ON THE OPTIMAL LIFE TEST PROCEDURES BASED ON A COST MODEL

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1. Introduction

J. D. Riley demonstrated that the minimum expected cost for the nonreplacement life test procedure is always less than or equal to that for the replacement procedure based on the cost model presented by B. Epstein. (see [1], [2], [3] and [4]) Namely, it has been concluded that the nonreplacement procedure is always more preferable than the replacement test procedure so long as that cost model is applicable. It seems to us, however, that there are many cases to which Epstein's cost model is not applicable because of the variation of the cost corresponding to the scale of test equipment. For example, there are many cases in which the cost for depreciation or the running cost corresponding to the scale of test equipment should be taken into consideration.

In this paper, a new cost model is presented which is applicable to such cases as stated above and includes Epstein's cost model as its special case. Moreover, it is proved that the replacement procedure may happen to be more preferable than the nonreplacement procedure based on our new model, which is different from the result obtaind by Riley. The optimal procedures (the optimum sample sizes) in our model are given in Table 1 and 2 for various values of parameters and the preassigned failure numbers.

2. Cost model

It is assumed that the time-to-failure distribution is exponential:

$$F(x)=1-\exp(-x/\theta), \qquad x>0,\ \theta>0. \tag{1}$$

Then, it is well known that the accuracy of the estimate based on the first r failure observations depends only on r, and not on n, the number of items placed on test, for a fixed mean life θ. Therefore the optimum procedures, for a given accuracy, are determined by the numbers of items simultaneously placed on test which minimize the expected total costs for the nonreplacement or replacement procedures.

We shall denote the costs for the nonreplacement and replacement procedures by C_{NR} and C_R, respectively. Let n_1 and n_2 be the numbers of items simultaneously placed on test in the nonreplacement and replacement procedures, respectively, and let c_1 be the cost per item by placing an item on test, c_2 the cost per unit time of waiting for the unit scale of the test equipment. Then, it is assumed that the total costs C_{NR} and C_R are represented as

(2) $$C_{NR}=c_1n_1+c_2n_1^{\alpha}x_r, \quad \text{for the nonreplacement case}$$

and

(3) $$C_R=c_1(n_2+r-1)+c_2n_2^{\alpha}x_r, \quad \text{for the replacement case,}$$

where r denotes the preassigned failure number, x_r the time of occurence of the rth failure, and α is some positive constant less than 1. (If $\alpha=0$, then this cost model is identical to the one presented by B. Epstein. However, from our standpoint, it is assumed that α is positive. Moreover, it is reasonable for α to be less than one from the economic point of view.) The expected total costs are represented as

(4) $$\bar{C}_{NR}=c_1n_1+c_2n_1^{\alpha}\theta\sum_{j=1}^{r}\frac{1}{n_1-j+1}=c_2\theta\left\{\beta n_1+n_1^{\alpha}\sum_{j=1}^{r}\frac{1}{n_1-j+1}\right\}$$

and

(5) $$\bar{C}_R=c_1(n_2+r-1)+c_2n_2^{\alpha}\frac{r\theta}{n_2}=c_2\theta\{\beta(n_2+r-1)+rn_2^{\alpha-1}\},$$

where $\beta=c_1/c_2\theta$. By differentiating the right-hand side of (5) with respect to n_2, we can obtain the minimum expected total cost $\bar{C}_R^*$ and the corresponding optimum n_2^* for fixed values of α, β and r. In practice, we obtain

(6) $$n_2^*=[r(1-\alpha)/\beta]^{1/(2-\alpha)}$$

and

(7) $$\bar{C}_R^*=c_2\theta\beta\left\{n_2^*\left(1+\frac{1}{1-\alpha}\right)+(r-1)\right\}.$$

However, it is rather complicated to obtain the optimum n_1^* and the minimum expected total cost $\bar{C}_{NR}^*$ from the relation (4) analytically. (But their approximate values are obtained by making use of the formula

(8) $$\sum_{k=1}^{n}\frac{1}{k}\doteqdot\log\left(n+\frac{1}{2}\right)+\gamma,$$

where γ is Euler's constant.) We computed exactly the values of the right-hand side of (4) disregarding the constant factor $c_2\theta$, and obtained the optimum n_1^* and minimum cost $\bar{C}_R^*$ for various values of α, β and r. These values are shown in Tables 1 and 2.

