FURTHER CONSIDERATION ON NORMAL RANDOM VARIABLE GENERATOR

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(Received Sep. 28, 1962)

Summary

In a previous paper [3] the author considered a computer technique for generating exponential random variables and its applications to the generation of random variables of other types. He suggested to combine the new exponential random variable generator with J. C. Butcher's method [1] to get normal random deviates. In this note, an improvement of the idea is shown with a remark on the rejection technique in the Monte Carlo method.

1. Remark on the rejection technique

The rejection technique for generating a random variable X with a probability density function

$$f(x)=Mk(x)g(x), \qquad a\leqq x\leqq b, \tag{1}$$

where $g(x)$ is also a probability density function on $[a, b]$, and

$$0\leqq k(x)\leqq 1, \qquad a\leqq x\leqq b,$$
$$M\geqq 1,$$

is explained by some authors as follows.

Let $\{U_i\}$ be a sequence of independent random variables distributed uniformly on (0,1), which may be quasi-random numbers obtained by a suitable method, and let $\{T_i\}$ be a sequence of those which are converted from quasi-random numbers so that they have the probability density function $g(t)$. Then generate U_1 and T_1. If

$$k(T_1)\geqq U_1 \tag{2}$$

accept T_1 as a random variable X. Otherwise, generate (U_2, T_2), $(U_3, T_3),\cdots$ until the inequality is satisfied.

In this procedure, however, there is an abuse of uniform random variables. In fact, we can fix the first U_1 to test the inequality (2), and we have to change only T until it is satisfied. Because, for a fixed value of $U_1=u$ the accepted T has the probability density function $g(t)$ truncated on a set

$$S_u = \{x\,;\,k(x) \geqq u\}\,. \tag{3}$$

Therefore, the probability that the X, the accepted T_i, falls on a measurable set $A \subset [a, b]$ is

$$\begin{aligned} P^X(X \in A) &= P^T(T_i \in A \mid T_i \text{ accepted}) \\ &= \int_0^1 P^T(T_i \in A \cap S_u)du \Big/ \int_0^1 P^T(T_i \in S_u)du \\ &= \int_0^1\int_A g(t)\chi(t\,;\,S_u)dt\,du \Big/ \int_0^1\int_a^b g(t)\chi(t\,;\,S_u)dt\,du \\ &= \int_A g(t)k(t)dt \Big/ \int_a^b g(t)k(t)\,dt \\ &= \int_A f(t)dt \end{aligned}$$

where $\chi(t\,;\,A)$ is the indicator function of a set A.

The probability of the acceptance of T_i is equal to M^{-1}, the same value for two cases, in which U's are changed and are not changed.

The remark is valid for more complicated rejection techniques. We consider here the mixture-rejection technique. Let us assume that the probability density function of X is decomposed as

$$\begin{aligned} &f(x) = \sum_{j=1}^{s} \alpha_j f_j(x) = \sum_{j=1}^{s} \alpha_j M_j k_j(x) g_j(x), \\ &\sum_{j=1}^{s} \alpha_j = 1. \end{aligned} \tag{4}$$

We generate the random variable $X^{(j)}$ with the density $f_j(x)$ by the above mentioned method with the frequency α_j. Then we obtain the sequence of random variables X's with the density $f(x)$. In this case, two schemes in Fig. 1 are possible. In both schemes, $\alpha_j M_j$ times of productions and tests of $X^{(j)}$ are necessary in average per one X for each procedure. Therefore, the first scheme saves the labor to select a

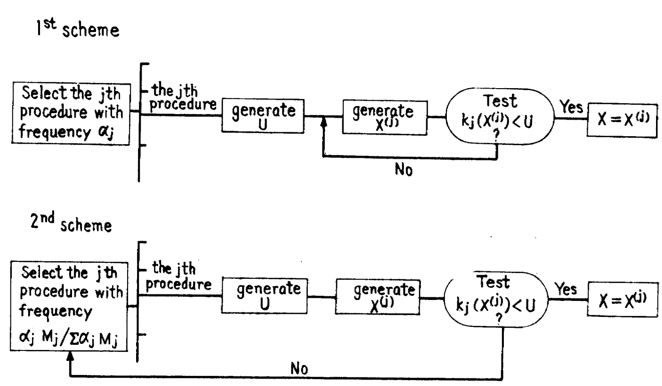

Fig. 1. Mixture-rejection technique.

procedure and to generate U when $X^{(j)}$ is rejected.

