distribution is that of integrating out the y's entering in the joint distribution of $\boldsymbol{B}$ and $\boldsymbol{Y}$. In the case of three non-zero means k_1, k_2 and k_3, the integral becomes much harder, as will be seen from the next section.

5. The case of three non-zero means

We assume $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ but the remaining k's are all zero. As in section 3, we can prove the result (3.4) for $i=4, \cdots, p$ but not for $i=1$, 2 and 3.

Keeping $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ fixed, (so that $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are also fixed) transform from $\boldsymbol{X}_3$ to $b_{31}, b_{32}, \eta_{33}, \cdots, \eta_{3n}$ by an orthogonal transformation in which

$$b_{31} = \boldsymbol{Y}_1' \boldsymbol{X}_3 \tag{5.1}$$

$$b_{32} = \boldsymbol{Y}_2' \boldsymbol{X}_3 \tag{5.2}$$

$$\eta_{33} = \frac{x_{33} - b_{31} y_{13} - b_{32} y_{23}}{(1 - y_{13}^2 - y_{23}^2)^{1/2}} \tag{5.3}$$

and $\eta_{34}, \cdots, \eta_{3n}$ are chosen suitably to complete the orthogonal transformation. As in the planar case, it can be verified by collecting the coefficients of $x_{31}, \cdots, x_{3n}$ in η_{33} that the requirements of an orthogonal transformation are fulfilled. Consequently $b_{31}, b_{32}, \eta_{33}, \cdots, \eta_{3n}$ are independent normal variables with unit variances. From (2.6)

$$\boldsymbol{X}_3 = b_{31} \boldsymbol{Y}_1 + b_{32} \boldsymbol{Y}_2 + b_{33} \boldsymbol{Y}_3$$

and therefore, from (5.3)

$$\eta_{33} = b_{33} y_{33} / (1 - y_{13}^2 - y_{23}^2)^{1/2} \, . \tag{5.4}$$

Since

$$E(\boldsymbol{X}_3') = [0, 0, k_3, 0, \cdots, 0] \, ,$$

we get

$$E(b_{31}) = y_{13}k_3 \ , \tag{5.5}$$

$$E(b_{32}) = y_{23}k_3 \ , \tag{5.6}$$

$$E(\eta_{33}) = k_3(1-y_{13}^2-y_{23}^2)^{1/2} \ . \tag{5.7}$$

As the transformation is orthogonal and as

$$[E(b_{31})]^2+[E(b_{32})]^2+[E(\eta_{33})]^2=k_3^2 \ ,$$

it is obvious that $\eta_{34}, \cdots, \eta_{3n}$ have zero means and therefore,

$$\begin{aligned} \eta_{34}^2+\cdots+\eta_{3n}^2 &= X_3'X_3-b_{31}^2-b_{32}^2-\eta_{33}^2 \\ &= b_{33}^2-\eta_{33}^2 \quad \text{(by 2.14)} \\ &= w \quad \text{say} \end{aligned} \tag{5.8}$$

is a χ^2 with $n-3$ degrees of freedom. The joint distribution of b_{31}, b_{32}, η_{33} and w when X_1 and X_2 are fixed is, therefore,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}(b_{31}-k_3y_{13})^2\right]db_{31}\cdot\frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}(b_{32}-k_3y_{23})^2\right]db_{32} \\ &\times\frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\{\eta_{33}-k_3(1-y_{13}^2-y_{23}^2)^{1/2}\}^2\right]d\eta_{33} \\ &\times\frac{1}{2^{(n-3)/2}\Gamma((n-3)/2)}w^{((n-3)/2)-1}\exp\left(-\tfrac{1}{2}w\right)dw \ . \end{aligned} \tag{5.9}$$

From (5.8) and (5.4)

$$w=b_{33}^2(1-y_{13}^2-y_{23}^2-y_{33}^2)/(1-y_{13}^2-y_{23}^2) \ . \tag{5.10}$$