3. Description of our assertion

We shall show that $\bar{C}_R^*$ can be smaller than $\bar{C}_{NR}^*$ if the value of parameter α is sufficiently near to 1 and $r \geqq 2$. (In case $r=1$, $\bar{C}_R^* \equiv \bar{C}_{NR}^*$ holds for any values of α and β.) At first, the following lemmas will be proved.

LEMMA 1. *For any positive integers $r(\geqq 2)$, $n^*(\geqq r)$ and any positive number α less than 1, there exists a positive number β such that $\bar{C}_{NR}$ attains its minimum $\bar{C}_{NR}^*$ when $n_1 = n^*$.*

PROOF. Differentiating $\bar{C}_{NR}$, given in (4), with respect to n_1, we obtain

$$(9) \qquad \frac{d\bar{C}_{NR}}{dn_1} = c_2\theta n_1^\alpha \left\{ \frac{\beta}{n_1^\alpha} + \frac{\alpha}{n_1} \sum_{j=1}^{r} \frac{1}{n_1 - j + 1} - \sum_{j=1}^{r} \frac{1}{(n_1 - j + 1)^2} \right\}.$$

Then, it is easily seen from (9) that $\bar{C}_{NR}$ attains its minimum when $n_1 = n^*$ if the value of β is given by

$$(10) \qquad \beta = n^{*\alpha} \left\{ \sum_{j=1}^{r} \frac{1}{(n^* - j + 1)^2} - \frac{\alpha}{n^*} \sum_{j=1}^{r} \frac{1}{n^* - j + 1} \right\},$$

which is always positive for any positive number $\alpha(<1)$, any positive integers $r(\geqq 2)$ and $n^*(\geqq r)$.

LEMMA 2. *For two fixed positive integers $r(\geqq 2)$ and $n(\geqq r)$, the inequalities*

$$(11) \qquad \frac{1}{r} \sum_{j=1}^{r} \left(1 - \frac{j-1}{n}\right)^{-1} < \left(1 - \frac{r-1}{n}\right)^{\alpha - 1}, \qquad 0 \leqq \alpha < \delta_{r,n}$$
$$(>) \qquad (\delta_{r,n} < \alpha \leqq 1),$$

hold for a suitably chosen positive number $\delta_{r,n}$ $(0 < \delta_{r,n} < 1)$.

PROOF. Putting $\alpha = 0$ or 1, the inequalities (11) are rewritten as

$$(12) \qquad \frac{1}{r} \sum_{j=1}^{r} \left(1 - \frac{j-1}{n}\right)^{-1} < \left(1 - \frac{r-1}{n}\right)^{-1}, \qquad \text{when } \alpha = 0,$$

and

$$(13) \qquad \frac{1}{r} \sum_{j=1}^{r} \left(1 - \frac{j-1}{n}\right)^{-1} > 1, \qquad \text{when } \alpha = 1,$$

which hold clearly for any positive integers $r(\geqq 2)$ and $n(\geqq r)$. Since the right-hand side of (11) is continuous and monotone decreasing func-