2. Normal random variable generator

We generate the absolute values of normal deviates, and consider a decomposition of the type of (4).

$$\sqrt{\frac{2}{\pi}}\exp(-x^2/2)=S(\xi)\cdot 1\cdot \chi(x\,;(0,\xi))/\xi$$
$$+\{G(\xi)-S(\xi)\}\cdot\frac{\sqrt{2/\pi}\,\xi-S(\xi)}{G(\xi)-S(\xi)}\cdot\frac{\exp(-x^2/2)-\exp(-\xi^2/2)}{1-\exp(-\xi^2/2)}\cdot\frac{\chi(x\,;(0,\xi))}{\xi}$$
$$+\{1-G(\xi)\}\cdot\frac{1}{\{1-G(\xi)\}\lambda}\sqrt{\frac{2}{\pi}}\exp((\lambda^2/2)-\lambda\xi)\cdot\exp(-(x-\lambda)^2/2)\cdot$$
$$\lambda\exp(-\lambda(x-\xi))\ \chi(x\,;(\xi,\infty)), \tag{5}$$

where

$$S(\xi)=\sqrt{\frac{2}{\pi}}\,\xi\exp(-\xi^2/2),\qquad G(\xi)=\sqrt{\frac{2}{\pi}}\int_0^{\xi}\exp(-x^2/2)dx,\qquad 0<\xi<\lambda,$$

$$\chi(x\,;(a,b))=\begin{cases}1, & x\in(a,b),\\ 0, & x\in(a,b).\end{cases}$$

It is a decomposition of the normal distribution into three distributions as Fig. 2. Now we investigate the procedures for generating the three distributions. We select each of them with frequencies $\alpha_1=S(\xi)$, $\alpha_2=G(\xi)-S(\xi)$, and $\alpha_3=1-G(\xi)$ respectively.

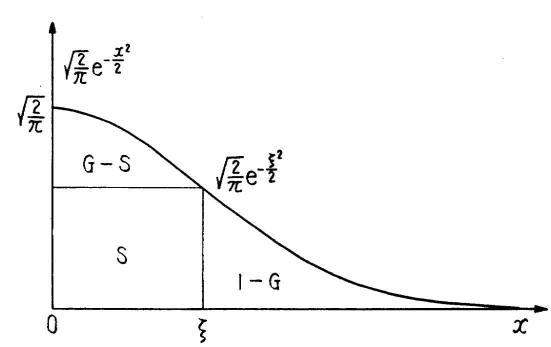

Fig. 2. Decomposition of normal density.

(1°) In the first procedure we generate a random variable distributed uniformly on $(0,\xi)$ to accept it as X.
(2°) Generate random variables U, uniform on $(0,1)$, and $X^{(2)}$, uniform on $(0,\xi)$, and accept $X^{(2)}$ if

$$\frac{\exp(-X^{(2)2}/2)-\exp(-\xi^2/2)}{1-\exp(-\xi^2/2)} \geqq U, \tag{6}$$

or equivalently

$$X^{(2)2}/2 \leqq -\log\{U(1-\exp(-\xi^2/2))+\exp(-\xi^2/2)\}. \tag{7}$$

It should be noticed that the random variable of the right-hand side of the inequality (7) has the probability density

$$\exp(-v)/(1-\exp(-\xi^2/2)), \qquad 0<v<\xi^2/2, \tag{8}$$

the exponential distribution with truncated upper tail.