Using (5.4) and (5.10), transform from η_{33} and w to b_{33} and y_{33} in the distribution (5.9). The Jacobian of transformation is

$$2b_{33}^2/(1-y_{13}^2-y_{23}^2)^{1/2}$$

and we have the distribution of b_{31}, b_{32}, b_{33} and y_{33} when X_1 and X_2 are fixed, in the form

$$\begin{aligned} &\exp\left\{-\tfrac{1}{2}k_3^2-\tfrac{1}{2}\sum_{i=1}^{3}b_{3i}^2+k_3\sum_{i=1}^{3}b_{3i}y_{i3}\right\}\cdot b_{33}^{n-3} \\ &\qquad\times\frac{(1-y_{13}^2-y_{23}^2-y_{33}^2)^{(n-5)/2}\,db_{31}\,db_{32}\,db_{33}\,dy_{33}}{(1-y_{13}^2-y_{23}^2)^{(n-4)/2}2^{(n-2)/2}\pi^{3/2}\Gamma((n-3)/2)} \ . \end{aligned} \tag{5.11}$$

This conditional distribution involves the conditioning variates X_1 and X_2 in the form y_{13} and y_{23} only. So, we now keep X_1 fixed and transform from X_2 to $b_{21}, Z_{22}, \cdots, Z_{2n}$ by a random orthogonal transformation in which

$$b_{21} = Y_1' X_2 \tag{5.12}$$

$$Z_{22} = \frac{x_{23} - b_{21} y_{13}}{(1-y_{13}^2)^{1/2}} = \frac{b_{22} y_{23}}{(1-y_{13}^2)^{1/2}} \tag{5.13}$$

and $Z_{23}, \cdots, Z_{2n}$ are suitably chosen to complete the orthogonal transformation. As in section 4, it can be verified that Z_{22} satisfies the requirements of an orthogonal transformation. Proceeding exactly in the same manner as in section 4, it can be proved that b_{21} is a normal variable with mean $k_2 y_{12}$ and unit variance. Z_{22} is a normal variable with mean

$$-k_2 y_{12} y_{13}/(1-y_{13}^2)^{1/2}$$

and unit variance and that

$$\begin{aligned} Z_{23}^2 + \cdots + Z_{2n}^2 &= X_2' X_2 - b_{21}^2 - Z_{22}^2 = b_{22}^2 - Z_{22}^2 \\ &= b_{22}^2 (1 - y_{13}^2 - y_{23}^2)/(1-y_{13}^2) \end{aligned} \tag{5.14}$$

is a non-central χ^2 with $n-2$ degrees of freedom and non-centrality parameter

$$k_2^2 (1 - y_{12}^2 - y_{13}^2)/(1-y_{13}^2) \ . \tag{5.15}$$

From these the joint distribution of b_{21}, b_{22} and y_{23} when X_1 is fixed, comes out as

$$\frac{1}{\pi} \exp\left\{ -\tfrac{1}{2}k_2^2 - \tfrac{1}{2}(b_{21}^2 + b_{22}^2) + k_2 b_{21} y_{12} - \frac{k_2 y_{12} y_{13} y_{23} b_{22}}{1-y_{13}^2} \right\}$$
$$\times \sum_{\beta=0}^{\infty} \frac{(k_2^2/2)^\beta (1-y_{12}^2-y_{13}^2)^\beta b_{22}^{n-2+2\beta} (1-y_{13}^2-y_{23}^2)^{((n-4)/2)+\beta}}{\beta!\, 2^{((n-2)/2)+\beta} (1-y_{13}^2)^{((n-3)/2)+2\beta} \Gamma[((n-2)/2)+\beta)} db_{21}\, db_{22}\, dy_{23} \ . \tag{5.16}$$

This conditional distribution involves the conditioning variates X_1 in the form y_{12} and y_{13}. From (2.6)

$$X_1 = b_{11} Y_1$$

and therefore,

$$y_{12} = x_{12}/b_{11} \ , \qquad y_{13} = x_{13}/b_{11} \qquad \text{and}$$

$$b_{11}^2 = x_{11}^2 + x_{12}^2 + x_{13}^2 + \cdots + x_{1n}^2 \ .$$

Also $x_{11}, \cdots, x_{1n}$ are independent normal variables with means $k_1, 0, \cdots, 0$ respectively and unit variances. This yields the joint distribution of y_{12}, y_{13} and b_{11} as

$$\frac{1}{\pi} \exp\left(-\tfrac{1}{2}k_1^2 - \tfrac{1}{2}b_{11}^2\right) \sum_{\alpha=0}^{\infty} \frac{(k_1^2/2)^\alpha}{\alpha!} \cdot \frac{b_{11}^{n-1+2\alpha} (1-y_{12}^2-y_{13}^2)^{((n-4)/2)+\alpha}}{\Gamma[((n-2)/2)+\alpha] \cdot 2^{((n-2)/2)+\alpha}} \ . \tag{5.17}$$