TABLE 1. Values of n_1^*, n_2^* and
for $\alpha=1/2$ ($\beta=1/500$, 1/200,

r \ β		n_i^*	1/500	n_i^*	1/200	n_i^*	1/100
2	NR	64	0.3800	35	0.5180	23	0.6565
	R	63	0.3800	34	0.5180	22	0.6564
3	NR	84	0.4993	47	0.6822	30	0.8671
	R	83	0.4993	45	0.6822	28	0.8670
4	NR	—	——	57	0.8294	37	1.0561
	R	97	0.6060	54	0.8203	34	1.0560
5	NR	—	——	67	0.9649	44	1.2306
	R	112	0.7042	63	0.9649	40	1.2306
6	NR	—	——	76	1.0920	50	1.3943
	R	126	0.7962	71	1.0921	45	1.3944
7	NR	—	——	85	1.2125	56	1.5498
	R	139	0.8833	79	1.2126	50	1.5500
8	NR	—	——	93	1.3276	61	1.6984
	R	152	0.9664	86	1.3277	54	1.6987
9	NR	—	——	—	——	67	1.8413
	R	164	1.0462	93	1.4383	59	1.8417
10	NR	—	——	—	——	72	1.9794
	R	175	1.1232	100	1.5450	63	1.9799
11	NR	—	——	—	——	77	2.1132
	R	186	1.1977	107	1.6483	67	2.1139
12	NR	—	——	—	——	—	——
	R	197	1.2701	113	1.7489	71	2.2441
13	NR	—	——	—	——	—	——
	R	207	1.3405	119	1.8467	75	2.3711
14	NR	—	——	—	——	—	——
	R	218	1.4111	125	1.9422	79	2.4951
15	NR	—	——	—	——	—	——
	R	227	1.4782	131	2.0356	83	2.6165
16	NR	—	——	—	——	—	——
	R	237	1.5419	137	2.1560	86	2.7353
17	NR	—	——	—	——	—	——
	R	246	1.6063	142	2.2166	90	2.8520
18	NR	—	——	—	——	—	——
	R	256	1.6694	148	2.3046	93	2.9665
19	NR	—	——	—	——	—	——
	R	265	1.7314	153	2.3911	97	3.0792
20	NR	—	——	—	——	—	——
	R	273	1.7924	159	2.4761	100	3.1900

(Remark) Values of n_1^* larger than 100 have not been calculated,

corresponding values of $\bar{C}^*_{NR}/c_2\theta$, $\bar{C}^*_R/c_2\theta$
1/100, 1/10, 1, 10, 100 and $r=2\sim20$)

n_i*	1/10	n_i*	1	n_i*	10	n_i*	100
6	1.4981	2	4.1213	2	22.121	2	202.12
5	1.4944	1	4.0000	1	22.000	1	202.00
8	2.0290	4	6.1667	3	33.175	3	303.18
6	2.0247	1	6.0000	1	33.000	1	303.00
10	2.5163	5	7.8696	4	44.167	4	404.17
7	2.5119	2	7.8284	1	44.000	1	404.00
13	2.9668	6	9.5518	5	55.106	5	505.11
9	3.0667	2	9.5355	1	55.000	1	505.00
15	3.3949	7	11.2143	6	66.001	6	606.00
10	3.3974	2	11.2426	1	66.000	1	606.00
17	3.8052	8	12.8588	7	76.860	7	706.86
11	3.8106	3	13.0415	1	77.000	1	707.00
19	4.2009	9	14.4869	8	87.687	8	807.69
12	4.2094	3	14.6188	1	88.000	1	808.00
20	4.5843	11	16.0409	9	98.487	9	908.49
13	4.5962	3	16.1962	1	99.000	1	909.00
22	4.9561	12	17.5537	10	109.262	10	1009.26
14	4.9726	3	17.7735	1	110.000	1	1010.00
24	5.3189	13	19.0578	11	120.016	11	1110.02
14	5.4399	3	19.3509	1	121.000	1	1111.00
—	——	14	20.5537	12	130.750	12	1210.75
15	5.6984	3	20.9282	1	132.000	1	1212.00
—	——	15	22.0420	13	141.466	13	1311.47
16	6.0500	3	22.5056	1	143.000	1	1313.00
—	——	16	23.5229	14	152.166	14	1412.17
17	6.3955	4	24.0000	1	154.000	1	1414.00
—	——	17	24.9970	15	162.852	15	1512.85
18	6.7355	4	25.5000	1	165.000	1	1515.00
—	——	18	26.4645	16	173.523	16	1613.52
19	6.0707	4	27.0000	1	176.000	1	1616.00
—	——	20	27.8907	17	184.182	17	1714.18
19	7.4001	4	28.5000	1	187.000	1	1717.00
—	——	21	29.3037	18	194.829	18	1814.83
20	7.7249	4	30.0000	1	198.000	1	1818.00
—	——	22	30.7124	19	205.464	19	1915.46
21	8.0461	4	31.5000	1	209.000	1	1919.00
—	——	23	32.1167	20	216.090	20	2016.09
22	8.3640	5	32.9443	1	220.000	1	2020.00

and this corresponding places are blank in Table 1 and 2.