The discussion in the papers [2, 3] shows the following fact.

Let N be 0-truncated Poisson random variable with parameter μ,

$$P(N=n)=(1-\exp(-\mu))^{-1}\exp(-\mu)\mu^n/n!, \qquad n=1, 2, \cdots \tag{9}$$

and let $U_1, U_2, \cdots$ be uniform random variables on $(0, 1)$. Then the variable

$$V^* = \mu \min(U_1, \cdots, U_N) \tag{10}$$

has the density

$$\exp(-v)/(1-\exp(-\mu)), \qquad 0<v<\mu.$$

Therefore the second procedure reduces to the following operations. Generate 0-truncated Poisson variate N with parameter $\xi^2/2$. Compute

$$V=\min(U_1, \cdots, U_N).$$

Generate a nuiform randon variable U, and accept $\xi U = X^{(2)}$ as X if

$$U^2 \leqq V.$$

The probability of the acceptance of ξU is

$$M_2^{-1} = \{G(\xi)-S(\xi)\}/\{\sqrt{2/\pi}\,\xi - S(\xi)\} \tag{11}$$

(3°) Generate the random variable $X^{(3)}$ with the density

$$\lambda \exp(-\lambda(x-\xi)), \qquad \xi<x<\infty, \tag{12}$$

and the uniform variable U.

Accept $X^{(3)}$, if

$$\exp\{-(X^{(3)}-\lambda)^2/2\} \geqq U,$$

or equivalently

$$(X^{(3)}-\lambda)^2/2 \leqq -\log U. \tag{13}$$

The right hand side of (13) is the standard exponential deviate.

Let Y_1 and Y_2 be standard exponential deviates. We take up $X^{(3)} = Y_2/\lambda+\xi$ as X, if

$$\frac{1}{2}\left(\frac{Y_2}{\lambda}+\xi-\lambda\right)^2 \leqq Y_1 . \tag{14}$$

In the papers [2, 3], the following technique was suggested for generating the standard exponential deviate.

Generate the geometric random variable M;

$$P(M=m)=pq^m, \qquad m=0, 1, \cdots ,$$
$$p=1-\exp(-\mu), \tag{15}$$

and the 0-truncated Poisson random variable N, with the probability distribution (9). Then

$$Y=\mu\{M+\min (U_1, \cdots, U_N)\} \tag{16}$$

is a standard exponential deviate. The relation

$$\mu M< Y<\mu(M+1) \tag{17}$$

saves slightly the time of the test (14).

If fact, if we put

$$p_1=P\left\{\frac{1}{2}\left(\frac{Y_2}{\lambda}+\xi-\lambda\right)^2 \leq \mu M\right\} ,$$
$$p_2=P\left\{\frac{1}{2}\left(\frac{Y_2}{\lambda}+\xi-\lambda\right)^2 \geq \mu(M+1)\right\} , \tag{18}$$
$$p_3=1-p_1-p_2 ,$$

then, under the condition that the third procedure is selected the test

$$\frac{1}{2}\left(\frac{Y_2}{\lambda}+\xi-\lambda\right)^2 <\mu M$$

or

$$\frac{1}{2}\left(\frac{Y_2}{\lambda}+\xi-\lambda\right)^2 >\mu(M+1) \tag{19}$$

continues $(1-p_3)^{-1}$ times in average. With the proportion $p_1/(1-p_3)$ Y_2 is accepted through the test (19), and with the proportion $p_3/(1-p_3)$, the further test (14) is necessary.

The probability of the acceptance of $X^{(3)}$ is

$$M_3^{-1}= \{1-G(\xi)\}\lambda \cdot \sqrt{\frac{\pi}{2}} \exp(-(\lambda^2/2)+\lambda\xi) \tag{20}$$

and the average number of the generation of Y is M_3.

3. Determining the values of the parameters

Now we determine the values of ξ and λ taking into account practical aspects of computer programing. We are interested in the use of binary system computers rather than decimal system ones.

