CERTAIN PROPERTIES OF THE GENERALIZED POWER SERIES DISTRIBUTION

BY G. P. PATIL*

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1. Introduction

Let $g(\theta)$ be a positive function admitting a power series expansion with non-negative coefficients for non-negative values of θ smaller than the radius of convergence of the power series:

$$g(\theta)=\sum_{z=0}^{\infty} a_z\theta^z \tag{1}$$

Noack (1950) defined a random variable Z taking non-negative integral values z with probabilities

$$\text{Prob}\{Z=z\}=\frac{a_z\theta^z}{g(\theta)} \qquad z=0,1,2,\cdots \tag{2}$$

He called the discrete probability distribution given by (2) a power series distribution (*psd*) and derived some of its properties relating its moments, cumulants, etc.

To be more general, we note that the set of values of an integral-valued random variable Z need not be the entire set of non-negative integers $(0,1,2,\cdots)$. For, let T be an arbitrary non-null subset of non-negative integers** and define the generating function

$$f(\theta)=\sum_{x\in T} a_x\theta^x$$

with $a_x>0$; $\theta\geqq 0$ so that $f(\theta)$ is positive, finite and differentiable. Then we can define a random variable X taking non-negative integral values in T with probabilities

$$P_x=\text{Prob}\{X=x\}=\frac{a_x\theta^x}{f(\theta)} \qquad x\in T \tag{3}$$

and call this distribution analogously a generalized power series distribution (*gpsd*). It may be noted that *gpsd* reduces to a *psd* when T is the entire set of non-negative integers. The properties established by

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** In fact, one can take T to be a countable subset of real numbers; for purposes of this paper, however, T is chosen to be a subset of non-negative integers.

Noack (1950) and Khatri (1959) for *psd* can be easily deduced for *gpsd* by following the same lines. Further, it can be easily seen that proper choice of T and $f(\theta)$ reduces the *gpsd*, in particular, to the binomial, negative binomial, Poisson and logarithmic series distributions and their truncated forms. Incidentally, it is obvious that a truncated *gpsd* is itself a *gpsd* in its own right and hence the properties that hold for a *gpsd* continue to hold for its truncated forms.

We add here that we call the set of admissible values of the parameter θ of the *gpsd* as the parameter space Θ of the *gpsd*. Roy and Mitra (1957) have studied the problem of the Uniformly Minimum Variance Unbiased Estimator for the parameter θ of the *psd*'s. Guttman (1958) also has studied a similar problem. The author (1957, 1959, 1961, 1962) has investigated certain problems of statistical inference associated with the *gpsd*'s. Here, we propose to record certain properties of the *gpsd* which may be of some use for statistical inference involved. Only those proofs are furnished which are somewhat interesting ; others are simply straightforward and simple.

2. Properties

Among other properties as shown by Noack (1950) for a *psd*, we have for a *gpsd* :

$$\mu = \theta \frac{d}{d\theta} \log f(\theta) \tag{4}$$

$$\mu_2 = \theta \frac{du}{d\theta} \tag{5}$$

and

$$\mu_2 = \mu + \theta^2 \cdot \frac{d^2}{d\theta^2} \log f(\theta) \tag{6}$$

where μ and μ_2 are the mean and the variance of the *gpsd* (3). Based on (3), (4), (5), and (6), the following properties for (3) follow.

P_1: If the parameter space Θ of a *gpsd* contains zero, then the range T of the *gpsd* contains zero and the corresponding random variable takes the value zero with positive probabilities for all θ in the parameter space ; and conversely.

P_2: The generating function and its logarithm, of a *gpsd* are monotone non-decreasing functions of θ.

P_3: The necessary and sufficient condition for the variance of a *gpsd* to equal its mean for every θ of its parameter space Θ is that the generating function be of the form

$$f(\theta)=\exp(k\theta+c)$$

where $k>0$ and c are arbitrary constants.

P_4: The equality of mean and variance is necessary and sufficient for a *gpsd* to become Poisson. (characterization of Poisson). This follows immediately when one observes that a positive constant multiple of the generating function $f(\theta)$ of a *gpsd* does not affect it, i.e., gives rise to the same original *gpsd*.

D_1: Let a *gpsd* be called super-Poisson and sub-Poisson respectively according as its variance is greater than or less than its mean value for every non-zero θ of its parameter space Θ.