TABLE 2. Values of n_1^*, n_2^* for $\alpha=2/3$ ($\beta=1/500$, 1/200,

r \ β		n_i^*	1/500	n_i^*	1/200	n_i^*	1/100
2	NR	—	—	41	0.7923	25	0.9482
	R	78	0.6261	39	0.7898	23	0.9433
3	NR	—	—	56	1.0786	35	1.2947
	R	106	0.8499	53	1.0737	32	1.2849
4	NR	—	—	69	1.3371	44	1.6138
	R	131	1.0556	66	1.3348	39	1.5995
5	NR	—	—	84	1.5899	52	1.9143
	R	155	1.2488	78	1.5802	46	1.8955
6	NR	—	—	97	1.8258	60	2.2007
	R	178	1.4326	89	1.8439	53	2.1773
7	NR	—	—	—	—	68	2.4758
	R	200	1.6090	100	2.0381	60	2.4481
8	NR	—	—	—	—	76	2.7418
	R	221	1.7792	111	2.2546	66	2.7096
9	NR	—	—	—	—	84	3.0000
	R	241	1.9442	121	2.4646	72	2.9634
10	NR	—	—	—	—	91	3.2515
	R	261	2.1048	131	2.6690	78	3.2105
11	NR	—	—	—	—	—	—
	R	280	2.2614	141	2.8684	84	3.4517
12	NR	—	—	—	—	—	—
	R	299	2.4146	150	3.0635	89	3.6877
13	NR	—	—	—	—	—	—
	R	318	2.5646	160	3.2546	95	3.9191
14	NR	—	—	—	—	—	—
	R	336	2.7118	169	3.4422	100	4.1462
15	NR	—	—	—	—	—	—
	R	354	2.8564	178	3.6266	106	4.3695
16	NR	—	—	—	—	—	—
	R	371	2.9987	187	3.8080	111	4.5892
17	NR	—	—	—	—	—	—
	R	388	3.1388	195	3.9866	116	4.8058
18	NR	—	—	—	—	—	—
	R	405	3.2769	204	4.1627	121	5.0192
19	NR	—	—	—	—	—	—
	R	422	3.4131	212	4.3365	126	5.2299
20	NR	—	—	—	—	—	—
	R	439	3.5475	221	4.5080	131	5.4380

and the corresponding values of $\bar{C}^*_{NR}/c_2\theta$, $\bar{C}^*_R/c_2\theta$
1/100, 1/10, 1, 10, 100 and $r=2\sim20$)