The function $\alpha_1=\sqrt{2/\pi}\,\xi\exp(-\xi^2/2)$ takes its maximum value at $\xi=1$, while α_2 is increasing and α_3 is decreasing function of ξ. Therefore, we should choose ξ larger than 1. If we insist on the generation of normal variates with variance 1, we have to choose such values of ξ as 1 or 2 to avoid multiplications $\xi\times U$ in procedures 1° and 2°. On many computers multiplication takes several times as long machine-time as addition. Therefore, one multiplication in the routine will off set the result of selection of such a value of ξ as minimizes, for example, the average number of uniform random variables consumed.

The value of λ affects on only M_3, and minimizes it when $\lambda=\xi/2+\sqrt{(\xi/2)^2+1}$. It must be also a simple value to avoid multiplications, and we put $\lambda=2$ for $\xi=1$ and 2, in view of Table 1. The value of the parameter μ in the third procedure is independent of ξ or μ. We,

TABLE 1.

M_3^{-1}: the acceptance probability in the 3rd procedure.

ξ	λ	M_3
1	1	1.525
	2	1.257
2	2	1.187
	4	4.384

TABLE 2.

Average numbers of uniform (U), Poisson (N), geometric (M) and exponential (Y) random variables consumed to generate one normal random variable in procedures 1°, 2° and 3°.

Operation		Frequency	$\xi=1,\ \lambda=2,\ \mu=1/2$	$\xi=2,\ \lambda=2,\ \mu=2$
1°	U	α_1	0.4839	0.2160
2°	N	α_2	0.1987	0.7385
	U_i	$\alpha_2 E(N)$	0.2526	1.7083
	U_i^2	$\alpha_2 M_2$	0.3139	1.3798
3°	Y_i	$\alpha_3 M_3$	0.3989	0.0540
	M	α_3	0.3173	0.0455
	N	$\alpha_3 p_3/(1-p_2)$	0.1249	0.0393
	U_i	$\alpha_3 p_3/(1-p_2)\cdot E(N)$	0.1587	0.0910

however, use the same sub-routine to generate truncated Poisson variable N in both the second and the third procedures. The average number of operations per one X in each procedures are listed in Table 2 for two sets of values of the parameters.

On a medium-size scientific computer HIPAC-103, if we get the quasi-random numbers by the mixed congruence method, the average machine time per one X is approximately 430 and 610 cycles (1 cycle $=40\ \mu s$) for $\xi=1$ and 2 respectively. In Fig. 3 the flow chart is shown with frequencies of each routes per one X for $\xi=1$.

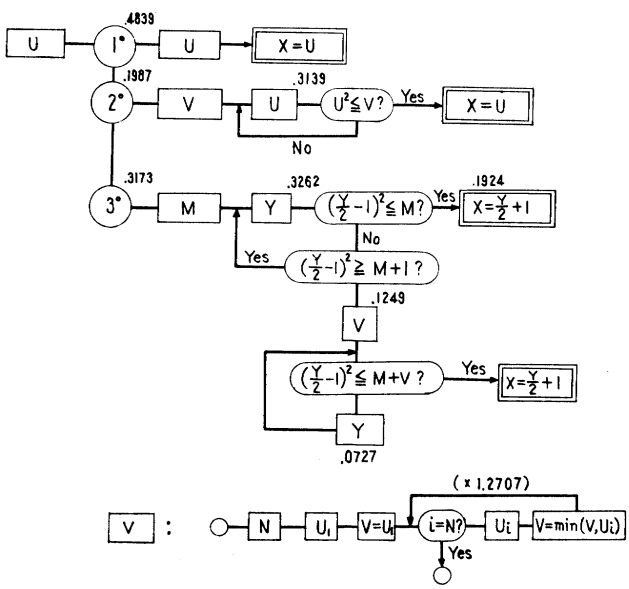

Fig. 3.

Acknowledgement

The author wishes to thank Miss A. Maruyama for her aids in preparing the paper.

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