P_5: The necessary and sufficient condition for a *gpsd* to be super-Poisson is that the generating function be of the form

$$f(\theta)=\exp(P(\theta)+R\theta+Q)$$

where Q and R are arbitrary constants and $P(\theta)$, along with its derivative, is a positive monotone increasing function of θ.

P_6: The necessary and sufficient condition for a *gpsd* to be sub-Poisson is that the generating function be of the form

$$f(\theta)=\exp(A(\theta)+B\theta+C)$$

where B and C are arbitrary constants and $A(\theta)$ is such that its derivative is a monotone decreasing function of θ.

P_7: The mean $\mu(\theta)$ of a *gpsd* is non-negative monotone non-decreasing function of θ.

PROOF. Consider the relation (5), which states

$$\mu_2(\theta)=\theta\frac{d}{d\theta}[\mu(\theta)]. \tag{7}$$

We know that $\mu_2(\theta)\geqq 0$, and also $\theta\geqq 0$. Therefore from (7) it follows that

$$\frac{d}{d\theta}[\mu(\theta)]\geqq 0, \text{ and also } \mu(\theta)\geqq 0.$$

Therefore $\mu(\theta)$ is a non-negative monotone non-decreasing function of θ.

P_8: The graph of the mean of a *gpsd* with parameter space containing zero is convex or concave or linear according as the *gpsd* is super-Poisson, sub-Poisson or Poisson and conversely.

PROOF. Suppose $\mu_2(\theta)>\mu(\theta)$. Then

$$\theta\frac{d}{d\theta}[\mu(\theta)]>\mu(\theta)$$

$$\frac{d}{d\theta}[\mu(\theta)]>\frac{\mu(\theta)}{\theta} \qquad \text{when } \theta\neq 0.$$

Also, as the *gpsd* is taken with parameter space containing zero, we can speak of $\mu(0)$ which is clearly $=0$. Hence, follows the convexity of the graph of the mean when the variance exceeds the mean.

On similar lines the rest of the statement can be very easily established.

P_9: The mean of a *gpsd* with parameter space containing zero is respectively a linear or convex or concave function of θ if and only if the generating function is respectively of the form of P_8, P_5, or P_6.

D_2: Let X be a *r.v.* taking values in T according to some probability law P_x, $x \in T$. Let c be the smallest integer in T. We define X to enjoy the property of proportions if the following holds:

(i) $P_c = 1/g(\theta)$ where $g(\theta)$ admits power series expansion in θ.

(ii) $P_{x+r}/P_x = b(x, r)\theta^r$; $b(x, r) \geqq 0$; x, $x+r \in T$; $r = 1, 2 \cdots$

P_{10}: The distribution of X is a *gpsd* if and only if X enjoys the property of proportions. (characterization of a *gpsd*).

UNIVERSITY OF MICHIGAN AND MCGILL UNIVERSITY

REFERENCES

[1] I. G. Guttman, "A note on a series solution of a problem in estimation," *Biometrika*, Vol. 45 (1958), pp. 565-567.

[2] C. G. Khatri, "On some properties of power series distribution," *Biometrika*, Vol. 46 (1959), pp. 486-490.

[3] A. Noack, "On a class of discrete random variables," *Annals of Mathematical Statistics*, Vol. 21 (1950), pp. 127-132.

[4] G. P. Patil, "On some problems of estimation in a class of discrete distributions," *Thesis for the Associateship of the Indian Statistical Institute*, (1957), p. 203.

[5] G. P. Patil, "Contributions to estimation in a class of discrete distributions," *Doctoral thesis at the University of Michigan*. (1959), p. 181.

[6] G. P. Patil, "Asymptotic bias and efficiency of ratio estimates for generalized power series distributions and certain applications," *Sankhya*, Vol. 23 (1961), pp. 269-280.

[7] G. P. Patil, "Maximum likelihood estimation for generalized power series distribution and its application to truncated binomial distribution," *Biometrika*, Vol. 49 (1962), pp. 222-232.

[8] G. P. Patil, "On homogeneity and combined estimation for generalized power series distribution and certain applications," *Biometrics*, Vol. 18 (1962), pp. 365-374.

[9] G. P. Patil, "Estimation by two-moments method for generalized power series distribution and certain applications," *Sankhya*, Vol. 24 (1962), pp. 201-214.

[10] J. Roy and S. K. Mitra, "Unbiased minimum variance estimation for generalized power series distribution and certain applications," *Sankhya*, Vol. 18 (1957), pp. 371-378.