n_i^*	1/10	n_i^*	1	n_i^*	10	n_i^*	100
6	1.8107	2	4.3811	2	22.381	2	202.38
4	1.7599	1	4.0000	1	22.000	1	202.00
8	2.5381	4	6.7298	3	33.814	3	303.81
6	2.4510	1	6.0000	1	33.000	1	303.00
11	3.2121	5	8.7525	4	35.250	4	405.25
7	3.0910	1	8.0000	1	44.000	1	404.00
14	3.8548	6	10.7878	5	56.677	5	506.68
8	3.7000	2	9.9685	1	55.000	1	505.00
16	4.4685	7	12.8288	6	68.090	6	608.09
10	4.2850	2	11.7622	1	66.000	1	606.00
18	5.0640	9	14.7501	7	79.488	7	709.48
11	4.8475	2	13.5559	1	77.000	1	707.00
21	5.6411	10	16.6327	8	90.871	8	810.87
12	5.3943	2	15.3496	1	88.000	1	808.00
23	6.2041	11	18.5175	9	102.240	9	912.24
13	5.9276	2	17.1433	1	99.000	1	909.00
25	6.7555	12	20.4032	10	113.595	10	1013.60
14	6.4491	3	18.9336	1	110.000	1	1010.00
28	7.2963	14	22.2382	11	124.937	11	1114.94
15	6.9603	3	20.6270	1	121.000	1	1111.00
30	7.8263	15	24.0314	12	136.265	12	1216.27
16	7.4622	3	22.3203	1	132.000	1	1212.00
32	8.3481	16	25.8254	13	147.582	13	1317.58
17	7.9559	3	24.0137	1	143.000	4	1313.00
34	8.8624	17	27.6195	14	158.888	14	1418.89
18	8.4420	3	25.7071	1	154.000	1	1414.00
36	9.3697	18	29.4136	15	170.182	15	1520.18
19	8.9214	3	27.4004	1	165.000	1	1515.00
38	9.8707	20	31.1582	16	181.406	16	1621.47
20	9.3944	4	29.0794	1	176.000	1	1616.00
40	10.3656	21	32.8896	17	192.741	17	1722.74
21	9.8618	4	30.7093	1	187.000	1	1717.00
43	10.8547	22	34.6210	18	204.005	18	1824.01
22	10.3239	4	32.3393	1	198.000	1	1818.00
45	11.3385	23	36.3523	19	215.261	19	1925.26
22	10.7808	4	33.9693	1	209.000	1	1919.00
47	11.8175	24	38.0832	20	226.508	20	2026.51
23	11.2327	4	35.5992	1	220.000	1	2020.00

tion in α for fixed r and n, it is easily seen that for any positive integers $r(\geqq 2)$ and $n(\geqq r)$ there exists a (unique) positive number $\delta_{r,n}$ such that the inequalities (11) hold. More explicitly, $\delta_{r,n}$ is given by

$$\delta_{r,n}=1+\log\left[\frac{1}{r}\sum_{j=1}^{r}\left(1-\frac{j-1}{n}\right)^{-1}\right]\Big/\log\left(1-\frac{r-1}{n}\right). \tag{14}$$

LEMMA 3. *For a fixed positive integer $r(\geqq 2)$ and for all positive integers $n(\geqq r)$, the inequalities*

$$\frac{1}{r}\sum_{j=1}^{r}\left(1-\frac{j-1}{n}\right)^{-1}<\left(1-\frac{r-1}{n}\right)^{\alpha-1}, \qquad 0\leqq\alpha<\underline{\delta}_r \tag{15}$$

$$(>) \qquad (\bar{\delta}_r<\alpha\leqq 1),$$

hold for some positive numbers $\underline{\delta}_r$ and $\bar{\delta}_r$.

PROOF. By Putting

$$\underline{\delta}_r=\inf_n \delta_{r,n}, \qquad \bar{\delta}_r=\sup_n \delta_{r,n},$$

where $\delta_{r,n}$ is given in (14), it is easily seen from lemma 2 that the assertion of this lemma is valid.

THEOREM 1. *Let $r(\geqq 2)$ and $n^*(\geqq r)$ be any positive integers. Then there exist infinitely many pairs of α and β such that the inequalities*

$$\bar{C}_{NR}^*>\bar{C}_R^*, \qquad \bar{C}_{NR}^*<\bar{C}_R^*, \tag{16}$$

hold and that $\bar{C}_{NR}(\bar{C}_R)$ attains its minimum at $n_1=n^*(n_2=n_2^*=n^*-r+1)$. For example, if α satisfies $\delta_{r,n}^*<\alpha\leqq 1$ $(0\leqq\alpha<\delta_{r,n}^*)$, then (16) necessarily holds, where

$$\delta_{r,n}^*=1+\log\left[\frac{1}{r}\sum_{j=1}^{r}\left(1-\frac{j-1}{n^*}\right)\right]\Big/\log\left(1-\frac{r-1}{n^*}\right). \tag{17}$$