Combining the results (5.11), (5.16) and (5.17), the density function of the joint distribution of b_{ij} $(i=1, 2, 3;\ j=1, \cdots, i)$ and y_{12}, y_{13}, y_{23}, y_{33} is

$$\exp\left\{-\tfrac{1}{2}\sum_{i=1}^{3}k_i^2-\tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{i}b_{ij}^2+k_2b_{21}y_{12}+k_3\sum_{i=1}^{3}b_{3i}y_{i3}-(k_2b_{22}y_{12}y_{13}y_{23}/(1-y_{13}^2))\right\}$$
$$\times\frac{b_{11}^{n-1}b_{22}^{n-2}b_{33}^{n-3}(1-y_{13}^2-y_{23}^2-y_{33}^2)^{(n-5)/2}}{\pi^{7/2}\Gamma((n-3)/2)\,2^{3(n-2)/2}}$$
$$\times\sum_{\alpha,\beta=0}^{\infty}\frac{(k_1^2/2)^{\alpha}(k_2^2/2)^{\beta}b_{11}^{2\alpha}b_{22}^{2\beta}(1-y_{12}^2-y_{13}^2)^{((n-4)/2)+\alpha+\beta}(1-y_{13}^2-y_{23}^2)^{\beta}}{\alpha!\beta!\Gamma[((n-2)/2)+\alpha]\Gamma[((n-2)/2)+\beta)\cdot(1-y_{13}^2)^{((n-3)/2)+2\beta}\,2^{\alpha+\beta}}\,. \tag{5.18}$$

To get the distribution of the b_{ij}'s alone, it is necessary to integrate out the variables y_{12}, y_{13}, y_{23} and y_{33}. Performing this and combining it with the independent distribution of the other b_{ij}'s $(j=1, 2, \cdots, i;\ i=4, \cdots, p)$ stated at the beginning of this section and then transforming from $\boldsymbol{B}$ to $\boldsymbol{S}$ as in section 3, we should be able to get the non-central Wishart distribution in the present case of three non-zero roots. However this is extremely tedious and also unnecessary because James (1955) has given the final distribution. He obtained it by an entirely different method. The distribution of $\boldsymbol{S}$ is

$$\frac{|S|^{(n-p-1)/2}\exp(-\tfrac{1}{2}t_rS)\,dS}{2^{np/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma((n+1-i)/2)}\exp\left(-\tfrac{1}{2}\sum_{1}^{3}k_i^2\right)\sum_{\alpha,\beta,\gamma=0}^{\infty}\frac{r_1^{\alpha}r_2^{\beta}r_3^{\gamma}}{4^{\alpha+2\beta+3\gamma}\alpha!\,\beta!\,\gamma!}$$
$$\times\frac{\Gamma(n/2)\Gamma((n-1)/2)}{\Gamma[(n/2)+\alpha+2\beta+3\gamma]\Gamma[((n-1)/2)+\beta+2\gamma]}$$
$$\times\frac{\Gamma[(n-2)/2]}{\Gamma[((n-2)/2)+\gamma]}\cdot\frac{(n+\alpha+2\beta+4\gamma-3)!}{(n+\alpha+2\beta+3\gamma-3)!} \tag{5.19}$$

where r_1, r_2, r_3 are the elementary symmetric functions of the latent roots of the 3×3 matrix

$$[k_ik_js_{ij}] \tag{5.20}$$

$(i, j=1, 2, 3)$.

Unfortunately, the distribution is so complicated that its usefulness is limited. However, for our purpose of investigating the interdependence of the b_{ij}'s and the y's, the method of random orthogonal transformations is sufficient. The method of random orthogonal transformation has great advantages in cases where it works easily. Thus in the linear case, the method gives the result immediately; certainly easier and nicer than with the method of Anderson and Girshick. In the planar case, trouble arises due to the dependence of $\boldsymbol{B}$ and $\boldsymbol{Y}$ and this becomes worse for more than two non-zero k's.

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