PROOF. By the above definition of n^*, we obtain from (4)

$$\bar{C}_{NR}^*=c_2\theta\left\{\beta n^*+n^{*\alpha-1}\sum_{j=1}^{r}\left(1-\frac{j-1}{n^*}\right)^{-1}\right\}. \tag{18}$$

(The existence of β satisfying (18) is seen by lemma 1.) By putting $n_2=n^*-r+1$ in (5), we obtain

$$\bar{C}_R=c_2\theta\left\{\beta n^*+n^{*\alpha-1}r\left(1-\frac{r-1}{n^*}\right)^{\alpha-1}\right\}. \tag{19}$$

Comparing (17) with (18), it is readily seen that $\bar{C}_R^*\leqq\bar{C}_R<\bar{C}_{NR}^*$ hold, only if the inequality

(20)
$$\sum_{j=1}^{r}\left(1-\frac{j-1}{n^*}\right)^{-1}>r\left(1-\frac{r-1}{n^*}\right)^{\alpha-1}$$

holds. By lemma 2, the inequality (20) holds if $\delta^*_{r,n}<\alpha\leqq 1$, where $\delta^*_{r,n}$ is given in (17). The inequality $\bar{C}^*_{NR}<\bar{C}^*_{R}$ can be proved in the same way as stated above.

THEOREM 2. *Let $r(\leqq 2)$ be any positive integer. Then there exist infinitely many pairs of α and β such that the inequalities (16) hold. Especialy, if α satisfies $\bar{\delta}^*_r<\alpha\leqq 1$ $(0\leqq\alpha<\underline{\delta}^*_r)$, then (16) necessarily hold, where*

(21)
$$\underline{\delta}^*_r=\inf_n \delta^*_{r,n}\,,\qquad \bar{\delta}^*_r=\sup_n \delta^*_{r,n}\,,$$

$\delta^*_{r,n}$ being given in (17).

PROOF. This theorem is obtained as an immediate consequence of theorem 1 and lemma 3.

Remark 1. Since it is easily seen after some calculations that the inequalities

(22)
$$1-\frac{\log (r+1)/2}{\log r}\leqq\underline{\delta}^*_r\leqq\bar{\delta}^*_r\leqq 1-\frac{1}{r}$$

hold, we obtain the following corollary:

Suppose that r, α and β are given. A sufficient condition that $\bar{C}^*_{NR}>\bar{C}^*_{R}$ is $\alpha>1-(1/r)$. A sufficient condition that $\bar{C}^*_{NR}<\bar{C}^*_{R}$ is that $\alpha<1-\log[(r+1)/2]/\log r$.

Remark 2. In the special case that $r=2$,

$$1-\frac{1}{r}=0.5,\ 1-\frac{\log (r+1)/2}{\log r}=0.432.$$

Therefore, $\bar{C}^*_{NR}>\bar{C}^*_{R}$ whenever $0.5<\alpha\leqq 1$, and $\bar{C}^*_{NR}>\bar{C}^*_{R}$ whenever $0<\alpha<0.432$.

4. Conclusion

Under the cost model, presented in this paper, the optimal procedures (the optimum sample sizes and the minimum expected costs) in estimating mean life θ have been investigated and the relating tables are given for the nonreplacement and replacement cases. Roughly speaking, it is shown that the nonreplacement procedure is more preferable than the replacement procedure when the parameter α is sufficiently near to zero, and that the replacement procedure is more preferable than the nonreplacement procedure when α is sufficiently near to one.

In practical applications, the parameters α and β must be determined at first, and then the optimal procedure can be determined based on our model. It seems to us that α is determined to be near to one in most applications, from the economic point of view. Then the replacement procedure is more preferable than the nonreplacement procedure in such cases.

If the mean life θ is not constant parameter but a random variable subject to some prior distribution, then it is necessary to replace θ by the expected mean life $\bar{\theta}$ in the above discussion.

We shall treat in the near future the optimal procedure for testing mean life θ by introducing the terminal loss in addition to the sampling loss stated above as the cost model. It is decisively important, we believe, in the life test procedures to obtain the optimum sample size and the failure number minimizing the total expected loss under the suitably chosen loss function.

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